CONSISTENT EXTENSIONS OF LINEAR FUNCTIONALS AND OF PROBABILITY MEASURES

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1. Introduction

Suppose we are given a family of fields $\mathscr{F}_\alpha$, $\alpha \in \mathscr{A}$, of subsets of a fixed set X, and for each α a finite measure m_α defined on $\mathscr{F}_\alpha$. Under what conditions will there exist a measure m, defined on a field $\mathscr{F}$ containing each $\mathscr{F}_\alpha$, and agreeing with each m_α on $\mathscr{F}_\alpha$? This problem is of some importance in probability theory (it arises in the theory of marginals), and has been studied, for instance, in [3], [5], [7], [14], [16]. The problem takes several forms: the measures considered can be positive or signed, dominated by a previously given measure or not, and countably or finitely additive. The countably additive cases are the most significant, but also the most difficult (see [1], [4], [8]); the most far reaching results here seem to be those of Kellerer [5], [6]. However, the finitely additive cases are also of interest (in fact, the case in which X itself is finite is of significance; see [16]), and the present paper deals almost entirely with them.

Our main object is to give a unified, simple, general treatment of the finitely additive cases, allowing arbitrarily many measures m_α (in the literature, only finitely many are usually considered). From this we shall obtain one theorem providing a countably additive extension, under additional topological assumptions. We shall also deal with a second general problem (see [16]): under what conditions on the fields $\mathscr{F}_\alpha$ will *every* consistent set of measures have a (finitely additive) consistent extension?

It is convenient to reformulate the finitely additive problems in a slightly more general form, which reduces them to linear algebra. Let L denote the set of all real valued functions on X; for each $\alpha \in \mathscr{A}$, let L_α be the subset of L consisting of all $\mathscr{F}_\alpha$ measurable step functions on X, and for each $f \in L_\alpha$, let

$$\phi_\alpha(f) = \int_X f \, dm_\alpha. \tag{1.1}$$

Then the measures m_α have a common extension to a (finitely additive) measure m if and only if the linear maps $\phi_\alpha: L_\alpha \to R$ have a common extension to a linear map ϕ of a linear space containing each L_α (which, as we shall see, may as well be taken to be L itself). Thus, throughout most of what follows, we shall be concerned with the problem of extending the linear maps ϕ_α. Since it makes things no harder, we permit the L_α to be arbitrary linear subspaces of an arbitrary linear space L, and allow the values of the linear maps ϕ_α to be in an arbitrary linear space M. For simplicity, all linear spaces are understood to be over the real numbers as groundfield (though in Sections 2 through 5 any other field would do as well; in Sections 6 and 7 we would require an ordered field); and the letters λ, μ, ν are reserved for real numbers, and δ, ε for positive real numbers.

The extension problem in its simplest form is dealt with in Section 2 (the content of which is essentially known). In Sections 3 through 5, we consider the second (and harder) problem in the form: under what conditions on the linear spaces L_α will *every* consistent system of linear maps ϕ_α have a common extension? We show that our results here apply to marginals (in the finitely additive signed case). In Section 6, we return to the first problem for the case of *positive* measures (reformulated in terms of positive linear maps), and we also cover the case in which the (positive) measure is to be dominated by a previously given measure. In Section 7, we deal with the second problem for positive measures, and for positive dominated ones. Finally, in Section 8, we consider briefly the positive, countably additive case. Throughout, all measures considered will be *finite*.

2. Extension of consistent linear maps

Let $\mathscr{L} = \{L_\alpha | \alpha \in \mathscr{A}\}$ be a nonempty family of linear subspaces of a linear space L and, for each $\alpha \in \mathscr{A}$, let ϕ_α be a linear map of L_α in a *fixed* linear space M. We say that the maps ϕ_α, $\alpha \in \mathscr{A}$, are *extendable* (or, more accurately, form an extendable family) if there exists a linear map $\phi: L \to M$ such that, for each $\alpha \in \mathscr{A}$, the restriction $\phi | L_\alpha = \phi_\alpha$. Obviously, a necessary condition for this is that the maps ϕ_α are *consistent* (that is, $\phi_\alpha | L_\alpha \cap L_\beta = \phi_\beta | L_\alpha \cap L_\beta$ for all $\alpha, \beta \in \mathscr{A}$) and we assume the consistency of the maps ϕ_α throughout what follows. They then combine to give a well-defined (but in general not linear!) map $\phi_U: U \to M$, where

$$U = \bigcup \mathscr{L} = \bigcup \{L_\alpha | \alpha \in \mathscr{A}\}. \tag{2.1}$$

We write $S = S(U)$ for the linear subspace of L spanned by U; generally, if $A \subset L$, $S(A)$ denotes the linear subspace of L spanned by A.

The question of when the consistent family $\{\phi_\alpha | \alpha \in \mathscr{A}\}$ is extendable has the following simple and essentially known answer (see [5], Sätze 2.1 and 2.2, and [12]).

THEOREM 2.1. *The following statements are equivalent (for a consistent family of linear maps ϕ_α, $\alpha \in \mathscr{A}$):*

(i) *the maps ϕ_α, $\alpha \in \mathscr{A}$, are extendable;*

(ii) *ϕ_U can be extended to a linear map $\phi: L \to M$;*

(iii) *ϕ_U can be extended to a linear map $\psi; S \to M$;*

(iv) *whenever $x_1, x_2, \cdots, x_n \in U$ and $\Sigma_{i=1}^n x_i = 0$, then $\Sigma_{i=1}^n \phi_U(x_i) = 0$.*

PROOF. Obviously (i) and (ii) are equivalent, and (ii) → (iii) → (iv). It is an easy matter of routine to verify that (iv) → (iii), while, finally, the implication (iii) → (ii) is the "pre-Hahn-Banach" theorem (see, for instance, [15], p. 140).

COROLLARY 2.1. *If* $|\mathscr{A}| = 2$, *then every consistent pair of maps* ϕ_1, ϕ_2 *is extendable.*

In fact, condition (iv) holds automatically in this case. For, suppose $x_i \in U = L_1 \cup L_2$ for $i = 1, 2, \cdots, n$, and that $x_1 + x_2 + \cdots + x_n = 0$. Write

$$y_1 = \sum \{x_i | x_i \in L_1\}, \qquad y_2 = \sum \{x_i | x_i \notin L_1\} \tag{2.2}$$

(with the usual convention that empty sums are 0). Then $y_j \in L_j$, $j = 1, 2$, and $y_1 + y_2 = 0$. Hence, $y_2 = -y_1 \in L_1$, so

$$\sum \{\phi_U(x_i) | i = 1, 2, \cdots, n\} = \phi_1(y_1) + \phi_2(y_2) = \phi_1(y_1 + y_2) = 0, \tag{2.3}$$

as required.

However, when $|\mathscr{A}| \geqq 3$, not every consistent system is extendable; in other words, condition (iv) is not vacuous. This is shown by the following simple example.

EXAMPLE 2.1. Let L be two dimensional Euclidean space R^2, M the real line R, and $\mathscr{L} = \{L_1, L_2, L_3\}$, where L_1, L_2, L_3 are the one dimensional subspaces of L spanned by the vectors (1, 0), (0, 1), (1, 1), respectively. Define ϕ_1 and ϕ_2 to be the constant maps sending L_1 and L_2 to 0, and define $\phi_3(x, x) = x$ $(x \in R)$. These maps are consistent, but clearly the only linear extension of ϕ_1 and ϕ_2 to L is 0, which disagrees with ϕ_3.

In the next sections we consider the less trivial problem: under what conditions on $\mathscr{L}$ is *every* consistent system $\{\phi_\alpha | \alpha \in \mathscr{A}\}$ of linear maps extendable?

3. $\mathscr{L}$ equivalence

3.1. We say that the family $\mathscr{L} = \{L_\alpha | \alpha \in \mathscr{A}\}$ of linear subspaces of the linear space L has the "M extension property," where M is a linear space, provided that every consistent system of linear maps $\phi_\alpha \colon L_\alpha \to M$ is extendable. It turns out that this property is independent of M (providing $M \neq \{0\}$); once this is proved, we shall refer to it merely as the "extension property." In this section, we introduce and study an equivalence relation which we shall use to characterize the M extension property. (For a completely different characterization in a special case, see [16].)

Notation. Let $\mathscr{F}$ denote the set of all finite sequences $(x_1, x_2, \cdots, x_n)$, where $x_i \in U = \cup \mathscr{L}$ for each $i = 1, 2, \cdots, n$ and where $n = 1, 2, \cdots$. If $F = (x_1, \cdots, x_n) \in \mathscr{F}$ and $\lambda \in R$, then λF denotes the sequence $(\lambda x_1, \cdots, \lambda x_n)$. We abbreviate $(-1)F$ to $-F$ as usual. If $F = (x_1, \cdots, x_n)$ and $G = (y_1, \cdots, y_m)$, then (F, G) denotes the sequence $(x_1, \cdots, x_n, y_1, \cdots, y_m)$. We define ΣF, where $F = (x_1, \cdots, x_n) \in \mathscr{F}$, to be the element $x_1 + x_2 + \cdots + x_n$ of L.

The elements $x_1, \cdots, x_n$ of L are said to be "$\mathscr{L}$ *related*" provided that there exists some $L_\alpha \in \mathscr{L}$ which contains all of them. Finally, a map $\phi \colon A \to M$, where

$A \subset L$, is called "$\mathscr{L}$ *linear*" provided that: whenever $x_1, \cdots, x_n$ are $\mathscr{L}$ related elements of A and $\lambda_1, \cdots, \lambda_n$ are real numbers such that $\Sigma_{i=1}^n \lambda_i x_i = 0$, then $\Sigma_{i=1}^n \lambda_i \phi(x_i) = 0$. Note that ϕ_U, in Section 2, is automatically $\mathscr{L}$ linear; conversely, every $\mathscr{L}$ linear map $\phi: U \to M$, where $U = \cup \mathscr{L}$, is expressible in the form ϕ_U for a consistent family of linear maps $\phi_\alpha: L_\alpha \to M$ (namely, $\phi_\alpha = \phi | L_\alpha$). Thus, in what follows, the $\mathscr{L}$ linearity of ϕ_U enables us to forget about the individual linear maps ϕ_α.

