ON CONTINUOUS COLLECTIONS OF MEASURES

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1. Introduction

In this paper we will consider a problem whose formulation in probability terms is essentially as follows: when can one construct a stochastic process $\{Z(t); t \in M\}$ having continuous paths and preassigned one dimensional distributions. We always take the state space X for the process to be a metric space and the parameter set M to be a compact topological space. An obvious necessary condition is that the preassigned distributions for the individual $Z(t)$ vary continuously with t. This condition is also sufficient [1], if X is complete metric and M is zero dimensional, the Cantor set for example. If M is anything else, for example an interval on the real line, the simple necessary condition is no longer sufficient as one easily sees, and further conditions on X and on the desired distributions for the $Z(t)$ are needed. We will give here a theorem which treats the case in which M is arbitrary.

First we must introduce some notation and a precise statement of the problem, which also makes it look more like the sort of thing one ordinarily considers.

If Y is a topological space, let $P(Y)$ denote the set of all probability measures on the Borel sets of Y. Let $C(Y)$ denote the continuous bounded real valued functions on Y. We give $C(Y)$ the uniform topology and $P(Y)$ the topology generated by the functions $\mu \to \int f d\mu, f \in C(Y)$. Given a measure space $(\Omega, \mathscr{F}, \mu)$ and a mapping φ from Ω to Y which is measurable relative to $\mathscr{F}$ and the Borel sets of Y let $\varphi\mu$ denote the measure on the Borel sets of Y defined by $\varphi\mu(A) = \mu(\varphi^{-1}(A))$. From now on let M and X be compact metric spaces and let $C(M, X)$ denote the continuous functions from M to X under the uniform topology. Each t in M defines, by evaluation at t, a continuous function from $C(M, X)$ to X. We denote this mapping simply by t, so that $tf = f(t)$ for $f \in C(M, X)$. Let μ be an element of $P(C(M, X))$; then in the notation we have just introduced, $t\mu$ defines an element of $P(X)$, namely, $t\mu(A) = \mu(\{f : f(t) \in A\})$. Moreover the mapping $t \to t\mu$ from M to $P(X)$ is continuous.

In this paper we shall consider the converse construction; that is, given a continuous function T from M to $P(X)$, when is there a measure μ in $P(C(M, X))$ such that $t\mu = T(t)$ for all $t \in M$. Note that any such T defines, via the formula $(T^*f)(t) = \int f dT(t)$, a continuous linear mapping from $C(X)$ to $C(M)$ such that $T^*(1) = 1 = \|T^*\|$; any such mapping T^* arises in this way. The mappings of this sort which are also multiplicative are those of the form $T^*f(t) = f(g(t))$ for some $g \in C(M, X)$, and so the existence of $\mu \in P(C(M, X))$ with $T(t) = t\mu$ is equivalent to the existence of an integral representation for T^* as an average of multiplicative operators. Such integral representations are objects of general interest. Finally one can check immediately that given a T, a measure μ of the desired sort exists if and only if there is a probability space $(\Omega, \mathscr{F}, P)$ and a function Z from $M \times \Omega$ to X such that $\omega \to Z(t, \omega)$ is $\mathscr{F}$ measurable for each $t \in M$, and $t \to Z(t, \omega)$ is continuous for each $\omega \in \Omega$ (that is, a stochastic process with continuous paths), and such that $P(Z(t) \in A) = T(t)(A)$ for all $t \in M$ and Borel sets $A \subset X$.

2. The main theorem

In the first few paragraphs we shall state our result and give a description of what we require for the proof. The remainder of the section shall be devoted to the rather complicated details. In Section 3 we shall discuss the necessity of our hypotheses. We always use M and X to denote compact metric spaces, and I to denote the closed unit interval.

