LIFTINGS COMMUTING WITH TRANSLATIONS

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1. Introduction

Let Z be a locally compact group and β a (left invariant) Haar measure on Z. We denote by $M_R^\infty(Z, \beta)$ the Banach algebra of all bounded real valued measurable functions defined on Z, endowed with the norm

$$f \rightsquigarrow \|f\|_\infty = \sup_{z \in Z} |f(z)|, \tag{1.1}$$

and by $C_R^b(Z)$ the subalgebra of all continuous bounded real valued functions defined on Z.

For two functions f and g, with domain Z, we write $f \equiv g$, whenever f and g coincide $\beta^\bullet$ almost everywhere.

We denote by $\mathscr{B}$ the *tribe* of all measurable parts of Z and by $\mathscr{B}_0$ the *clan* of all $A \in \mathscr{B}$ satisfying $\beta^\bullet(A) > +\infty$.

A mapping $\rho: M_R^\infty(Z, \beta) \to M_R^\infty(Z, \beta)$ is a lifting of $M_R^\infty(Z, \beta)$ if:

(I) $\rho(f) \equiv f$;

(II) $f \equiv g$ implies $\rho(f) = \rho(g)$;

(III) $\rho(1) = 1$;

(IV) $f \geqq 0$ implies $\rho(f) \geqq 0$;

(V) $\rho(\alpha f + \beta g) = \alpha\rho(f) + \beta\rho(g)$;

(VI) $\rho(fg) = \rho(f)\rho(g)$.

For $s \in Z$ and $f: Z \to R$ we denote $\gamma(s)f$ the mapping $z \rightsquigarrow f(s^{-1}z)$ of Z into R. A lifting ρ of $M_R^\infty(Z, \beta)$ commutes with (the left translations of) Z if

(VII) $\rho(\gamma(s)f) = \gamma(s)\rho(f)$

for all $s \in Z$ and $f \in M_R^\infty(Z, \beta)$.

In the paper [1], published in the Proceedings of the Fifth Berkeley Symposium, it has been shown that *for every locally compact group Z there exists a lifting of $M_R^\infty(Z, \beta)$ commuting with Z* and that such a lifting is strong. In the next two sections we shall give two applications of this result.

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If ρ is a lifting of $M_R^\infty(Z, \beta)$ then, for every $A \in \mathscr{B}$, $\rho(\varphi_A)$ is a characteristic function of a set $\rho(A) \in \mathscr{B}$. This way we define a mapping of $\mathscr{B}$ into $\mathscr{B}$ (which we denote again by ρ and call lifting of $\mathscr{B}$). For the properties of $\rho: \mathscr{B} \to \mathscr{B}$ corresponding to (I)—(VI) we refer to [2], p. 35. We notice that a lifting $\rho: M_R^\infty(Z, \beta) \to M_R^\infty(Z, \beta)$ commutes with Z if and only if $\rho(sA) = s\rho(A)$ for all $s \in Z$ and $A \in \mathscr{B}$.

In what follows *we denote by ρ a lifting of $M_R^\infty(Z, \beta)$ commuting with Z.*

2. A problem of A. B. Simon

For each set $S \in \mathscr{B}$ let $L_S^1(Z, \beta)$ be the vector space of all $\tilde{f} \in L^1(Z, \beta)$ satisfying $\tilde{f} = \tilde{f}\tilde{\varphi}_S$. (The equivalence relations we consider in $L_R^1(Z, \beta)$ and $\mathscr{B}$ are defined with respect to $\beta^\bullet$. The corresponding canonical mappings are denoted $f \twoheadrightarrow \tilde{f}$ and $A \twoheadrightarrow \tilde{A}$, respectively.) Notice that if S', S'' are measurable parts of Z then $L_{S'}^1(Z, \beta) = L_{S''}^1(Z, \beta)$ if and only if $S' \,\Delta\, S''$ is β negligible.

For any sets $A \in \mathscr{B}_0$ and $B \in \mathscr{B}_0$ we define $f_{\tilde{A},\tilde{B}}: Z \to R$ by

$$f_{\tilde{A},\tilde{B}}(x) = \varphi_A * \varphi_B(x) = \beta(x^{-1}A \cap B^{-1}), \tag{2.1}$$

for $x \in Z$. Then $f_{\tilde{A},\tilde{B}}$ is a well-defined *continuous* mapping of Z into R. Hence

$$U_{\tilde{A},\tilde{B}} = \{x \mid f_{\tilde{A},\tilde{B}}(x) \neq 0\} \tag{2.2}$$

is an *open* part of Z.

It has been shown in [5] that the following assertions are equivalent:

(i) $L_S^1(Z, \beta)$ is a subalgebra of $L^1(Z, \beta)$;
(ii) $\tilde{\varphi}_A * \tilde{\varphi}_B \in L_S^1(Z, \beta)$ for all $\beta^\bullet$ integrable sets A, B contained in S;
(iii) $\beta^\bullet(S - U_{\tilde{A},\tilde{B}}) = 0$ for all $\beta^\bullet$ integrable sets A, B contained in S.