3.2. Now consider the following relations on $\mathscr{F}$: FR_1G means that G is a permutation of F; FR_2G means that $F = (x_1, \cdots, x_{n-1}, x_n)$ and $G = (x_1, \cdots, x_{n-1}, y_1, \cdots, y_r)$, where $x_n, y_1, \cdots, y_r$ are $\mathscr{L}$ related and $x_n = y_1 + \cdots + y_r$. Here n, r are arbitrary positive integers. FR_3G means that GR_2F.

We say that F, G are $\mathscr{L}$ *equivalent*, and write $F \sim G$, provided there exists a finite sequence $H_1, H_2, \cdots, H_m \in \mathscr{F}$ (where $m \geqq 2$) such that $F = H_1$, $G = H_m$, and, for each $i = 1, \cdots, m - 1$, we have $H_iR_jH_{i+1}$ for some $j = 1, 2, 3$. Clearly $\mathscr{L}$ equivalence is just the equivalence relation generated by R_1 and R_2.

We shall make frequent use of the following easily verified properties of $\mathscr{L}$ equivalence, where it is understood that $F, G, H \in \mathscr{F}$ and $\lambda \in R$:

(a) if $F = (x_1, \cdots, x_n)$, and $x_{i_1}, x_{i_2}, \cdots, x_{i_r}$ are $\mathscr{L}$ related and $x_{j_1}, \cdots, x_{j_s}$ are the other elements of F, then $F \sim (x_{j_1}, \cdots, x_{j_s}, x_{i_1} + x_{i_2} + \cdots + x_{i_r})$;

(b) if $F \sim G$ then $\lambda F \sim \lambda G$;

(c) $\big(F, (0, 0, \cdots, 0)\big) \sim F$;

(d) $(F, G) \sim (G, F)$;

(e) if $F_i \sim G_i$, $i = 1, \cdots, m$, then $(F_1, F_2, \cdots, F_m) \sim (G_1, G_2, \cdots, G_m)$;

(f) if $F \sim G$, then $\Sigma F = \Sigma G$ (the converse is *not* true; see Theorem 3.1 and Example 5.2);

(g) $F \sim G$ if and only if $(F, -G) \sim (0)$; in particular, $(H, -H) \sim (0)$ always;

(h) if $(F, H) \sim (G, H)$, then $F \sim G$ (and conversely, by (e)).

Of these assertions, only (g) and (h) need be proved here. We first establish the special case of (g), that $(H, -H) \sim (0)$. Suppose $H = (h_1, \cdots, h_n)$; then

$$
\begin{aligned}
(3.1)\qquad & (H, -H)R_1(h_1, -h_1, h_2, -h_2, \cdots, h_n, -h_n) \\
& R_3(h_1, -h_1, \cdots, h_{n-1}, -h_{n-1}, 0) \\
& R_3(h_1, -h_1, \cdots, h_{n-1}, -h_{n-1}) \cdots R_3(h_1, -h_1)R_3(0).
\end{aligned}
$$

From (e), it now follows that

(i) $(F, H, -H) \sim F$.

Now if $F \sim G$ then, from (e), $(F, -G) \sim (G, -G) \sim (0)$, from the special case of (g) just established. Conversely, if $(F, -G) \sim (0)$, then by (i) we have $F \sim (F, -G, G) \sim (0, G) \sim G$, by (e) and (c). This proves (g), and (h) follows easily from (g) and (i).

The following lemma permits us to express the $\mathscr{L}$ linearity of ϕ_U in terms of $\mathscr{L}$ equivalence.

LEMMA 3.1. *Let ψ be a map from $U = \cup \mathscr{L}$ to M. The following statements are equivalent:*

(i) *ψ is $\mathscr{L}$ linear;*

(ii) *if $(\lambda_1 x_1, \lambda_2 x_2, \cdots, \lambda_n x_n) \sim (0)$, where $x_i \in U$ and $\lambda_i \in R$, $i = 1, 2, \cdots, n$, then $\Sigma_{i=1}^n \lambda_i \psi(x_i) = 0$;*

(iii) *if $(\lambda_1 x_1, \cdots, \lambda_n x_n) \sim (\mu_1 y_1, \cdots, \mu_m y_m)$, where the x and the y are in U and the λ and the μ are real, then $\Sigma_{i=1}^n \lambda_i \psi(x_i) = \Sigma_{j=1}^m \mu_j \psi(y_j)$.*

Suppose (ii) holds, and suppose that $x_1, \cdots, x_n$ are $\mathscr{L}$ related elements of U and $\lambda_1, \cdots, \lambda_n$ are real numbers such that $\Sigma_i \lambda_i x_i = 0$. Then $\lambda_1 x_1, \cdots, \lambda_n x_n$ are also $\mathscr{L}$ related; hence, $(\lambda_1 x_1, \cdots, \lambda_n x_n) R_3 (0)$, so (ii) gives $\Sigma_i \lambda_i \psi(x_i) = 0$, and (i) follows.

If (i) holds, we easily check for each of the three relations $R_j, j = 1, 2, 3$, that if $(\lambda_1 x_1, \cdots, \lambda_n x_n) R_j (\mu_1 y_1, \cdots, \mu_m y_m)$, then $\Sigma_i \lambda_i \psi(x_i) = \Sigma_j \mu_j \psi(y_j)$; and from this (iii) follows.

Finally, the implication (iii) $\rightarrow$ (ii) is trivial.

We can now state the first main theorem.

THEOREM 3.1. *The following assertions are equivalent:*

(i) *$\mathscr{L}$ has the M extension property, for some $M \neq \{0\}$;*

(ii) *$\mathscr{L}$ has the M extension property for all M;*

(iii) *for each $F \in \mathscr{F}$ such that $\Sigma F = 0$, we have $F \sim (0)$.*

REMARKS. The converse of (iii) always holds by (f), so (iii) could be restated: $\Sigma F = 0$ is equivalent to $F \sim (0)$. It has already been shown (Example 2.1) that (ii) does not hold automatically, in general; thus, the requirement in (iii) is not vacuous. However, it follows from Corollary 2.1 that, when $|\mathscr{A}| = 2$, then $\{L_1, L_2\}$ always has the M extension property for all M, so that (iii) must hold automatically in this case.

PROOF. Assume (iii) holds, and let $x_1, \cdots, x_n$ be elements of U such that $x_1 + \cdots + x_n = 0$. Then $(x_1, \cdots, x_n) \sim (0)$, by (iii). Since ϕ_U is $\mathscr{L}$ linear, Lemma 3.1 (ii) (with all the $\lambda = 1$) shows $\Sigma_i \phi_U(x_i) = 0$. By Theorem 2.1, ϕ_U can be extended to a linear map $\phi: L \rightarrow M$, so (ii) follows.

The implication (ii) $\rightarrow$ (i) being trivial, all that remains is to prove (i) $\rightarrow$ (iii). This we do in the next section, after developing some further machinery.

4. $\mathscr{L}$ span and $\mathscr{L}S$ linearity

4.1. To complete the proof of Theorem 3.1, we introduce two further notions. First, if A is a nonempty subset of $U = \cup \mathscr{L}$, its "$\mathscr{L}$ span" $\mathscr{L}S(A)$ is the set of all $y \in U$ such that, for some $x_1, \cdots, x_n \in A$ and $\lambda_1, \cdots, \lambda_n \in R$ (and some positive integer n), $(y) \sim (\lambda_1 x_1, \cdots, \lambda_n x_n)$. The following properties of the $\mathscr{L}$ span are easily verified:

(j) $A \subset \mathscr{L}S(A) \subset S(A) \cap U$ (from (f));

(k) if $y \in \mathscr{L}S(A)$ and $\lambda \in R$, then $\lambda y \in \mathscr{L}S(A)$ (from (b));

(l) if $x_1, \cdots, x_n$ are $\mathscr{L}$ related elements of $\mathscr{L}S(A)$, then $x_1 + \cdots + x_n \in \mathscr{L}S(A)$ (from (a) and (e)).

Together, (k) and (l) give:

(m) for each $\alpha \in \mathscr{A}$, $L_\alpha \cap \mathscr{L}S(A)$ is a linear subspace of L_α;

(n) $\mathscr{L}S(\mathscr{L}S(A)) = \mathscr{L}S(A)$ (from (j) and (e)).

Caution. $\mathscr{L}S(A)$ is *not*, in general, a linear space; it will be so only when it coincides with the span $S(A)$.

4.2. Next, with A as above, suppose ψ is a map of A into the linear space M. We say that ψ is *$\mathscr{L}S$ linear* (on A) if, whenever $x_1, \cdots, x_n \in A$ and $\lambda_1, \cdots, \lambda_n \in R$ are such that $(\lambda_1 x_1, \cdots, \lambda_n x_n) \sim (0)$, then $\Sigma_{i=1}^n \lambda_i \psi(x_i) = 0$. The following easily proved properties will be relevant:

(o) if ψ is the restriction to A of a linear map of L into M, then ψ is $\mathscr{L}S$ linear;

(p) if ψ is $\mathscr{L}S$ linear, then ψ is $\mathscr{L}$ linear (from (a)); the converse of (p) holds when $A = U$, from Lemma 3.1, but not in general;

(q) if ψ and θ are $\mathscr{L}S$ linear on A, then so is $\lambda\psi + \mu\theta$, $\lambda, \mu \in R$;

(r) if $0 \in A$ and ψ is $\mathscr{L}S$ linear, then $\psi(0) = 0$.

LEMMA 4.1. *If $\varnothing \neq A \subset U$, and if $\psi: A \to M$ is $\mathscr{L}S$ linear, then ψ can be extended to an $\mathscr{L}S$ linear map Ψ: $\mathscr{L}S(A) \to M$.*

If $y \in \mathscr{L}S(A)$, we have $(y) \sim (\lambda_1 x_1, \cdots, \lambda_n x_n)$, where the x are in A (and the λ in R). Define

$$\Psi(y) = \sum_{i=1}^n \lambda_i \psi(x_i). \tag{4.1}$$

This is well defined, for if we also have $(y) \sim (\mu_1 z_1, \cdots, \mu_m z_m)$, where the z are in A, then by (g) we have $(\lambda_1 x_1, \cdots, \lambda_n x_n, -\mu_1 z_1, \cdots, -\mu_m z_m) \sim (0)$, so that the $\mathscr{L}S$ linearity of ψ shows that $\Sigma_i \lambda_i \psi(x_i) = \Sigma_j \mu_j \psi(z_j)$.