THEOREM 2.1. *Let X be connected and locally connected. Let $T : M \to P(X)$ be continuous and have the property that for each $t \in M$ the support of the measure $T(t)$ is all of X. Then there is a $\mu \in P(C(M, X))$ such that $t\mu = T(t)$ for all $t \in M$.*

LEMMA 2.1. *Theorem 2.1 is valid in the case $X = I$.*

PROOF. Let $\Omega = (0, 1)$, let $\mathscr{F}$ = Borel sets of $(0, 1)$, and let P = Lebesgue measure on $\mathscr{F}$. Let F_t be the distribution function of $T(t)$, that is, $F_t(x) = T(t)([0, x])$ for $x \in [0, 1]$, and for $\omega \in \Omega$ define $Z(t, \omega) = \inf \{x : F_t(x) \geqq \omega\}$. Then the following facts are easy to check: (a) $\omega \to Z(t, \omega)$ is $\mathscr{F}$ measurable and $P\{\omega : Z(t, \omega) \in A\} = T(t)(A)$ for all $t \in M$ and Borel sets $A \subset X$; (b) for each $\omega \in \Omega$ the mapping $t \to Z(t, \omega)$ is continuous at each point r such that $\{x : F_r(x) = \omega\}$ consists of at most one point. Since the support of $T(t)$ is by hypothesis all of I, it follows that $t \to Z(t, \omega)$ is continuous for all ω. In view of the remarks at the end of Section 1 the proof is complete.

For the next statement let $P_0(X)$ denote the subspace of $P(X)$ consisting of those measures whose support is all of X.

THEOREM 2.2. *Let X be connected and locally connected. Then there is a continuous function $\tilde{\varphi}$ from $P_0(X)$ to $P_0(I)$ and a continuous function φ from I to X such that $\varphi(\tilde{\varphi}\mu) = \mu$ for all μ in $P_0(X)$.*

REMARKS. First of all, Theorem 2.1 is an immediate consequence of Lemma 2.1 and Theorem 2.2. Indeed if φ and $\tilde{\varphi}$ are as in Theorem 2.2 and T is continuous from M into $P_0(X)$, then $\tilde{T}(t) = \tilde{\varphi}(T(t))$ defines a continuous function $\tilde{T}$ from M into $P_0(I)$. By Lemma 2.1, there is a measure $\tilde{\mu}$ in $P(C(M, I))$ such that $t\tilde{\mu} = \tilde{T}(t)$. Now $\varphi : I \to X$ defines a mapping, also called φ, from $C(M, I)$ to $C(M, X)$ by $\varphi f(t) = \varphi(f(t)), f \in C(M, I)$. If we define $\mu \in P(C(M, X))$ by $\mu = \varphi\tilde{\mu}$, then it is trivial to check that μ satisfies the conclusion of Theorem 2.1. Secondly, we should comment on the hypotheses on X and on the long proof of Theorem 2.2:

according to a theorem of Hahn and Mazurkiewicz, X is compact metric, connected, and locally connected if and only if there is a continuous function φ from I onto X. Now simply from the existence of such a φ, it follows immediately that given any $\theta \in P(X)$ there is a $\tilde{\theta} \in P(I)$ such that $\varphi\tilde{\theta} = \theta$. But unless φ has some additional properties it will not be true that $\tilde{\theta}$ can be chosen to depend on θ in a continuous manner. Much of the detail in the proof which follows stems from this fact.

Before coming to the proof of Theorem 2.2 we must develop some additional facts. A *finite partition of unity on* X is an indexed collection $F = \{f_1, \cdots, f_n\}$ of nonnegative continuous functions (not identically 0) on X such that $\Sigma_{i=1}^n f_i = 1$. It is possible that $f_i \neq f_j$ for some $i = j$; but we wish to regard functions with different indices as different elements of the partition. It will simplify the reading if this possibility is ignored. We shall first construct some partitions of unity with additional properties.

LEMMA 2.2. *There is a sequence* $\{F_n\}_{n \geqq 0}$ *of finite partitions of unity and a sequence* $\{\pi_n\}_{n \geqq 0}$ *of mappings,* π_n *mapping* F_{n+1} *onto* F_n*, such that* (*i*) *for each* n *and* $f \in F_n$ *the support of* f *is a connected set of diameter no greater than* $1/n$; (*ii*) *for each* n *and* $f \in F_n$ *we have* $f = \Sigma\, g_i$*, the sum being over those* $g \in F_{n+1}$ *such that* $\pi_n(g) = f$.

