THEORY OF AGING ELEMENTS

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1. Introduction

At the present time, the exponential law is the fundamental law used in reliability theory. The popularity of this law is explained principally by the fact that almost all problems occurring in reliability theory are solved incomparably more simply if it is assumed that all the random variables in the problem are distributed according to an exponential law. The same problems for arbitrary laws are either not solved in closed form, or lead to formulas which are awkward and not readily usable.

On the other hand, it is known that the random variables used in reliability (time of faultless operation, recovery time, time to find the imperfection, and so on) have a distribution radically different from the exponential distribution, for many systems and elements. In these cases the solutions obtained under the assumption that all the laws are exponential will not describe the processes we are interested in as accurately as desired. How does one get out of this blind alley?

In the first place, methods worked out in similar disciplines, principally in queueing theory and renewal theory, methods which permit the solution of problems for cases when part of or even all the distribution laws are arbitrary, may be used in reliability theory. Unfortunately, the class of such problems is quite narrow.

Another very promising direction is to search for approximate formulas in the proofs of any limit theorems from which other approximate formulas may, in turn, be obtained. Thus, for example, it can be proved that a large number of rarely recurring events generate a Poisson flow, and therefore, the time prior to the first appearance of an event is distributed according to an exponential law. Here it should also be noted that as a rule, a rigorous proof of such limit theorems requires great efforts.

The third possible direction, to one aspect of which this paper is devoted, is that some natural physical conditions may be imposed on the distribution laws encountered in reliability, and estimates for the different reliability characteristics may be sought in the given class of laws.

2. The hypothesis of aging

As is known, the exponential distribution of the time of faultless operation of an element has a simple physical meaning—the probability of failure of such an element in a given time interval is independent of how much time the element has spent working prior to this, but depends only on the length of the interval. (By the word "element," we shall understand any unit which is considered as a single whole, independently of the reliability of its component parts.) In other words, the reliability of the element does not change with time; the element does not age. Analytically, this is expressed by the fact that the local reliability characteristic, the hazard rate, is constant for an exponential law $\lambda(t) = -P'(t)/P(t) = \lambda = \text{const.}$ where $P(t)$ is the probability of faultless operation during the time t. It is more natural physically to assume that the reliability of the element can only decrease with time.

In conformity with this, let us introduce the following definition. An element is said to be an aging element if its hazard rate increases monotonely (in the nonstrict sense): that is, for any t_1 and t_2, $(0 < t_1 < t_2)$,

$$\lambda(t_1) \leqq \lambda(t_2). \tag{1}$$

If the derivative $\lambda'(t)$ exists, this condition may be rewritten thus: $\lambda'(t) \geqq 0$. Henceforth, for convenience, we shall speak of an "aging random variable," and "aging law," as well as of an "aging element." Apparently the majority of elements are aging elements. This is indicated by the results of numerous tests conducted on elements of various types. It is true for many elements that the hazard rate is initially high. However, this does not contradict the hypothesis of aging, but only says that the batch of tested elements was not homogeneous; there was a group of defective items in it which failed in the initial period of the test and thereby distorted the resulting hazard rate curve.

The fact that almost all laws used in reliability theory are aging laws is an important argument in favor of the hypothesis of aging. Let us show this.

(a) The Weibull law $P(t) = e^{-\lambda t^{\alpha}}$ is an aging law for $\alpha \geqq 1$, since $\lambda(t) = \alpha\lambda t^{\alpha-1}$ increases monotonely. In particular, the aging law for $\alpha = 1$ becomes the exponential law.

(b) The normal law

$$P(t) = c \int_{\frac{t-T_0}{\sigma}}^{\infty} e^{-\frac{1}{2}x^2}, \tag{2}$$

(the constant c is determined by the condition that $P(0) = 1$). The hazard rate $\lambda(t)$ equals

$$\lambda(t) = \frac{1}{\sigma} \frac{e^{-\left(\frac{t-T_0}{2\sigma^2}\right)^2}}{\int_{\frac{t-T_0}{\sigma}}^{\infty} e^{-\frac{1}{2}x^2}\, dx}. \tag{3}$$