To verify that Ψ is $\mathscr{L}S$ linear on $\mathscr{L}S(A)$, suppose $y_1, \cdots, y_k \in \mathscr{L}S(A)$ and $(\mu_1 y_1, \cdots, \mu_k y_k) \sim (0)$. For each $i = 1, \cdots, k$, we have $(y_i) \sim (\lambda_{i,1} x_{i,1}, \cdots, \lambda_{i,n_i} x_{i,n_i})$, where the x are in A, and $\Sigma_i \mu_i \Psi(y_i) = \Sigma_{i,j} \mu_i \lambda_{i,j} \psi(x_{i,j})$. But, by (e),

$$\begin{aligned}(\mu_1 \lambda_{1,1} x_{1,1}, \cdots, \mu_1 \lambda_{1,n_1} x_{1,n_1}, \mu_2 \lambda_{2,1} x_{2,1}, \cdots, \mu_k \lambda_{k,n_k} x_{k,n_k})& \\ \sim (\mu_1 y_1, \cdots, \mu_k y_k) \sim (0);&\end{aligned} \tag{4.2}$$

hence, the fact that ψ is $\mathscr{L}S$ linear on A shows that $\Sigma_i \mu_i \Psi(y_i) = 0$, as required.

REMARK. It is easily seen that the $\mathscr{L}S$ linear extension Ψ here is *unique*.

LEMMA 4.2. *Let A be a nonempty subset of U, and suppose $x_0 \in U \backslash \mathscr{L}S(A)$, $m_0 \in M$. Write $B = A \cup \{x_0\}$, and let ψ be any $\mathscr{L}S$ linear map of A into M. Then the map $\psi^*: B \to M$, defined to agree with ψ on A and to make $\psi^*(x_0) = m_0$, is $\mathscr{L}S$ linear on B.*

Suppose $y_1, \cdots, y_k \in B$ and $(\lambda_1 y_1, \cdots, \lambda_k y_k) \sim (0)$; we must show that $\Sigma_i \lambda_i \psi^*(y_i) = 0$. If none of the y is x_0, this follows from the $\mathscr{L}S$ linearity of ψ. If all of them are x_0, we see from (f) that $(\lambda_1 + \cdots + \lambda_k) x_0 = 0$; but $x_0 \neq 0$ since $x_0 \notin \mathscr{L}S(A)$; thus, $\Sigma_i \lambda_i = 0$ and again the result follows. So we may assume, without loss of generality, that $y_1, \cdots, y_r$ are different from x_0 and that $y_{r+1}, \cdots, y_k$ all $= x_0$, where $0 < r < k$. Then we have

$$(0) \sim (\lambda_1 y_1, \cdots, \lambda_k y_k) \sim (\lambda_1 y_1, \cdots, \lambda_r y_r, \mu x_0), \tag{4.3}$$

where $\mu = \lambda_{r+1} + \cdots + \lambda_k$. If $\mu \neq 0$, an application of (b) and (g) would make $(x_0) \sim (\nu_1 y_1, \cdots, \nu_r y_r)$, where $\nu_i = -\lambda_i/\mu$, so that $x_0 \in \mathscr{L}S(A)$, a contradiction. So $\mu = 0$, and it follows by (c) that $(\lambda_1 y_1, \cdots, \lambda_r y_r) \sim (0)$. Thus,

$$(4.4) \qquad \lambda_1\psi^*(y_1) + \cdots + \lambda_k\psi^*(y_k) = \lambda_1\psi(y_1) + \cdots + \lambda_r\psi(y_r) + \mu m_0 = 0,$$

as required.

LEMMA 4.3. *Let ψ be an $\mathscr{L}S$ linear map of A into M, where $\varnothing \neq A \subset U$. Then there exists an $\mathscr{L}S$ linear map Ψ of U into M which extends ψ. Moreover, given $x_0 \in U\backslash\mathscr{L}S(A)$ and $m_0 \in M$, we may choose Ψ so that $\Psi(x_0) = m_0$.*

For, using Zorn's lemma, we obtain a maximal $\mathscr{L}S$ linear extension Ψ of ψ, mapping, say, B in M, where $A \subset B \subset U$ and B is maximal. By Lemma 4.2, $\mathscr{L}S(B) = U$; by Lemma 4.1, $B = \mathscr{L}S(B)$; hence, $B = U$ as asserted.

PROOF OF THEOREM 3.1, CONCLUDED. Suppose that Theorem 3.1 (iii) is false (we must show that Theorem 3.1 (i) is also false). There is then a smallest positive integer n for which n elements $x_1, \cdots, x_n$ of U exist such that $x_1 + \cdots + x_n = 0$ but $(x_1, \cdots, x_n)$ is *not* $\sim(0)$. Clearly, $n \geqq 2$ here (it is not hard to see that in fact $n \geqq 3$). We first show:

(1) every $n - 1$ of $x_1, \cdots, x_n$ are linearly independent.

For if, say, $x_1 = \lambda_2 x_2 + \cdots + \lambda_{n-1}x_{n-1}$, write $y_i = (1 + \lambda_i)x_i$ for $i = 2, 3, \cdots, n - 1$, and $y_n = x_n$. Then each $y_i \in U$ and $y_2 + \cdots + y_n = 0$. Because of the minimality of n, it follows that $(y_2, \cdots, y_n) \sim (0)$. Again, write $z_1 = x_1$ and $z_i = -\lambda_i x_i$ for $i = 2, \cdots, n - 1$; a similar argument shows that $(z_1, \cdots, z_{n-1}) \sim (0)$. Thus, by (g), we have

$$(4.5) \qquad (z_1, z_2, \cdots, z_{n-1}, y_2, \cdots, y_{n-1}, y_n) \sim (0).$$

Now, for $i = 2, \cdots, n - 1$, we observe that y_i and z_i are $\mathscr{L}$ related and $y_i + z_i = x_i$. Thus, from the foregoing $\mathscr{L}$ equivalence we deduce $(x_1, \cdots, x_{n-1}, x_n) \sim (0)$, a contradiction; and (1) is established.

Next we show that, for all real $\lambda_1, \cdots, \lambda_{n-1}$,

(2) $(\lambda_1 x_1, \cdots, \lambda_{n-1}x_{n-1}, x_n)$ is *not* $\sim(0)$.

For if (2) were false, (f) would give $\lambda_1 x_1 + \cdots + \lambda_{n-1}x_{n-1} + x_n = 0$ and, hence, $\Sigma_{i=1}^{n-1} (\lambda_i - 1)x_i = 0$. This contradicts (1) unless $\lambda_1 = \lambda_2 = \cdots = \lambda_{n-1} = 1$, but in that case (2) holds by hypothesis.

Now put $A = \{x_1, x_2, \cdots, x_{n-1}\}$, a nonempty subset of U. From (2), $x_n \notin \mathscr{L}S(A)$. Choose any linear map $\phi: L \to M$ (for instance, the zero map), and any $m_0 \in M$ different from $\phi(x_n)$ (this is where we use the hypothesis $M \neq \{0\}$). Let ψ be the restriction $\phi\,|A$; by (o). ψ is $\mathscr{L}S$ linear on A. By Lemma 4.3 there exists an $\mathscr{L}S$ linear map $\Psi: U \to M$ which extends ψ, and which is such that $\Psi(x_n) = m_0$. In particular, by (p), Ψ is $\mathscr{L}$ linear on U, so that if Theorem 3.1 (i) holds, Ψ has an extension to a linear map $\Psi^*: L \to M$. Since $x_1 + \cdots + x_n = 0$, we see that

$$(4.6) \qquad \phi(x_n) = -\sum_{i=1}^{n-1} \phi(x_i) = -\sum_{i=1}^{n-1} \Psi^*(x_i) = \Psi^*(x_n) = \Psi(x_n) = m_0,$$

a contradiction.

5. Simpler conditions

5.1. As Theorem 3.1 shows, if the family $\mathscr{L}$ of linear subspaces of the linear space L has the M extension property for *one* nontrivial linear space M, it has this property for *all* and, accordingly, we refer to it as "the extension property" in what follows. (Note also that, by Theorem 2.1, the extension property is in a sense independent of L, too, since L can be replaced by $S(\bigcup \mathscr{L})$.) The characterization of the extension property provided by Theorem 3.1 is somewhat complicated (though, as we shall see, quite usable), and in the present section we consider two plausible simplifications of it. It turns out that one of these is necessary and the other is sufficient, and that no more is true in general. We conclude the section by applying the simpler sufficient condition to the problem of marginals (in the case of finitely additive signed measures).

THEOREM 5.1. *Consider the following statements.*

(i) *Whenever* $L_0, L_1, \cdots, L_k \in \mathscr{L}$ (*where* k *is a positive integer*), $L_0 \cap S(L_1 \cup \cdots \cup L_k) = S((L_0 \cap L_1) \cup \cdots \cup (L_0 \cap L_k))$.

(ii) $\mathscr{L}$ *has the extension property.*

(iii) *For each* $\alpha \in \mathscr{A}$,

$$L_\alpha \cap S(\bigcup \{L_\beta \mid \beta \in \mathscr{A}\backslash\{\alpha\}\}) = S(L_\alpha \cap \bigcup \{L_\beta \mid \beta \in \mathscr{A}\backslash\{\alpha\}\}). \tag{5.1}$$

Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii); *but in general neither implication is reversible.*

PROOF. For the proof that (i) $\Rightarrow$ (ii), suppose that (i) holds but (ii) fails; then, by Theorem 3.1, there exist $x_0, x_1, \cdots, x_k \in U$ such that $x_0 + \cdots + x_k = 0$ and $(x_0, \cdots, x_k)$ is not $\sim (0)$. We may suppose that k here is as small as possible; note that $k \geqq 2$. We have $x_i \in L_{\alpha(i)}$ for some $\alpha(i) \in \mathscr{A}$, $i = 0, \cdots, k$, and then

$$x_0 = -(x_1 + \cdots + x_k) \in L_{\alpha(0)} \cap S(L_{\alpha(1)} \cup \cdots \cup L_{\alpha(k)}). \tag{5.2}$$

From (i) we can, therefore, write $x_0 = y_1 + \cdots + y_k$, where $y_i \in L_{\alpha(0)} \cap L_{\alpha(i)}$, $i = 1, \cdots, k$. Thus, $x_0, y_1, \cdots, y_k$ are $\mathscr{L}$ related, and we have

(1) $(x_0, -y_1, \cdots, -y_k) \sim (0)$.