PROOF. Let $\mathscr{V}_0 = \{X\}$ and for each $n \geqq 1$ let $\mathscr{V}_n$ be a finite open cover of X by open connected subsets each of diameter no greater than $2^{-(n+2)}$. Since X is locally connected such covers exist. Now given any subset A of X, define $s_n(A)$ to be the union of those sets $V \in \mathscr{V}_n$ such that $V \cap A \neq \phi$, and define $s_{n,\infty}(A)$ to be $\bigcup_{k=1}^{\infty} s_{n+k} \cdots (s_n(A))$. Then $s_{n,\infty}(A)$ is open, it is connected if A is connected, and if A has diameter not exceeding ε then $s_{n,\infty}(A)$ has diameter not exceeding $\varepsilon + 2^{-n}$. In addition it is easy to check that $s_{n,\infty}(A \cup B) = s_{n,\infty}(A) \cup s_{n,\infty}(B)$. Now define a sequence of covers $\{\mathscr{U}_n\}$ of X by taking $\mathscr{U}_n$ to consist of the sets $s_{n+1,\infty}(V)$ as V ranges over the elements of $\mathscr{V}_n$. By what we have just said, each $\mathscr{U}_n$ is a finite cover of X by open connected sets each of diameter no greater than $1/n$ and in addition each set in $\mathscr{U}_n$ is the union of the sets in $\mathscr{U}_{n+1}$ which are contained in it.

Now to construct the partitions of unity and mappings take F_0 to consist of the function 1. Suppose F_n has been constructed and for each $f \in F_n$, $\{f > 0\}$ is an element $U(f)$ of $\mathscr{U}_n$. Let $Z_1, \cdots, Z_{n(f)}$ denote the elements of $\mathscr{U}_{n+1}$ contained in $U(f)$ and define

$$g_{f,i} = f \frac{f_{Z_i}}{f_{Z_1} + \cdots + f_{Z_{n(f)}}}, \tag{2.1}$$

where f_V denotes any nonnegative continuous function such that $\{f_V > 0\}$ is the open subset V of X. We take F_{n+1} to consist of the functions $g_{f,i}$, $f \in F_n$, $i \leqq n(f)$, and we define π_n by $\pi_n g_{f,i} = f$. This defines inductively sequences with the desired properties, so the proof is complete. We note once again that the same function may appear more than once in a partition.

Let $F = \{f_1, \cdots, f_n\}$ be an indexed collection of continuous functions on X. By the *1-complex associated with* F we mean a linearly independent set of n points, which we call $f_1, \cdots, f_n$ in a Euclidean space of high enough dimension and closed line segments, denoted $[f_i, f_j]$, joining those points f_i and f_j for which the function $f_i f_j$ is not identically zero. The 1-complex is topologized as a subset of the Euclidean space. Clearly the choice of Euclidean space and linearly independent points is unimportant, so we will not specify these further. It will be convenient at times to denote the point f_i also by $[f_i, f_i]$. The symbol (f_i, f_j) will denote the line segment $[f_i, f_j]$ with the end points deleted. Any two segments in a 1-complex do not meet, or else meet in a single point which is an end point of each. Frequently we refer to the points f as vertices. Now we pick once and for all a sequence $\{F_n\}$ of finite partitions of unity, and a sequence $\{\pi_n\}$ of mappings satisfying the requirements of Lemma 2.2. Let X_n denote the 1-complex associated with F_n. We shall use $\pi_n \colon F_{n+1} \to F_n$ to define a mapping also called π_n, from X_{n+1} to X_n as follows: for a vertex f in X_{n+1} we set $\pi_n(f) = (\pi_n(f))$ and we map a line segment $[f_i, f_j]$ in X_{n+1} linearly onto the line segment $[\pi_n(f_i), \pi_n(f_j)]$ in X_n. We call attention to the following fact whose proof is immediate from the properties of $\{F_n\}$ and $\{\pi_n\}$.