In order to prove that the normal law is an aging law, let us find the derivative $\lambda'(t)$:

$$\frac{\lambda'(t)}{\lambda(t)} = \frac{1}{\sigma}\left[\frac{e^{-\left(\frac{t-T_0}{2\sigma^2}\right)^2}}{\int_{\frac{t-T_0}{\sigma}}^{\infty} e^{-\frac{1}{2}x^2}\, dx} - \frac{t-T_0}{\sigma}\right]. \tag{4}$$

Let us introduce a simple inequality. If $z > 0$, then

$$(5) \qquad \int_z^\infty e^{-\frac{1}{2}x^2}\,dx = \int_z^\infty \frac{1}{x}\,d[-e^{-\frac{1}{2}x^2}] < \frac{1}{z}\int_z^\infty d(-e^{-\frac{1}{2}x^2}) = \frac{1}{z}\,e^{-\frac{1}{2}z^2}.$$

Hence, it follows that the square bracket in equality (4) is positive for $t > T_0$. If $t \leqq T_0$, then it is all the more positive. Thus $\lambda(t) > 0$, and the normal law is an aging law.

(c) The gamma distribution

$$(6) \qquad P(t) = \int_{\lambda t}^\infty \frac{x^{\alpha-1}}{\Gamma(\alpha)}\,e^{-x}\,dx.$$

Let us show that this is an aging law for $\alpha \geqq 1$:

$$(7) \qquad \lambda(t) = \frac{\lambda^\alpha t^{\alpha-1}\,e^{-\lambda t}}{\int_{\lambda t}^\infty x^{\alpha-1}\,e^{-x}\,dx} = \frac{\lambda}{\int_{\lambda t}^\infty \left(\frac{x}{\lambda t}\right)^{\alpha-1} e^{-(x-\lambda t)}\,dx}.$$

Let us make the change of variable $x = \lambda t + z$ in the integral. Then

$$(8) \qquad \lambda(t) = \frac{\lambda}{\int_0^\infty \left(1 + \frac{z}{\lambda t}\right)^{\alpha-1} e^{-z}\,dz},$$

from which it is seen that for $\alpha \geqq 1$ the integral decreases monotonely as t increases, and the whole fraction increases monotonely. Thus, the gamma distribution is an aging distribution.

3. Operation on aging variables

Some operations on aging variables again lead to aging variables.

(a) If ξ and η are independent aging variables, then $\zeta = \min(\xi, \eta)$ is an aging variable.

Let $P_1(t) = P\{\xi > t\}$, $P_2(t) = P\{\eta > t\}$. Then

$$(9) \qquad P(t) = P\{\zeta > t\} = P_1(t)\cdot P_2(t).$$

Let us express the probabilities $P_1(t)$, $P_2(t)$, and $P(t)$ in terms of their hazard rate

$$(10) \qquad \begin{aligned} P_1(t) &= \exp\left\{-\int_0^t \lambda_1(t)\,dt\right\}, \\ P_2(t) &= \exp\left\{-\int_0^t \lambda_2(t)\,dt\right\}, \\ P(t) &= \exp\left\{-\int_0^t \lambda(t)\,dt\right\}. \end{aligned}$$

It then follows from equality (9) that $\lambda(t) = \lambda_1(t) + \lambda_2(t)$, and since $\lambda_1(t)$ and $\lambda_2(t)$ increase monotonely, their sum $\lambda(t)$ also increases monotonely, that is, ζ is an aging variable. From this property it follows by induction that the minimum of any number of independent aging variables is an aging variable.

This property has a simple physical interpretation. If a system consisting of independent elements operates up to the failure of the first of its elements, then the time of its faultless operation is the minimum of the times of faultless operation of the elements. Hence it follows that if all the elements in a system are aging elements, the system itself, considered as a single element, is an aging element.

(b) If $\xi_1, \xi_2, \cdots, \xi_n$ are independent, identically distributed aging variables, then $\eta = \max(\xi_1, \xi_2, \cdots, \xi_n)$ is an aging variable.

Let $F(t) = P\{\xi_k < t\}$, $k = 1, 2, \cdots, n$, and $\varphi(t) = P\{\eta < t\}$. Then $\varphi(t) = F^n(t)$. Since the ξ_k are aging,

$$\lambda'(t) = \left|\frac{F'(t)}{1 - F(t)}\right|' = \frac{(1 - F)F'' + F'^2}{(1 - F)^2} \geqq 0 \tag{11}$$

or $(1 - F)F'' + F'^2 \geqq 0$.