Again, for each $i = 1, \cdots, k$, we have that x_i, y_i are $\mathscr{L}$ related, and $(x_1 + y_1) + \cdots + (x_k + y_k) = 0$. The minimality of k shows that $(x_1 + y_1, \cdots, x_k + y_k) \sim (0)$, from which we obtain

(2) $(x_1, \cdots, x_k, y_1, \cdots, y_k) \sim (0)$.

By applying (c) and (e) to (1) and (2), we obtain

$$(x_1, \cdots, x_k, y_1, \cdots, y_k, -y_1, \cdots, -y_k, x_0) \sim (0), \tag{5.3}$$

and thence, $(x_0, x_1, \cdots, x_n) \sim (0)$, a contradiction.

For (ii) $\Rightarrow$ (iii), assume that Theorem 5.1 (ii) holds, and write

$$\begin{aligned} M_\alpha &= L_\alpha \cap S(\bigcup \{L_\beta \mid \beta \in \mathscr{A}\backslash\{\alpha\}\}), \\ N_\alpha &= S(L_\alpha \cap \bigcup \{L_\beta \mid \beta \in \mathscr{A} - \{\alpha\}\}); \end{aligned} \tag{5.4}$$

we must prove $M_\alpha = N_\alpha$, and since $N_\alpha \subset M_\alpha$, trivially, it is enough to prove $M_\alpha \subset N_\alpha$.

It will be convenient to introduce the following notation. For each $F = (x_1, \cdots, x_k) \in \mathscr{F}$ and $\alpha \in \mathscr{A}$, write

$$\Sigma_\alpha(F) = \Sigma\,\{x_i \mid x_i \in L_\alpha, i = 1, 2, \cdots, k\}. \tag{5.5}$$

Clearly (for F, $G \in \mathscr{F}$), we have
(3) $\Sigma_\alpha(F, -G) = \Sigma_\alpha(F) - \Sigma_\alpha(G)$.
We shall show:
(4) if $F \sim G$, then $\Sigma_\alpha(F, -G) \in N_\alpha$.

Because of (3), it is enough to verify (4) in the three cases FR_jG, $j = 1, 2, 3$, separately (see Section 3.2). The case $j = 1$ (permutation) is trivial. If FR_2G, say $F = (x_1, \cdots, x_k)$, $G = (x_1, \cdots, x_{k-1}, x_{k+1}, \cdots, x_{k+r})$, where $x_k = x_{k+1} + \cdots + x_{k+r}$ and the elements x_{k+i}, $i = 0, 1, \cdots, r$, are $\mathscr{L}$ related; say they all belong to L_β. If $\beta = \alpha$, clearly, $\Sigma_\alpha(F, -G) = 0 \in N_\alpha$; and if $\beta \neq \alpha$, it is also easy to see that $\Sigma_\alpha(F, -G) \in N_\alpha$. This verifies (4) in the R_2 case; and the R_3 case follows. Thus (4) is established.

Now suppose $x \in M_\alpha$; then $x \in L_\alpha$ and there exist $y_1, \cdots, y_n$ such that $x = y_1 + \cdots + y_n$, $y_i \in L_{\beta(i)}$, and $\beta(i) \neq \alpha$, $i = 1, 2, \cdots, n$. We may suppose the y so numbered that $y_1, \cdots y_r \notin L_\alpha$ and $y_{r+1}, \cdots, y_n \in L_\alpha$, where $0 \leqq r \leqq n$. Clearly, $y_{r+1} + \cdots + y_n \in N_\alpha$, so if $z = y_1 + \cdots + y_r$ it will be enough to prove that $z \in N_\alpha$ also. Since $\mathscr{L}$ has the extension property, Theorem 3.1 (iii) shows that $(z, -y_1, \cdots, -y_r) \sim (0)$. Thus, by (4), $\Sigma_\alpha(z, -y_1, \cdots, -y_r) = 0$. That is, $z = 0$, and $x \in N_\alpha$, as required.

Example 5.1. To see that (ii) does not imply (i) in Theorem 5.1, let $\mathscr{L}$ be any system of linear subspaces L_α of a linear space L, which does not have the extension property (for instance, Example 2.1), and consider $\mathscr{L}^* = \mathscr{L} \cup \{L\}$. Trivially, $\mathscr{L}^*$ has the extension property (ii) (for $L \in \mathscr{L}^*$). But if (i) were true of $\mathscr{L}^*$, it would be true of $\mathscr{L}$ also, so that (from Theorem 5.1) $\mathscr{L}$ would have the extension property, a contradiction.

Example 5.2. To see that (iii) does not imply (ii) in Theorem 5.1, take $L = R^3$ and let $\mathscr{L}$ consist of the lines L_1, L_2, L_3 spanned by the vectors $(1, 0, 0)$, $(0, 1, 0)$, $(1, 1, 0)$, respectively, together with the planes L_4, L_5, L_6, where L_{i+3} is spanned by L_i and $(0, 0, 1)$, $i = 1, 2, 3$. One readily verifies that (iii) holds; but the extension property fails for the same reason as in Example 2.1.

5.2. *Application.* Consider a product space $X = \Pi\,\{X_a \mid a \in A\}$. For each $B \subset A$, we write $X_B = \Pi\,\{X_a \mid a \in B\}$ and $B' = A \backslash B$; thus X can be regarded as $X_B \times X_{B'}$. For $x \in X$, we write x_B for the projection of x in X_B; thus, we can write $x = (x_B, x_{B'})$. (With the usual convention that points of X are maps from A to $\bigcup_a X_a$, $x_B = x \mid B$.) A function $f\colon X \to R$ is a "cylinder function on B" if $f(x_B, x_{B'})$ is independent of $x_{B'}$.

Now suppose that a nonempty family $\mathscr{T}$ of subsets of A is given, and that for each $T \in \mathscr{T}$ a linear space L_T of real valued functions on X is given, in such a way that:

(s) L_T consists of cylinder functions on T, $T \in \mathscr{T}$, and

(t) given $T, T_1, \cdots, T_n \in \mathscr{T}$, and $f_i \in L_{T_i}$, $i = 1, \cdots, n$, there exists $y_{T'} \in X_{T'}$ such that the function $g_i: X \to R$, defined by $g_i(x) = f_i(x_T, y_{T'})$, is in $L_T \cap L_{T_i}$, $i = 1, \cdots, n$. (Note that, in (t), g_i is automatically a cylinder function on $T \cap T_i$.)

This situation arises, for example, in the theory of marginals (see [5]), where each X_a has a given field $\mathscr{B}_a$ of measurable subsets specified; we put $\mathscr{B}_T$ for the field (of subsets of X) generated by the cylinders on the sets in $\mathscr{B}_a$, $a \in T$, and L_T consists of all the $\mathscr{B}_T$ measurable T cylinder functions. (In this case, (t) is satisfied automatically by choosing $y_{T'}$ arbitrarily in $X_{T'}$.) The problem there is whether a consistent system of measures on the various spaces $(X_T, \mathscr{B}_T)$, $T \in \mathscr{T}$, can be extended to a measure on X. Kellerer ([5], Satz 2.2) shows that the answer is "yes" when we are dealing with a finite number of countably additive signed measures. The following theorem is a partial generalization; it allows the number $|\mathscr{T}|$ of fields to be arbitrary, but produces only a finitely additive (signed) measure.

THEOREM 5.2. *Under conditions* (s) and (t), *the family* $\{L_T | T \in \mathscr{T}\}$ *has the extension property.*

(As remarked in 5.1, it is not necessary to specify L precisely; any linear space containing all the L_T as subspaces would do, for instance, the space of all real functions on X.)

PROOF. We show that Theorem 5.1 (i) applies. Let $f \in L_T \cap S(L_{T_1} \cup \cdots \cup L_{T_k})$, where $T, T_i \in \mathscr{T}$, $i = 1, \cdots, k$. Then there exist functions $f_i \in L_{T_i}$ such that $f = f_1 + \cdots + f_k$. Applying condition (t), we obtain functions $g_i \in L_T \cap L_{T_i}$, where $g_i(x) = f_i(x_T, y_{T'})$. Clearly, $f = g_1 + \cdots + g_k$ (since f is a cylinder function on T). Thus $f \in S(L_T \cap L_{T_1} \cup \cdots \cup L_T \cap L_{T_k})$. This proves one inclusion between the sets in Theorem 5.1 (i); the reverse inclusion holds in any case, so the proof is complete.

6. Extending positive maps

6.1. *Positive extensions.* In order to cover the case of *positive* (that is, nonnegative) finitely additive measures, we introduce the notion of (partial) order into the previous setup (Section 2). Accordingly, we make the following standing assumptions. We suppose that L and M are given p.o. (partially ordered) linear spaces over the reals (for simplicity; any other ordered field would do). For the definition and basic properties of p.o. linear spaces, see for instance [11], Chapter 1. We assume that M is order complete (that is, every nonempty subset which is bounded above has a least upper bound), and that L has a given order unit e (that is, $e > 0$ and for each $x \in L$ there exists $\lambda \in R$ such that $\lambda e \geqq x$). (In the application to measures, as described in the introduction, where L consists of all real functions on the space X, e will be the constant function 1.) As before, we suppose given a family $\mathscr{L} = \{L_\alpha | \alpha \in \mathscr{A}\}$ of linear subspaces of $\mathscr{L}$, and for each $\alpha \in \mathscr{A}$ a linear map $\phi_\alpha: L_\alpha \to M$; but we now suppose that each ϕ_α is *positive* in the sense: if $x \in L_\alpha$ and $x \geqq 0$, then $\phi_\alpha(x) \geqq 0$. We say that the maps

ϕ_α are "*positively extendable*" if there exists a *positive* linear map $\phi: L \to M$ such that $\phi | L_\alpha = \phi_\alpha$, $\alpha \in \mathscr{A}$. As before, it is necessary for this that the maps ϕ_α be consistent, and we assume the consistency of the maps ϕ_α throughout what follows. Then they combine to give a well defined map $\phi_U: U \to M$, where $U = \cup \mathscr{L}$. Note that ϕ_U is now "*positively $\mathscr{L}$ linear*" in the sense: it is $\mathscr{L}$ linear (see *Notation*, Section 3.1), and its restriction to each L_α is positive.

The question of when the maps ϕ_α are positively extendable is answered by the following analog of Theorem 2.1.