LEMMA 2.3. *Each X_n is connected; $\pi_n \colon X_{n+1} \to X_n$ is onto. For each segment (or point) $[f, g] \subset X_n$, $\pi_n^{-1}([f, g])$ is a connected 1-complex contained in X_{n+1}.*

Let $J = [a, b]$ be a closed interval on the line and let Y be a 1-complex associated with functions $f_1, \cdots, f_n$. A mapping $\varphi \colon J \to Y$ is called *piecewise linear* if there is a finite subset V of J with a and b in V such that for each $v \in V$, $\varphi(v)$ is a vertex of Y and φ is linear on each interval between adjacent members of V. Now suppose that Y is connected. Then one proves easily that given two vertices f_i and f_j of Y there is a piecewise linear mapping of J *onto* Y such that $\varphi(a) = f_i$ and $\varphi(b) = f_j$. In fact one sees without difficulty that if Y is connected and f_i and f_j are vertices, then a piecewise linear $\varphi \colon J \to Y$ exists with the following properties:

(a) φ is onto Y, $\quad \varphi(a) = f_i, \quad \varphi(b) = f_j$;

(b) for each vertex or segment $[f_r, f_s]$ of Y, $\varphi^{-1}([f_r, f_s])$ is a union of nondegenerate closed intervals in J;

(c) $\varphi([a, a']) = f_i, \quad \varphi([b', b]) = f_j$

where $a' = a + \frac{1}{3}(b - a)$, $b' = b - \frac{1}{3}(b - a)$ (property (c) is needed only for a slight, technical reason and might better be ignored).

Now to avoid trivialities assume that Y consists of more than a single point. Any piecewise linear φ with properties (a), (b), (c) breaks up J in a natural way into nondegenerate nonoverlapping closed intervals $J_1, \cdots, J_r$ and $I_1, \cdots, I_{r-1}$ such that φ maps each J onto a vertex in Y and each I_k linearly onto a nondegenerate segment in Y. We assume the numbering has been done so that (with an obvious notation) $J_1 \leqq I_1 \leqq J_2 \leqq \cdots \leqq I_{r-1} \leqq J_r$. If we break each of $J_2, \cdots, J_{r-1}$ into a left half $J_{i,1}$, and a right half $J_{i,2}$, then J is broken up into units $J_1 I_1 J_{2,1}$, $J_{2,2} I_2 J_{3,1}$, $J_{3,2} I_3 J_{4,1}$, $\cdots$, $J_{r-1,2} I_{r-1} J_r$, and in each unit φ is constant on the J and linear and nonconstant on the I. Given any φ satisfying (b) and any decomposition of J into nondegenerate intervals $J_1 \leqq I_1 \leqq J_{2,1} \leqq$

$J_{2,2} \leqq I_2 \leqq J_{3,1} \leqq \cdots$, such that φ is constant on the J and nonconstant linear on the I, we shall refer to the setup as a *piecewise linear mapping with a specification of units* and we shall refer to the triples $J_{i,2}I_iJ_{i+1,1}$ as *units*. The specific decomposition we just gave is not the only one possible; for example, we could have broken up the intermediate J into unequal parts. However any φ satisfying (a), (b), (c) together with the specific decomposition we gave will be denoted by $\varphi[J, Y, f_i, f_j]$.