A set $S \in \mathscr{B}$ is said to be *almost stable* if there is $S' \in \mathscr{B}$ satisfying

$$S'S' \subset S', \qquad \beta^\bullet(S' \,\Delta\, S) = 0. \tag{2.3}$$

It is easy to see that if $S \in \mathscr{B}$ is almost stable, then $L_S^1(Z, \beta)$ is a subalgebra of $L^1(Z, \beta)$. However the following problem remained open for several years.

(A. B. Simon) *Decide whether or not $S \in \mathscr{B}$ is almost stable if $L_S^1(Z, \beta)$ is a subalgebra.*

The problem was solved by a delicate and (relatively) lengthy analysis by T.-S. Liu, who has shown that S is almost stable if $L_S^1(Z, \beta)$ is a subalgebra (see for instance [4]). We shall represent below a short proof of this result, using the existence of a lifting commuting with Z.

THEOREM 2.1. *If $S \in \mathscr{B}$ and $L_S^1(Z, \beta)$ is a subalgebra then*

$$\rho(S)\rho(S^{-1})^{-1} \subset \rho(S). \tag{2.4}$$

PROOF. Let $a \in \rho(S)$ and $b \in \rho(S^{-1})^{-1}$; then $b^{-1} \in \rho(S^{-1})$. Now let V be a compact neighborhood of a and $A = \rho(V) \cap \rho(S)$; let W be a compact neighborhood of b^{-1} and $B = \rho(W) \cap \rho(S^{-1})$. Then $B' = B^{-1}$ is $\beta^\bullet$ integrable and contained in $\rho(S^{-1})^{-1} \equiv S$. Hence

$$U_{\tilde{A},\tilde{B}'} \subset \rho(U_{\tilde{A},\tilde{B}'}) \subset \rho(S). \tag{2.5}$$

(Recall that ρ is strong and $U_{\tilde{A},\tilde{B}'}$ open.) But

$$(ab)^{-1}A \cap (B')^{-1} = b^{-1}a^{-1}A \cap B \supset \{b^{-1}\} \cap B = \{b^{-1}\} \neq \varnothing \tag{2.6}$$

and

$$\rho\big((ab)^{-1}A \cap (B')^{-1}\big) = (ab)^{-1}\rho(A) \cap \rho(B) = (ab)^{-1}A \cap B \neq \varnothing \tag{2.7}$$

so that

$$\beta\big((ab)^{-1}A \cap (B')^{-1}\big) \neq 0. \tag{2.8}$$

We deduce that $ab \in U_{\tilde{A},\tilde{B}'}$, that is, $ab \in \rho(S)$. Since $a \in \rho(S)$ and $b \in \big(\rho(S^{-1})\big)^{-1}$ were arbitrary, the theorem is proved.

THEOREM 2.2. *If $S \in \mathscr{B}$ and $L^1_S(Z, \beta)$ is a subalgebra, then S is almost stable.*

PROOF. Let $T = \rho(S) \cap \rho(S^{-1})^{-1}$. Then

$$\rho(S)T \subset \rho(S)\rho(S^{-1})^{-1} \subset \rho(S). \tag{2.9}$$

Hence if

$$S' = T \cup TT \cup \cdots \cup TT \cdots T \cup \cdots, \tag{2.10}$$

we deduce $T \subset S' \subset \rho(S)$. Since $T \equiv \rho(S) \equiv S$ and since S' is stable, the theorem is proved.

3. Approximate identities

It is well known that for every locally compact group there exist approximate identities yielding mean convergence. Using the existence of liftings commuting with translations we can show that for every locally compact group there exist approximate identities yielding pointwise convergence. This result, which is stated precisely below, can be obtained as a particular case of certain derivation theorems. (See [3]. Particular forms of these theorems give examples of pointwise convergence of martingales indexed by noncountable directed sets.)

Denote by B^∞_c the algebra of all mappings of Z into R which have *compact support*, are *bounded* and *Borel measurable*. An approximate identity of Z (of type B^∞_c) is a filter basis $\mathscr{F}$ on B^∞_c having the following properties:

(i) $A \in \mathscr{F}$ and $h \in A$ implies $h \geqq 0$ and $\int_Z h\, d\beta = 1$;

(ii) for every $V \in V(e)$ there is $A \in \mathscr{F}$ such that $h \in A$ implies $\operatorname{supp} h \subset V$.

For every measure ν on Z we denote $(\nu, \gamma)(z)$ the mapping $g \rightsquigarrow \nu * g(z)$ of B^∞_c into R and $(\nu, \delta)(z)$ the mapping $g \rightsquigarrow g * \nu(z)$ of B^∞_c into R.

THEOREM 3.1. *There exist approximate identities $\mathscr{I}$ and $\mathscr{D}$ of Z (of type B^∞_c) such that if $\nu = f \cdot \beta + \sigma$ where f is locally β integrable and σ singular with respect to β, then*

$$\lim_{\mathscr{I}} (\nu, \gamma)(z) = f(z) \text{ and } \lim_{\mathscr{D}} (\nu, \delta)(z) = f(z), \tag{3.1}$$

$\beta^\bullet$ almost everywhere.

A complete proof of this result will appear in [3].

Added in proof. The solution to Simon's problem presented here was obtained several years ago. It was presented in a seminar given at Northwestern University during 1969. In a paper in *Proc. Japan Acad.* (January, 1970), T. Shimizu gave, by a somewhat different method, again using liftings, a solution to the same problem.

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