THEOREM 6.1. *The following statements are equivalent (for a consistent family of positive linear maps ϕ_α, $\alpha \in \mathscr{A}$):*

(i) *the maps ϕ_α, $\alpha \in \mathscr{A}$, are positively extendable;*

(ii) *ϕ_U can be extended to a positive linear map $\phi: L \to M$;*

(iii) *ϕ_U can be extended to a positive linear map $\psi: S(U) \to M$;*

(iv) *whenever $x_1, x_2, \cdots, x_n \in U$ and $\Sigma_{i=1}^n x_i \geqq 0$, then $\Sigma_{i=1}^n \phi_U(x_i) \geqq 0$.*

REMARK. In the case of two maps $(|\mathscr{A}| = 2)$, this theorem is due to Guy [3]. For related results (especially in the case of marginals) see [7], [14], Theorem 7.1, and [16], p. 156. The condition (iv) has been used, in a somewhat different connection, by von Neumann ([10], p. 90). It should be remarked that, even in the case of marginals, where the existence of an extension for *signed* measures is automatic (see Section 5.2), some condition is needed to ensure the existence of a *positive* extension for positive measures; see [16], p. 147 for an example.

PROOF. Obviously (i) and (ii) are equivalent, and (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv). It is an easy matter of routine to verify that (iv) $\Rightarrow$ (iii) (note that (iv) here implies condition (iv) of Theorem 2.1). Finally, the implication (iii) $\Rightarrow$ (ii) follows easily from the following (essentially known) modification of the Hahn-Banach theorem.

LEMMA 6.1. *Suppose X, Y are p.o. linear spaces, where X has an order unit e and Y is order complete. Let X' be a linear subspace of X containing e, and let ϕ' be a positive linear map of X' in Y. Then ϕ' can be extended to a positive linear map of X in Y.*

When $Y = R$ this is in [9], p. 119; the argument carries over to the general case without difficulty.

EXAMPLE 6.1. We observe that, even when $|\mathscr{A}| = 2$, not every consistent system of positive linear maps ϕ_α, $\alpha \in \mathscr{A}$, is positively extendable—in other words, condition (iv) in Theorem 6.1 is not vacuous. Moreover, it can even happen that *two positive linear maps ϕ_1, ϕ_2 are extendable without being positively extendable.* To see this, take $L = R^3$, $M = R$, ordering L so that $(x_1, x_2, x_3) \geqq 0$ if and only if all of $x_1, x_2, x_3 \geqq 0$. Take $e = (1, 1, 1)$; let L_1 be the two dimensional subspace of L spanned by e and $(1, 1, 0)$; and let L_2 be the subspace spanned similarly by e and $(1, 0, 1)$. Define ϕ_1 on L_1 by $\phi_1(x, x, z) = 8x + 4z$, and ϕ_2 on L_2 by $\phi_2(x, y, x) = 3x + 9y$. Then ϕ_1 and ϕ_2 are consistent, and have a unique common linear extension ϕ to L, where $\phi(x, y, z) = -x + 9y + 4z$. Clearly, ϕ is not positive.

6.2. *Dominated extension.* Suppose that, in the setup described in 6.1, we are given a positive linear map $\theta: L \to M$ which "dominates" each ϕ_α in the sense: if $x \in L_\alpha$ and $x \geqq 0$, then $\theta(x) \geqq \phi_\alpha(x)$. Under what conditions will there exist a positive extension ϕ of the ϕ_α which is likewise dominated by θ (that is, $\theta(x) \geqq \phi(x)$ for all positive $x \in L$)? This question is answered by the expected modification of Theorem 6.1.

THEOREM 6.2. *The following statements are equivalent:*

(i) *the maps ϕ_α, $\alpha \in \mathscr{A}$, are positively extendable to a map dominated by θ;*

(ii) *ϕ_U can be extended to a positive linear map $\phi: L \to M$ which is dominated by θ;*

(iii) *ϕ_U can be extended to a positive linear map $\psi: S(U) \to M$ which is dominated by θ;*

(iv) *whenever $x_1, x_2, \cdots, x_n \in U$ and $\Sigma_{i=1}^n x_i \geqq 0$, then $0 \leqq \Sigma_{i=1}^n \phi_U(x_i) \leqq \Sigma_{i=1}^n \theta(x_i)$.*

REMARK. This is the analog of a characterization due to Kellerer ([5], Satz 4.3) in the case of marginals (with a finite number of countably additive measures).

The proof is essentially the same as that of Theorem 6.1. Instead of Lemma 6.1, we use the following variation of the Hahn-Banach theorem, proved by a slight sharpening of the proof of that lemma.

LEMMA 6.2. *Under the hypotheses of Lemma 6.1, suppose that a linear map $\theta: X \to Y$ is given such that $0 \leqq \phi'(x) \leqq \theta(x)$ for each positive $x \in X'$. Then ϕ' can be extended to a linear map $\phi: X \to Y$ such that $0 \leqq \phi(x) \leqq \theta(x)$ for each positive $x \in X$.*

7. The positive extension property

7.1. With the same assumptions and notation as in 6.1, we say that $\mathscr{L}$ has the *positive M extension property* providing every consistent family of positive linear maps $\phi_\alpha: L_\alpha \to M$ is positively extendable. As with the M extension property (3.1), it turns out that this is independent of M (if $M \neq \{0\}$), as well as (in a sense) of L (since L could be replaced by $S(U)$, from Theorem 6.1). Once this is proved (Theorem 7.1) we refer to it simply as the "positive extension property". First we need some further notation. If $F = (x_1, \cdots, x_n) \in \mathscr{F}$, we say that F is "*positive*" if there exists $G = (y_1, \cdots, y_m) \in \mathscr{F}$ such that $F \sim G$ and $y_j \geqq 0, j = 1, \cdots, m$. We shall make use of the following obvious properties:

(u) if x_i and y_i are $\mathscr{L}$ related and $x_i \geqq y_i$, $i = 1, \cdots, n$, and if $(y_1, \cdots, y_n)$ is positive, then so is $(x_1, \cdots, x_n)$;

(v) if $F_i \in \mathscr{F}$ and is positive, and if $\lambda_i \geqq 0, i = 1, \cdots, n$, then $(\lambda_1 F_1, \cdots, \lambda_n F_n)$ is positive;

(w) if $F \in \mathscr{F}$ is positive, then $\Sigma F \geqq 0$;

(x) if $F \sim G$ and G is positive, then so is F.

THEOREM 7.1. *Under the assumptions of* 6.1, *the following statements are equivalent:*

(i) *$\mathscr{L}$ has the positive M extension property, for some $M \neq \{0\}$;*

(ii) *$\mathscr{L}$ has the positive M extension property for every* (*order complete*) *M;*

(iii) *if $F \in \mathscr{F}$ and $\Sigma F \geqq 0$, then $(F, \varepsilon e)$ is positive for all $\varepsilon > 0$.*

REMARK. When L is finite dimensional, a characterization along entirely different lines has been given by Vorob'ev [16].

PROOF. Trivially (ii) $\Rightarrow$ (i). To prove (iii) $\Rightarrow$ (ii), suppose $\phi_U: U \to M$ is positively $\mathscr{L}$ linear (Section 6.1) and that (iii) holds. By Theorem 6.1 it is enough to prove that, given $x_1, \cdots, x_n \in U$ such that $\Sigma_i x_i \geqq 0$, then $\Sigma_i \phi_U(x_i) \geqq 0$. For each $\varepsilon > 0$, $(x_1, \cdots, x_n, \varepsilon e)$ is positive (by (iii)); that is, there exist $y_1, \cdots, y_m \in U$ such that each $y_j \geqq 0$ and $(x_1, \cdots, x_n, \varepsilon e) \sim (y_1, \cdots, y_m)$. Lemma 3.1 shows that

$$(7.1) \qquad \phi_U(x_1) + \cdots + \phi_U(x_n) + \varepsilon\phi_U(e) = \phi_U(y_1) + \cdots + \phi_U(y_m) \geqq 0.$$

Since this is true for all $\varepsilon > 0$, it follows that $\phi_U(x_1) + \cdots + \phi_U(x_n) \geqq 0$, as required.

Before completing the proof by showing that (i) $\Rightarrow$ (iii), we need some further notation and some lemmas.

7.2. Suppose ψ is a map of U into M. It is easy to see that

(1) ψ is positively $\mathscr{L}$ linear if and only if, whenever $x_1, \cdots, x_n$ are $\mathscr{L}$ related elements of U and $\lambda_1 x_1 + \cdots + \lambda_n x_n \geqq 0$, then $\lambda_1\psi(x_1) + \cdots + \lambda_n\psi(x_n) \geqq 0$.

Now suppose, more generally, that ψ is a map of A into M, where A is an arbitrary subset of U. We say that ψ is *positively $\mathscr{L}S$ linear* (on A) if, whenever $x_1, \cdots, x_n \in A$ and $(\lambda_1 x_1, \cdots, \lambda_n x_n)$ is positive, then $\lambda_1\psi(x_1) + \cdots + \lambda_n\psi(x_n) \geqq 0$.

When $A = U$, these notations are equivalent, as we state in Lemma 7.1.

LEMMA 7.1. *If $\psi: U \to M$ is positively $\mathscr{L}S$ linear* (*on U*), *then ψ is positively $\mathscr{L}$ linear; and conversely.*

For if $x_1, \cdots, x_n$ are as in (1), then $(\lambda_1 x_1, \cdots, \lambda_n x_n) \sim (\lambda_1 x_1 + \cdots + \lambda_n x_n)$ and is, therefore, positive; thus, the conclusion of (1) follows. The converse is proved by an argument similar to the proof of Lemma 3.1.

REMARK. If $\psi: A \to M$ is positively $\mathscr{L}S$ linear, then clearly ψ is both $\mathscr{L}S$ linear (Section 4.2) and positive on A. The converse is false in general, though it is true if $A = U$.