Now we return to the partitions F_n and associated 1-complexes X_n and define a sequence $\{\varphi_n\}$ of piecewise linear mappings of I onto X_n with certain additional properties. We are going to assume that for each n and $f \in F_n$, $\pi_n^{-1}(f)$ consists of more than one function. This involves no loss of generality for we may replace each element $g \in F_{n+1}$ by two elements, each being the function $\frac{1}{2}g$. Define φ_0 by $\varphi_0(x) = 1$ for all $x \in I$ (of course 1 denotes the single point of X_0). Suppose $\varphi_n : I \to X_n$ has been defined so that φ_n satisfies property (b) with $J = I$ and $Y = X_n$, and that we have specified a decomposition $J_1 \leqq I_1 \leqq J_{2,1} \leqq J_{2,2} \leqq \cdots$, of $[0, 1]$ into units. Let J denote any interval of type J or I in the decomposition and recall that by Lemma 2.3, $\pi_n^{-1}[\varphi_n(J)]$ is a connected 1-complex contained in X_{n+1}. To define φ_{n+1} pick any points f_1 and $f_1 \in \pi_n^{-1}[\varphi_n(J_1)]$ and define φ_{n+1} on J_1 to be any $\varphi[J_1, \pi_n^{-1}[\varphi_n(J_1)], f_1, f_2]$. Then pick any $g \in \pi_n^{-1}[\varphi_n(J_{2,1})]$ and define φ_{n+1} on I_1 *to* be any $\varphi[I_1, \pi_n^{-1}[\varphi_n(I_1)], f_2, g]$. Note that f_2 and g are indeed both in $\pi_n^{-1}[\varphi_n(I_1)]$ so this makes sense. We continue in this manner until φ_{n+1} is defined on all of I. We take the units for φ_{n+1} to be those arising from the decomposition of the segments making up the units for φ_n. It is clear that each unit for φ_n is a finite union of units for φ_{n+1}, and φ_{n+1} satisfies (b) with $J = I$ and $Y = X_{n+1}$ and

(d) if $[f, g]$ is a vertex or segment in X_n and J is one of the intervals making up $\varphi_n^{-1}([f, g])$ then $\varphi_{n+1}(J) = \pi_n^{-1}([f, g])$.

Thus we have defined inductively a sequence $\{\varphi_n\}$ of piecewise linear maps. This sequence will be held fixed from now on. Consider the units specified for φ_n and let ε_n denote the length of the longest one. From (c) it follows easily that $\varepsilon_n \to 0$ as $n \to \infty$.

Now we shall use $\{\varphi_n\}$ to define a continuous function φ from I onto X. Given n and $z \in X_n$ set

$$X_n(z) = \begin{cases} \overline{\{f > 0\}} & \text{if } z \text{ is a vertex } f, \\ \overline{\{f > 0\}} \cup \overline{\{g > 0\}} & \text{if } z \in (f, g). \end{cases} \tag{2.2}$$

Each $X_n(z)$ is a closed subset of X having diameter no greater than $2/n$. For $a \in I$ define $\Phi_n(a) = X_n(\varphi_n(a))$. Using property (d) the reader will verify without difficulty that $\Phi_{n+1}(a) \subset \Phi_n(a)$. We define φ by $\varphi(a) = \bigcap_n\{\Phi_n(a)\}$. It is simple to check that φ is a continuous function from I onto X. Note for later use that if K is a closed subset of X then $\{a : \Phi_n(a) \subset K\}$ is a closed subset of I. In addition suppose O is an open subset of X and $\{K_n\}$ is an increasing sequence of compact subsets of O such that every compact subset of O is contained in one of the K_n. Then the sequence $\{a : \Phi_n(a) \subset K_n\}$ increases to $\varphi^{-1}(O)$.

Finally we are ready to define the mapping $\tilde{\varphi}$ required for Theorem 2.2. Given a measure $\mu \in P(X)$ define a measure $\mu_n \in P(X_n)$ as follows: μ_n puts mass $\int f^2 d\mu$ at each vertex $f \in X_n$ and on each open segment (f, g), μ_n is linear Lebesgue measure of mass $\int 2fgd\mu$. It follows immediately from the definition of $\pi_n : X_{n+1} \to X_n$ that for each n, we have $\pi_n \mu_{n+1} = \mu_n$. Clearly the support of μ_n is all of X_n if the support of μ is all of X. In addition it is important to note that if μ varies continuously in $P(X)$ then $\mu_n(A)$ varies continuously for every Borel subset A of X_n. We shall suppose from now on that $\mu \in P_0(X)$.