In order to show that η is an aging variable, it is necessary to prove the inequality $(1 - \varphi)\varphi'' + \varphi'^2 \geqq 0$. Let us transform the left side of this inequality:

$$\begin{aligned}(12)\quad (1 - \varphi)\varphi'' + \varphi'^2 &= (1 - F^n)[n(n-1)F^{n-2}F'^2 + nF^{n-1}F''] + n^2F^{2n-2}F'^2 \\ &\geqq [n(n-1)F^{n-2} + nF^{2n-2}]F'^2 - (1 + F + \cdots + F^{n-1})nF^{n-1}F'^2 \\ &= [1 - F + 1 - F^2 + \cdots + 1 - F^{n-1} + (n-1)F^n]nF^{n-2}F'^2 \geqq 0.\end{aligned}$$

Thus the variable η is aging. Let us note that this property is generally untrue for nonidentically distributed variables ξ_k.

The proved property also has a simple physical meaning. If n identical elements are connected in parallel so that the failure of this whole group sets in when the last of the n elements fails, the lifetime of such a group is then equal to the maximum of the lifetimes of its elements. Hence, if all the elements in the group are aging elements, then the whole group, considered as a single element, will also be aging. In brief, a hot standby consisting of aging elements is itself an aging element.

(c) The sum of aging variables is an aging variable [2]. This property has the following interpretation: if all elements in a cold standby are aging elements, the standby group is itself aging.

Properties (a), (b), and (c) lead us to the following deduction. A system with hot and cold standbys in which all the elements are aging elements, is itself an aging element.

(d) Let us consider the flow of failures of an element which at the time of its failure instantly and completely recovers every time. Such a flow is called a renewal flow. In practice it is often important to know the probability that an element will operate faultlessly in a time interval $(t, t + \tau)$.

If η_t denotes the time which has passed from time t to the first failure, then the probability of faultless operation $P_t(\tau)$ in the time $(t, t + \tau)$ will equal $P_t(\tau) = P\{\eta_t > \tau\}$.

It may be shown that as $t \to \infty$ the variable η_t has the limit distribution

$$\lim_{t\to\infty} P\{\eta_t > \tau\} = P\{\eta > \tau\} = \frac{1}{T_0}\int_\tau^\infty [1 - F(x)]\,dx. \tag{13}$$

Here $F(x)$ is the distribution law of the time of faultless operation, and T_0 is the mean time of faultless operation of the element.

Let us designate the variable η as the residual time of faultless operation of the element. Let us prove that if an element is aging, the residual time η is an aging variable.

Let $\varphi(t) = (1/T_0)\int_0^t [1 - F(x)]\,dx$ denote the distribution law for η. Let $\lambda(t)$ and $\lambda_1(t)$ denote the element hazard rate and the residual time hazard rate:

$$\lambda(t) = \frac{F'(t)}{1 - F(t)};\ \lambda_1(t) = \frac{\varphi'(t)}{1 - \varphi(t)} = \frac{1 - F(t)}{\int_t^\infty [1 - F(x)]\,dx}. \tag{14}$$

Furthermore, we have the inequalities

(15)

$$1 - F(t) = \int_t^\infty F'(x)\,dx = \int_t^\infty \lambda(x)[1 - F(x)]\,dx \geqq \lambda(t)\int_t^\infty [1 - F(x)]\,dx,$$

from which it follows that $\lambda_1(t) \geqq \lambda(t)$.

Now, let us evaluate the logarithmic derivative

$$\frac{\lambda_1'(t)}{\lambda_1(t)} = \frac{1 - F(t)}{\int_t^\infty [1 - F(x)]\,dx} - \frac{F'(t)}{1 - F(t)} = \lambda_1(t) - \lambda(t) \geqq 0; \tag{16}$$

that is, $\lambda_1'(t) \geqq 0$, and therefore η is an aging variable. It will be shown below how to use this property to estimate $\varphi(t)$, the distribution function of the residual time.