LEMMA 7.2. *If $\varnothing \neq A \subset U$, and if $\psi: A \to M$ is positively $\mathscr{L}S$ linear, then ψ can be extended to a positively $\mathscr{L}S$ linear map $\Psi: \mathscr{L}S(A) \to M$.*

By Lemma 4.1, ψ can be extended to an $\mathscr{L}S$ linear map $\Psi: \mathscr{L}S(A) \to M$. We verify that Ψ is positively $\mathscr{L}S$ linear. Suppose, then, that $x_1, \cdots, x_n \in \mathscr{L}S(A)$, and that $(\lambda_1 x_1, \cdots, \lambda_n x_n)$ is positive. Since $x_i \in \mathscr{L}S(A)$, we have $(x_i) \sim (\mu_{i,1}y_{i,1}, \cdots, \mu_{i,m_i}y_{i,m_i})$, $i = 1, \cdots, n$, where the y are in A; hence

$$(7.2) \qquad (\lambda_1 x_1, \cdots, \lambda_n x_n) \sim (\lambda_1\mu_{1,1}y_{1,1}, \cdots, \lambda_1\mu_{1,m_1}y_{1,m_1}, \cdots, \lambda_n\mu_{n,m_n}y_{n,m_n}),$$

which is, therefore, also positive. Because Ψ is $\mathscr{L}$ linear by (p), Lemma 3.1 gives

$$(7.3) \qquad \sum_i \lambda_i\Psi(x_i) = \sum_{i,j} \lambda_i\mu_{i,j}\Psi(y_{i,j}) = \sum_{i,j} \lambda_i\mu_{i,j}\psi(y_{i,j}) \geqq 0,$$

since ψ is positively $\mathscr{L}S$ linear, completing the proof.

REMARK. As in Lemma 4.1, the map Ψ here is unique.

LEMMA 7.3. *Suppose $e \in A \subset U$ and $x_0 \in U \backslash A$, and let ψ be a positively $\mathscr{L}S$ linear map of A into M. Then ψ can be extended to a positively $\mathscr{L}S$ linear map $\psi^*: B \to M$, where $B = A \cup \{x_0\}$.*

REMARK. In contrast to Lemma 4.2, the value of $\psi^*(x_0)$ here cannot be assigned arbitrarily, even if $x_0 \notin \mathscr{L}S(A)$.

PROOF. Because of Lemma 7.2, we may replace A by $\mathscr{L}S(A)$; this allows us to use a somewhat simpler notation in view of (k).

Let $\mathscr{G}$ be the set of all finite sequences $\{g_1, \cdots, g_n\}$ of elements of A for which $(g_1, \cdots, g_n, -x_0)$ is positive; similarly, let $\mathscr{H}$ be the set of all finite sequences $\{h_1, \cdots, h_m\}$ of elements of A for which $(-h_1, \cdots, -h_m, x_0)$ is positive. We first note that neither $\mathscr{G}$ nor $\mathscr{H}$ is empty. In fact, if N is a large enough integer, we have $Ne \geqq x_0$; also Ne and x_0 are $\mathscr{L}$ related (because each L_α contains e); hence $(Ne, -x_0) \sim (Ne - x_0)$ and is positive. Thus, the one term sequence $\{Ne\} \in \mathscr{G}$; and similarly $\{-Ne\} \in \mathscr{H}$.

Next we show that, for all $\{g_1, \cdots, g_n\} \in \mathscr{G}$ and $\{h_1, \cdots, h_m\} \in \mathscr{H}$,

(2) $\psi(g_1) + \cdots + \psi(g_n) \geqq \psi(h_1) + \cdots + \psi(h_m)$.

In fact, (v) shows that $(g_1, \cdots, g_n, -x_0, -h_1, \cdots, -h_m, x_0)$ is positive, and thus $(g_1, \cdots, g_n, -h_1, \cdots, -h_m)$ is also positive by (x). Our assumption on ψ now gives

$$\psi(g_1) + \cdots + \psi(g_n) - \psi(h_1) - \cdots - \psi(h_m) \geqq 0, \tag{7.4}$$

establishing (2). Define

$$\begin{aligned} q &= \inf \{\psi(g_1) + \cdots + \psi(g_n) | \{g_1, \cdots, g_n\} \in \mathscr{G}\}, \\ r &= \sup \{\psi(h_1) + \cdots + \psi(h_m) | \{h_1, \cdots, h_m\} \in \mathscr{H}\}, \end{aligned} \tag{7.5}$$

(using the fact that M is order complete). From (2), we see that q, r indeed exist and that $r \leqq q$. Choose any $m_0 \in M$ such that $r \leqq m_0 \leqq q$ (for instance, $m_0 = q$), and extend ψ to a map $\psi^*: B \to M$ by setting $\psi^*(x_0) = m_0$. We verify that ψ^* is positively $\mathscr{L}S$ linear.

Suppose, then, that $(\lambda_1 f_1, \cdots, \lambda_n f_n)$ is positive, where $f_1, \cdots, f_n \in B = A \cup \{x_0\}$. We must prove

(3) $\lambda_1 \psi^*(f_1) + \cdots + \lambda_n \psi^*(f_n) \geqq 0$.

If none of $f_1, \cdots, f_n$ is x_0, this follows from the assumption that ψ is positively $\mathscr{L}S$ linear (on A). If all of them are x_0, we have from (w) that $\lambda x_0 \geqq 0$, where $\lambda = \lambda_1 + \cdots + \lambda_n$, and have to prove that $\lambda \psi^*(x_0) \geqq 0$. If $\lambda = 0$ there is nothing to prove. If $\lambda > 0$, we have $x_0 \geqq 0$, so that $(0, x_0)$ is positive and $\{0\} \in \mathscr{H}$; thus $0 \leqq r \leqq m_0 = \psi^*(x_0)$, as required. Finally, if $\lambda < 0$ the argument is similar, using $\mathscr{G}$.

Thus, we may assume $f_1, \cdots, f_k$ different from x_0 and $f_{k+1}, \cdots, f_n$ all equal to x_0, where $1 \leqq k < n$. Write $\mu_0 = \lambda_{k+1} + \cdots + \lambda_n$; then

$$(\lambda_1 f_1, \cdots, \lambda_n f_n) \sim (\lambda_1 f_1, \cdots, \lambda_k f_k, \mu_0 x_0), \tag{7.6}$$

and thus

(4) $(\lambda_1 f_1, \cdots, \lambda_k f_k, \mu_0 x_0)$ is positive.

Again there are three cases, depending now on μ_0.

First, if $\mu_0 = 0$, (3) follows from (4) and the fact that ψ is positively $\mathscr{L}S$ linear on A.

Second, if $\mu_0 > 0$, write $\mu_i = -\lambda_i/\mu_0$, $i = 1, \cdots, k$; from (4), $(-\mu_1 f_1, \cdots, -\mu_k x_k, x_0)$ is positive, and thus $(\mu_1 f_1, \cdots, \mu_k f_k) \in \mathscr{H}$ (since each $\mu_i f_i \in A$; this is the point of replacing A by $\mathscr{L}S(A)$ at the beginning). Hence, $\mu_1 \psi(f_1) + \cdots + \mu_k \psi(f_k) \leqq r \leqq m_0 = \psi^*(x_0)$, from which (3) follows.

Third, if $\mu_0 < 0$, the argument is similar.

LEMMA 7.4. *Let A be a nonempty subset of U, and ψ a positively $\mathscr{L}S$ linear map of A into M. Then there exists a positively $\mathscr{L}$ linear map Ψ of U into M which extends ψ.*

This follows straightforwardly from the foregoing lemmas and Zorn's lemma.

7.3. Now we conclude the proof of Theorem 7.1. Assume that Theorem 7.1 (iii) fails; we show that Theorem 7.1 (i) fails with $M = R$.

For some positive integer n, there exist $f_1, \cdots, f_n \in U$ such that $f_1 + \cdots + f_n \geqq 0$, and $\varepsilon > 0$, such that $(f_1, \cdots, f_n, \varepsilon e)$ is not positive. We take n to be as small as possible; note that $n \geqq 2$ from (v). We are going to apply Lemma 7.4 to $A = \{f_1, \cdots, f_n, e\}$, but the definition of a suitable ψ requires some preparation.

Consider the subset K of R^{n+1} consisting of all points $p = (\lambda_1, \cdots \lambda_{n+1})$ for which $(\lambda_1 f_1, \cdots, \lambda_n f_n, \lambda_{n+1} e)$ is positive. It is easy to see, using (v), that K is a convex cone. Let q be the point $(1, \cdots, 1, 0) \in R^{n+1}$. We show that

(5) $q \notin \bar{K}$ (the closure of K).

For, take N to be an integer large enough to make $-Ne \leqq f_i \leqq Ne$, $i = 1, \cdots, n$, and put $\delta = \varepsilon/(Nn + 1)$. If (5) were false, there would exist $\mu_1, \cdots, \mu_{n+1}$ such that $|\mu_i| < \delta$, $i = 1, \cdots, n + 1$, and $(1 + \mu_1, \cdots, 1 + \mu_n, \mu_{n+1}) \in K$; that is,

(6) $F = ((1 + \mu_1)f_1, \cdots, (1 + \mu_n)f_n, \mu_{n+1} e)$ is positive.

Now $\delta Ne - \mu_i f_i \geqq 0$, $i = 1, \cdots, n$, from the choice of N and the fact that $e > 0$. Hence,

(7) $G = (\delta Ne - \mu_1 f_1, \cdots, \delta Ne - \mu_n f_n, (\delta - \mu_{n+1})e)$ is positive.

Accordingly, (F, G) is positive. But e and f_i are $\mathscr{L}$ related for each i; hence, we may perform an $\mathscr{L}$ equivalence on (F, G) by combining each pair $(1 + \mu_i)f_i$, $\delta Ne - \mu_i f_i$, into the single term $f_i + \delta Ne$. We can also replace the pair $\mu_{n+1} e$, $(\delta - \mu_{n+1})e$ by the single term δe. Thus, $(F, G) \sim H = (f_1 + \delta Ne, \cdots, f_n + \delta Ne, \delta e)$. Now we apply a further $\mathscr{L}$ equivalence to H, replacing the pair $f_1 + \delta Ne$, δe first by their sum, and then this sum by the pair f_1, $\delta(N + 1)e$. Applying this principle n times, we obtain $H \sim (f_1, \cdots, f_n, \delta(Nn + 1)e)$, which therefore $\sim (F, G)$ and is positive. From (u) and the choice of δ, it follows that $(f_1, \cdots, f_n, \varepsilon e)$ is positive, contrary to the hypothesis. Thus (5) is proved.

From (5), q and K can be separated by a hyperplane through the origin; that is, there exist real numbers $\theta_1, \cdots, \theta_{n+1}$ such that

(8) $\theta_1 + \cdots + \theta_n < 0 \leqq \theta_1\lambda_1 + \cdots + \theta_{n+1}\lambda_{n+1}$ for all $(\lambda_1, \cdots, \lambda_{n+1}) \in K$.