We are going to define a sequence $\{\mu^n\}$ of measures in $P_0(I)$ such that $\varphi_n \mu^n = \mu_n$ and having additional properties. For each vertex $f \in X_1$, let μ^1 put linear Lebesgue measure of mass $\mu_1(\{f\})$ on the set $\varphi_1^{-1}(\{f\})$ (recall this set is a union of nondegenerate intervals) and for each open segment (f, g) contained in X_1, let μ^1 put linear Lebesgue measure of mass $\mu_1((f, g))$ on the set $\varphi_1^{-1}((f, g))$. Suppose μ^n has been defined and has the following properties (which do indeed hold in case $n = 1$): (i) $\varphi_n \mu^n = \mu_n$, (ii) for each closed interval J appearing in any unit for φ_n; μ^n restricted to J is a nonzero multiple of Lebesgue measure on J; and (iii) for each J as in (b) $\mu^n(J)$ varies continuously with μ. We shall describe how to construct μ^{n+1} so that (i), (ii), and (iii) hold at the $(n + 1)$st stage.

As described in the construction of φ_n we have I broken up into units $I_1 \leqq \cdots \leqq I_r$ and each I_i is made up of three nondegenerate closed intervals $I_{i,1} \leqq I_{i,2} \leqq I_{i,3}$ such that φ_n is constant on $I_{i,1}$ and $I_{i,3}$ and φ_n is linear and nonconstant on $I_{i,2}$. Let us fix an i, and denote I_i by J and $I_{i,1}, I_{i,2}, I_{i,3}$ by J_1 J_2, J_3. Suppose that $\varphi_n(J_1) = f$ and $\varphi_n(J_3) = g$. Let θ_1 be the measure that puts, for each vertex $f' \in \pi_n^{-1}(\{f\})$, linear Lebesgue measure of mass $\mu_{n+1}(\{f'\})/\mu_n(\{f\})$ on $J \cap \varphi_{n+1}^{-1}(\{f'\})$, and puts, for each $(f', f'') \subset \pi_n^{-1}(\{f\})$, linear Lebesgue measure of mass $\mu_{n+1}((f', f''))/\mu_n(\{f\})$ on $J \cap \varphi_{n+1}^{-1}((f', f''))$. Define θ_3 similarly using g in place of f. Define θ_2 to be the measure that puts, for each interval $(f', g') \subset \pi_n^{-1}((f, g))$, linear Lebesgue measure of mass $\mu_{n+1}((f', g'))/\mu_n((f, g))$ on the set $J \cap \varphi_{n+1}^{-1}((f', g'))$. Finally, recalling that $J = I_i$, put

$$\mu_i^{n+1} = \mu^n(J_1)\theta_1 + \mu^n(J_2)\theta_2 + \mu^n(J_3)\theta_3, \tag{2.3}$$

and set

$$\mu^{n+1} = \mu_1^{n+1} + \cdots + \mu_r^{n+1}. \tag{2.4}$$

It is a slightly tedious but straightforward matter to verify that μ^{n+1} satisfies properties (i) through (iii). In addition we have the important fact that if J is one of the units for φ_n, then $\mu^{n+1}(J) = \mu^n(J)$. Now the construction of $\tilde{\varphi}$ is practically complete—if J is a unit for φ_n then $\mu^m(J) = \mu^n(J)$ for $m \geqq n$. (Recall a unit for φ_n is a union of units for φ_{n+1}). Since the units for φ_n approach 0 in length as $n \to \infty$, it follows that as $n \to \infty$, μ^n approaches a measure which we define to be $\tilde{\varphi}\mu$. From (ii) it follows that the support of $\tilde{\varphi}\mu$ is all of I, and from (iii) it follows that $\tilde{\varphi}\mu$ varies continuously with μ. The last thing we must check is that $\varphi(\tilde{\varphi}\mu) = \mu$. Call $\tilde{\varphi}\mu = \nu$. Since ν and μ are both probability measures it is enough to check the $\nu(\varphi^{-1}(O)) \leqq \mu(O)$ for every open subset O of X. Let K_n consist of those points in X distant from $X - O$ by at least $1/n$, and let $U_n = \{a : \Phi_n(a) \subset K_n\}$. If A is a compact subset of $\varphi^{-1}(O)$ then $A \subset U_n$ for all large n. Since μ^n converges to ν it follows that