Now, because of the minimality of n, no two of $f_1, \cdots, f_n$ can be $\mathscr{L}$ related, for if f_1, f_2 were $\mathscr{L}$ related we could have started with the $n-1$ elements $f_1 + f_2, f_3, \cdots, f_n$ instead of $f_1, \cdots, f_n$. In particular, no two of them are equal, and (since $n \geqq 2$) none of them is e. Writing $f_{n+1} = e$, we have that $f_1, \cdots, f_{n+1}$ are all different. Thus, we can define a map $\psi: A \to R$, where $A = \{f_1, \cdots, f_{n+1}\}$, by: $\psi(f_i) = \theta_i$, $i = 1, \cdots, n+1$.

We shall show that ψ is positively $\mathscr{L}S$ linear. Suppose $x_1, \cdots, x_m \in A$ and that $(\mu_1 x_1, \cdots, \mu_m x_m)$ is positive; we must show

(9) $\mu_1\psi(x_1) + \cdots + \mu_m\psi(x_m) \geqq 0$.

For each $i = 1, \cdots, n+1$ write $J_i = \{j \mid 1 \leqq j \leqq m, x_j = f_i\}$; of course, J_i may be empty. Let $\lambda_i = \Sigma \{\mu_j \mid j \in J_i\}$ (with the convention that an empty sum is 0). Then

(10) $(\mu_1 x_1, \cdots, \mu_m x_m) \sim (\lambda_1 f_1, \cdots, \lambda_{n+1} f_{n+1})$,

because if $J_i \neq \varnothing$ we may apply an $\mathscr{L}$ equivalence (see (a)) to replace the terms $\mu_j x_j$, $j \in J_i$, i fixed, by their sum, $\lambda_i f_i$. (If $J_i = \varnothing$, we appeal instead to (c).)

Thus, $(\lambda_1 f_1, \cdots, \lambda_{n+1} f_{n+1})$ is positive, that is, $(\lambda_1, \cdots, \lambda_{n+1}) \in K$. By (8), $\lambda_1\psi(f_1) + \cdots + \lambda_{n+1}\psi(f_{n+1}) \geqq 0$, and therefore,

$$\sum_j \mu_j\psi(x_j) = \sum_i \sum \{\mu_j\psi(f_i) \mid j \in J_i\} = \sum_i \lambda_i\psi(f_i) \geqq 0, \tag{7.7}$$

as required.

By Lemma 7.4, ψ can be extended to a positively $\mathscr{L}$ linear map $\Psi: U \to R$. However, Ψ can *not* be extended to a positive linear map $\Psi^*: L \to R$; for such an extension would make $\Psi^*(f_1 + \cdots + f_n) = \theta_1 + \cdots + \theta_n < 0$, whereas $f_1 + \cdots + f_n \geqq 0$. This completes the proof of Theorem 7.1.

EXAMPLE 7.1. It is tempting to conjecture that, in Theorem 7.1, condition (iii) could be replaced by the simpler (and stronger) condition:

(iii′) *if* $F \in \mathscr{F}$ *and* $\Sigma F \geqq 0$, *then* F *is positive.*

However, this is not possible in general, even if $\mathscr{L}$ consists of only two linear spaces. This is shown by the following example. Take $L = m$, the space of all bounded sequences $x = \{\xi_1, \xi_2, \cdots\}$ of real numbers, ordered coordinatewise (that is, $x \geqq 0$ means $\xi_n \geqq 0$ for all $n = 1, 2, \cdots$). Put $e = \{1, 1, \cdots\}$ (this in an order unit), $a = \{1, 1/2, \cdots, 1/n, \cdots\}$, $b = \{2, 3/4, \cdots, 1/n + 1/n^2, \cdots\}$, and let L^0 denote the subspace of L consisting of the sequences which have only finitely many terms $\neq 0$. We define L_1 to be the span $S(L^0 \cup \{e\} \cup \{a\})$, and L_2 to be $S(L^0 \cup \{e\} \cup \{b\})$; $\mathscr{L}$ will consist of L_1 and L_2. The standing hypotheses in Section 6.1 are clearly satisfied.

We show that $\mathscr{L}$ has the positive extension property by verifying condition (iii) of Theorem 7.1. Suppose, then, that $x^1, x^2, \cdots, x^n \in U = L_1 \cup L_2$, that $x^1 + \cdots + x^n \geqq 0$, and that $\varepsilon > 0$; we must show that $(x^1, \cdots, x^n, \varepsilon e)$ is positive. Writing $y^1 =$ sum of the x^i in L_1, $y^2 =$ sum of the others, we see that this is equivalent to showing that $F = (y^1, y^2, \varepsilon e)$ is positive, where $y^1 \in L_1$, $y^2 \in L_2$, and $y^1 + y^2 \geqq 0$. We have $y^1 = \lambda_1 a + \mu_1 e + p$, $y^2 = \lambda_2 b + \mu_2 e + q$, where $p, q \in L^0$. Thus, $F \sim (\mu_1 e, \lambda_1 a, p, \mu_2 e, \lambda_2 b, q, \varepsilon e)$ and, therefore,

(1) $F \sim G = \big((\mu + \varepsilon)e, \lambda_1 a, \lambda_2 b, r\big)$, where $\mu = \mu_1 + \mu_2$ and $r = p + q \in L^0$.
Take a positive integer N so large that
(2) $\varepsilon N > 4(|\lambda_1| + |\lambda_2|)$,
(3) the nth coordinate of r is 0 for all $n \geqq N$.
Since $y^1 + y^2 \geqq 0$, we have
(4) $\mu + \lambda_1/n + \lambda_2(1/n + 1/n^2) \geqq 0$ for all $n \geqq N$.
Now for each $x = \{\xi_1, \xi_2, \cdots\} \in L$, write

$$x' = \{\xi_1, \cdots, \xi_N, 0, 0, \cdots\},$$
$$x'' = x - x' = \{0, \cdots, 0, \xi_{N+1}, \xi_{N+2}, \cdots\}. \tag{7.8}$$

Note that if $x \in L_j$, $j = 1, 2$, then $x' \in L^0 \subset L_1 \cap L_2$, $x'' \in L_j$, so that x, x', x'' are $\mathscr{L}$ related for each $x \in U$. Thus, on writing

$$s = r + (\mu + \varepsilon)e' + \lambda_1 a' + \lambda_2 b', \tag{7.9}$$

we obtain

(5) $G \sim H = \big((\mu + \varepsilon)e'', \lambda_1 a'', \lambda_2 b'', s\big)$ where $s \in L^0$.

Further, from (3), we have

(6) the nth coordinate of s is 0 for all $n \geqq N$.

Again, e'' and a'' are $\mathscr{L}$ related, as are e'' and b''. Thus, on writing $\delta = \varepsilon/2$, we obtain

(7) $H \sim K = (\mu e'', \delta e'' + \lambda_1 a'', \delta e'' + \lambda_2 b'', s)$.

We assert that each of the terms of K is $\geqq 0$, which then shows that F is positive, as required. By making $n \to \infty$ in (4), we see that $\mu \geqq 0$ and, thus, $\mu e'' \geqq 0$. From (2), it easily follows that $\delta e'' + \lambda_1 a''$ and $\delta e'' + \lambda_2 b''$ are $\geqq 0$. Finally, since $\Sigma K = \Sigma F \geqq 0$, we see that, in particular, each of the first N coordinates of s is $\geqq 0$. Thus, from (6), $s \geqq 0$, and F is positive.

Nevertheless, (iii′) does not hold. For consider $F = (-a, b)$. Clearly, $\Sigma F \geqq 0$, but we claim that F is not positive. For otherwise we obtain, as before, $F \sim (y^1, y^2)$, where $y^j \in L_j$ and $y^j \geqq 0$, $j = 1, 2$. As before, we write $y^1 = \lambda_1 a + \mu_1 e + p$, $y^2 = \lambda_2 b + \mu_2 e + q$, where $p, q \in L^0$. Thus, the nth coordinate if $y^1 + y^2$ is

$$\mu_1 + \mu_2 + \frac{\lambda_1 + \lambda_2}{n} + O(n^{-2}), \tag{7.10}$$

for large n. But $y^1 + y^2 = \Sigma F = b - a$, and its nth coordinate is $O(n^{-2})$. This proves that $\mu_1 + \mu_2 = 0 = \lambda_1 + \lambda_2$. However, the fact that $y^1 \geqq 0$ shows that $\mu_1 \geqq 0$, and $\lambda_1 \geqq 0$ if $\mu_1 = 0$; similarly, $\mu_2 \geqq 0$, and $\lambda_2 \geqq 0$ if $\mu_2 = 0$. Thus, we have $\lambda_1 = \lambda_2 = \mu_1 = \mu_2 = 0$, so that $b - a = p + q$. But the nth coordinate of $p + q$ is 0 if n is large enough, while that of $b - a$ is not, giving the desired contradiction.

It would be interesting to know whether (iii′) can replace (iii), in Theorem 7.1, when L is finite dimensional.

EXAMPLE 7.2. There is no implication between the extension property and the positive extension property, in general. That the extension property does not imply the positive extension property has been remarked in Theorem 6.1; see also Example 6.1 and the remark in Theorem 3.1. What is perhaps more surprising is that the positive extension property does not imply the extension property. This is shown by the following example.

Let L, L^0, e have the same meaning as in Example 7.1, and select three elements d^1, d^2, d^3 of L, no two of them linearly dependent, such that their nth coordinates do not equal zero but tend to zero as $n \to \infty$, and $d^1 + d^2 + d^3 = 0$; this is easily done. Define $L_j = S(L^0 \cup \{e\} \cup \{d^j\}), j = 1, 2, 3$; $\mathscr{L}$ will consist of L_1, L_2 and L_3. Consider the following linear maps $\phi_j \colon L_j \to R$, $j = 1, 2, 3$: $\phi_1(x) = 0$ for all $x \in L_1$, $\phi_2(x) = 0$ for all $x \in L_2$, and, for each $p \in L^0$, $\phi_3(\lambda p + \mu e + \nu d^3) = \nu$.

It is easy to see that $L_1 \cap L_2 = L_2 \cap L_3 = L_3 \cap L_1 = S(L^0 \cup \{e\})$, so that the maps ϕ_1, ϕ_2, ϕ_3 form a consistent system. However, they have no common linear extension because $d^1 + d^2 + d^3 = 0$, whereas $\phi_1(d^1) + \phi_2(d^2) + \phi_3(d^3) = 1$. Thus, the extension property fails here. Nevertheless, the positive extension property holds, as we now show by verifying condition (iii) of Theorem 7.1.

Suppose $H \in \mathscr{F}$ and $\Sigma H \geqq 0$. As before we may assume $H = (x^1, x^2, x^3)$, where $x^j \in L_j$; thus, $x^j = \lambda_j p^j + \mu_j e + \nu_j d^j$, where $p^j \in L^0$, $j = 1, 2, 3$. Since $x^1 + x^2 + x^3 \geqq 0$, we see (by considering the nth coordinates and making $n \to \infty$) that $\mu_1 + \mu_2 + \mu_3 = \mu$, say, $\geqq 0$. Given $\varepsilon > 0$, put $\delta = \varepsilon/3$, and pick N large enough. We use the same decomposition $x = x' + x''$ as in Example 7.1; thus $p^{j\prime\prime} = 0, j = 1, 2, 3$. The same reasoning as in Example 7.1 will show that

$$(H, \varepsilon e) \sim (\mu e'', \delta e'' + \lambda_1 d^{1\prime\prime}, \delta e'' + \lambda_2 d^{2\prime\prime}, \delta e'' + \lambda_3 d^{3\prime\prime}, s), \tag{7.11}$$

where each of the terms appearing on the right is $\geqq 0$, as required.

7.4. *Dominated positive extension.*

THEOREM 7.2. *If $\mathscr{L}$ has the positive extension property, then $\mathscr{L}$ has the dominated positive extension property.*

By this we mean that if linear maps $\phi_\alpha \colon L_\alpha \to M$ and $\theta \colon L \to M$ are given such that $0 \leqq \phi(x) \leqq \theta(x)$ for each positive $x \in L_\alpha$, $\alpha \in \mathscr{A}$, and $0 \leqq \theta(x)$ for each positive $x \in L$, then the ϕ_α can be extended to a linear map $\phi \colon L \to M$ such that $0 \leqq \phi(x) \leqq \theta(x)$ for each positive $x \in L$.

By Theorem 6.2 it is enough to prove that, if $x_1, \cdots, x_n \in U$ and $\Sigma_i x_i \geqq 0$ then $0 \leqq \Sigma_i \phi_U(x_i) \leqq \Sigma_i \theta(x_i)$. By Theorem 7.1, we know that $(x_1, \cdots, x_n, \varepsilon e)$ is positive for each $\varepsilon > 0$, say $(x_1, \cdots, x_n, \varepsilon e) \sim (y_1, \cdots, y_m)$, where $y_j \in U$ and $y_j \geqq 0, j = 1, \cdots, m$. By Lemma 3.1, we have

$$\sum_i \phi_U(x_i) + \varepsilon \phi_U(e) = \sum_j \phi_U(y_j) \geqq 0, \tag{7.12}$$

and also

$$\sum_i \phi_U(x_i) + \varepsilon \phi_U(e) \leqq \sum_j \theta(y_j) = \sum_i \theta(x_i) + \varepsilon \theta(e). \tag{7.13}$$

On making $\varepsilon \to 0$, the result follows.

COROLLARY 7.1. *Suppose that L has the positive extension property, and that linear maps θ_*, $\theta^*: L \to M$ such that $\theta_*(x) \leqq \phi_\alpha(x) \leqq \theta^*(x)$ for each positive $x \in L_\alpha$, $\alpha \in \mathscr{A}$, are given. Then there exists a linear map $\phi: L \to M$, extending the ϕ_α, such that $\theta_*(x) \leqq \phi(x) \leqq \theta^*(x)$ for each positive $x \in L$.*

It would be interesting to know whether, conversely, the dominated positive extension property implies the positive extension property.

8. Countably additive measures

8.1. In this section we turn to the difficult but important problem of extending a consistent family of countably additive measures to a countably additive measure. We consider only positive finite measures.

There is a sense in which the preceding considerations suffice here, too, provided we are willing to enlarge the space X on which the measure is to be defined. We have obtained criteria which are necessary and sufficient for there to exist a *finitely* additive common extension, say m, on a field $\mathscr{F}$ of measurable sets. Now there always exists a space $\tilde{X} \supset X$, with a σ-field $\mathscr{B}$ of subsets of $\tilde{X}$ and a *countably* additive measure $\tilde{m}$ on $\mathscr{B}$, such that each $F \in \mathscr{F}$ is of the form $X \cap B$ for some $B \in \mathscr{B}$, and for each such B we have $\tilde{m}(B) = m(F)$. To see this, we take $\tilde{X}$ to be the Stone representation space of F (see [13], p. 24); the points of $\tilde{X}$ are the ultrafilters of F, and we identify $x \in X$ with the ultrafilter consisting of those elements of $\mathscr{F}$ which contain x. Each $F \in \mathscr{F}$ determines a set $\tilde{F} \subset \tilde{X}$ (consisting of those ultrafilters on $\mathscr{F}$ to which F belongs); the family of all such $\tilde{F}$ is a field $\mathscr{F}^*$ of subsets of X which is isomorphic to $\mathscr{F}$ (under the correspondence $F \to \tilde{F}$; note that $F = \tilde{F} \cap X$). We take $\mathscr{B}$ to be the Borel field generated by $\mathscr{F}^*$, and we define $\tilde{m}$ on $\mathscr{F}^*$ by: $\tilde{m}(\tilde{F}) = m(F)$, $F \in \mathscr{F}$. It is easily seen (by using the Stone topology on $\tilde{X}$, which makes X compact and $\mathscr{F}^*$ consist of the open-closed sets) that if $\tilde{F}_1, \tilde{F}_2, \cdots$, are disjoint elements of $\mathscr{F}^*$ whose union is also in $\mathscr{F}^*$, then $\tilde{m}(\bigcup_i \tilde{F}_i) = \Sigma_i \tilde{m}(\tilde{F}_i)$; thus, by a standard theorem, $\tilde{m}$ has a countably additive extension to all of $\mathscr{B}$, as asserted. Under reasonable topological conditions on X, one can even embed X topologically in $\tilde{X}$; see [2].

8.2. But it is more interesting to try to obtain a countably additive extension without enlarging the space. In this direction we have the following fairly simple theorem.

THEOREM 8.1. *Let m_α, $\alpha \in \mathscr{A}$, be a positive, finite, finitely additive measure defined on a field $\mathscr{F}_\alpha$ of subsets of a fixed Hausdorff space X, and assume that each m_α is "regular" in the sense that, for each $F \in \mathscr{F}_\alpha$,*

$$m_\alpha(F) = \sup \{m_\alpha(K) \mid K \in \mathscr{F}_\alpha, K \subset F, K \text{ compact}\}. \tag{8.1}$$

Suppose further that, whenever f_i is an m_{α_i} measurable real valued step function on X, $i = 1, \cdots, n$, such that $\Sigma_{i=1}^n f_i(x) \geqq 0$ for all $x \in X$, then

$$\sum_{i=1}^{n} \int_X f_i \, dm_{\alpha_i} \geqq 0. \tag{8.2}$$

Then there exists a countably additive (positive, finite) measure μ, defined on the Borel field generated by $\cup_\alpha F_\alpha$, which extends all the m_α simultaneously.

REMARK. Though we have not explicitly required the m_α to be countably additive, or consistent, it follows easily from (8.2) that they are consistent, and from (8.1) that they have countably additive extensions.

PROOF. By Theorem 6.1, the measures m_α have a common extension to a finitely additive (positive, finite) measure μ which, for convenience, we restrict to the field $\mathscr{F}$ generated by $\cup_\alpha \mathscr{F}_\alpha$. We need only show that μ can be extended to a countably additive measure on $\mathscr{B}$, the σ-field generated by $\mathscr{F}$. The first step is to show that, given $F \in \mathscr{F}$ and $\varepsilon > 0$, there exist a compact $K \subset F$ and open $G \supset F$ such that

(1) $K, G \in \mathscr{F}$ and $\mu(K) + \varepsilon > \mu(F) > \mu(G) - \varepsilon$.

Suppose first that F is of the form $A_1 \cap A_2 \cap \cdots \cap A_n$, where $A_i \in \mathscr{F}_{\alpha_i}$, $i = 1, \cdots, n$. Using (8.1) (applied both to A_i and to its complement), we find compact $K_i \in \mathscr{F}_{\alpha_i}$ and open $G_i \in \mathscr{F}_{\alpha_i}$ such that

$$\mu(K_i) + \frac{\varepsilon}{n} > \mu(A_i) > \mu(G_i) - \frac{\varepsilon}{n}. \tag{8.3}$$

Put $K = \cap_i K_i$, $G = \cap_i G_i$; it is easy to see that (1) holds in this case.

But if F is an arbitrary member of $\mathscr{F}$, it is well known that F is expressible as a union of disjoint sets $F_1, \cdots, F_m$ each of the form just considered. Since (1) holds for each F_j, it follows that (1) holds for F.

The rest is standard, but for completeness we sketch the argument. It is enough to prove that if $F_1, F_2, \cdots$ are disjoint members of $\mathscr{F}$ whose union F is also in $\mathscr{F}$, then $\mu(F) = \Sigma_n \mu(F_n)$. One inequality is trivial. For the other, given $\varepsilon > 0$, take open $G_n \in \mathscr{F}$ such that $G_n \supset F_n$ and $\mu(G_n \backslash F_n) < \varepsilon/2^{n+1}$, and take compact $K \in \mathscr{F}$ such that $K \subset F$ and $\mu(F\backslash K) < \varepsilon/2$. Then K is covered by the sets $G_1, G_2, \cdots$, and, hence, $K \subset G_1 \cup \cdots \cup G_n$ for some n. It follows that $\mu(K) \leqq \Sigma_{i=1}^n \mu(G_i)$, whence $\mu(F) \leqq \Sigma_n \mu(F_n) + \varepsilon$, for all $\varepsilon > 0$.

8.3. We remark, in conclusion, that in the case of marginals, and with only finitely many measures involved, Kellerer ([5], Satz 2.2) has shown (by explicit construction) that a consistent family of countably additive signed measures always has a countably additive common extension. It would be desirable to have an abstract theorem which includes this.

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