$$\nu(\varphi^{-1}(O)) \leqq \liminf \mu^n(U_n). \tag{2.5}$$

But from the definition of $\Phi_n(a)$ it follows that

$$\mu^n(U_n) \leqq \sum_i \int f_i^2 \, d\mu + 2 \sum_{i<j} \int f_i f_j \, d\mu, \tag{2.6}$$

where the sum involves only functions f in F_n such that f vanishes outside O. Consequently $\mu^n(U_n) \leqq \mu(O)$, so the proof is complete.

3. Comment on the hypotheses

Assuming that X is compact metric, the condition that X be connected and locally connected is in fact necessary for the conclusion of Theorem 2.1 as the following slightly stronger statement shows.

THEOREM 3.1. *If for every continuous mapping $T : I \to P_0(X)$ there is a $\mu \in P(C(I, X))$ such that $t\mu = T(t)$ for all $t \in I$, then X is connected and locally connected.*

PROOF. It is immediate that X must be arcwise connected. Indeed if x_1 and x_2 are points of X, and for $0 \leqq t \leqq 1$ we set

$$T(t) = \tfrac{3}{4}\{(1 - t)\varepsilon_{x_1} + t\varepsilon_{x_2}\} + \tfrac{1}{4}\theta, \tag{3.1}$$

where ε_x denotes unit mass at the point x and θ is any measure in $P_0(X)$, then T is a continuous mapping of I into $P_0(X)$. If μ is a measure on $C(I, X)$ with $t\mu = T(t)$, then the two sets $\{f : f(0) = x_1\}$ and $\{f : f(1) = x_2\}$ each have μ measure no less than $\frac{3}{4}$. Thus their intersection is nonempty and so there is a curve in X joining x_1 and x_2.

As to the local connectedness, it is an exercise in general topology to show that a space X is locally connected if it has the following property: given any point $x \in X$ and neighborhood U of x there is a neighborhood V of x such that for each point $y \in V$ there is a continuous function $f : I \to U$ with $f(0) = x$ and $f(1) = y$. Consequently if X is not locally connected there is a point $x \in X$, a neighborhood U of x and a sequence $\{x_n\}$ of points approaching x such that any continuous function from an interval $[a, b]$ to X having x and some x_n in its range must also take on values outside of U. In particular if

$$A_n = \left\{f \in C(I, X) : f(0) = x, f\left(\frac{1}{n}\right) = x_n\right\} \tag{3.2}$$

then $\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_n$ is the empty set. To define $T(t)$ pick a measure $\theta \in P_0(X)$, define $T(0)$ to be $\frac{3}{4}\varepsilon_x + \frac{1}{4}\theta$ and

$$T(t) = \tfrac{3}{4}n(n - 1)\left\{\left(\frac{1}{n - 1} - t\right)\varepsilon_{x_n} + \left(t - \frac{1}{n}\right)\varepsilon_{x_{n-1}}\right\} + \tfrac{1}{4}\theta \tag{3.3}$$

if $1/n \leqq t \leqq 1/(n-1)$. Then T maps I continuously into $P_0(X)$. If μ is any probability measure on $C(I, X)$ then $\mu(\bigcup_{n=k}^{\infty} A_n) \to 0$ as $k \to \infty$. But if μ is such that $t\mu = T(t)$ for all t, then $\mu(A_k) \geqq \frac{1}{2}$ for all k. Thus there can be no such μ and the proof is complete.

REFERENCE

[1] R. M. Blumenthal and H. H. Corson, "On continuous collections of measures," *Ann. Inst. Fourier (Grenoble)*, Vol. 20 (1970), pp. 193–199.