4. Estimate of the reliability of aging elements

We now show how to use the hypothesis of aging to estimate various reliability parameters. The general meaning of all the estimates presented below is the following: many equalities connecting the reliability parameters of elements subject to an exponential law turn into appropriate inequalities in the case of an aging law. These inequalities indeed yield an estimate of the reliability. It is essential to note in this connection that the reliability parameters are estimated just from that aspect which the meaning of the problem requires. Thus, for example, it is natural to estimate the failure probability from above, and the mean time of faultless operation and the probability of faultless operation from below.

(a) Let us first consider the simplest of such problems: to give a lower bound for $P(t)$, the probability of faultless operation of an aging element, if the mean time of faultless operation T_0 is known. Let us introduce the following notation. Let $\Lambda(t) = \int_0^t \lambda(x)\,dx$; then $P(t) = e^{-\Lambda(t)}$, and the mean time

$$T_0 = \int_0^\infty P(t)\,dt = \int_0^\infty e^{-\Lambda(t)}\,dt. \tag{17}$$

Since $\Lambda''(t) = \lambda'(t) \geqq 0$, the function $\Lambda(t)$ is convex downward. Let $\varphi(x)$ denote the inverse function to $\Lambda(t)$, and let us make the change of variable $x = \Lambda(t)$, $t = \varphi(x)$ in the integral

$$T_0 = \int_0^\infty e^{-x}\,d\varphi(x) = \int_0^\infty \varphi(x)\,e^{-x}\,dx \tag{18}$$

(the second integral is obtained by integration by parts).

The function $\varphi(x)$ is convex upward, and hence, its graph lies below any tangent. In particular, $\varphi(x) \leqq \varphi(1) + \varphi'(1)(x - 1)$. From the inequality it follows that

$$T_0 \leqq \int_0^\infty [\varphi(1) + \varphi'(1)(x - 1)]\,e^{-x}\,dx = \varphi(1), \tag{19}$$

and this is equivalent to the inequality $\Lambda(T_0) \leqq 1$.

On the other hand, any arc of the graph of $\Lambda(t)$ lies below its chord. In particular, for $t_1 < t_2$,

$$\Lambda(t_1) \leqq \frac{\Lambda(t_2)}{t_2}\,t_1. \tag{20}$$

Hence, letting $t_1 = t$ and $t_2 = T_0$, and taking into account the inequality derived above, we obtain

$$\Lambda(t) \leqq \frac{\Lambda(T_0)}{T_0}\,t \leqq \frac{t}{T_0}, \tag{21}$$

from which follows the final estimate

$$P(t) = \exp\{-\Lambda(t)\} \geqq \exp(-t/T_0) \qquad \text{for} \quad t \leqq T_0. \tag{22}$$

This inequality has a simple meaning, if we know the mean lifetime of an aging element and we calculate the probability of faultless operation by an exponential law, we thereby underestimate the reliability as compared with the true reliability. It is true that inequality (22) is valid only for $t \leqq T_0$ but, as a rule, this condition is satisfied.

(b) It follows from inequality (20) that

$$P(t_1) \geqq [P(t_2)]^{t_1/t_2}. \tag{23}$$

It is convenient to use this inequality when we test for a large time t_2, but we desire to estimate the reliability in a lesser time t_1.

If the estimate of the probability of faultless operation in time t_2 is known, as a result of testing, to be $P(t_2) \geqq p$, then it follows at once from inequality (23) that the estimate of faultless operation in time t_1 is

$$P(t_1) \geqq (p)^{t_1/t_2}. \tag{23'}$$

Note that in the general case when no restrictions are imposed on the function $P(t)$ the estimate of $P(t_1)$ coincides with the estimate of $P(t_2)$ because, theoretically, the case $P(t_1) = P(t_2)$ is possible. Hence, inequality (23′) gives a large improvement in the reliability estimate.

(c) Let us prove some inequalities for the moments of aging random variables. Let

$$(24)\quad m_n = M\xi^n = \int_0^\infty t^n \, d[1 - e^{-\Lambda(t)}] = \int_0^\infty \varphi^n(x)\, e^{-x}\, dx, \qquad n = 1, 2, \cdots,$$

where $\varphi(x)$ is the inverse function of $\Lambda(t)$. In particular,

$$(25)\qquad m_1 = T_0 = \int_0^\infty \varphi(x)\, e^{-x}\, dx; \qquad \sigma^2 = D\xi = m_2 - m_1^2.$$

Let us show that the inequality

$$(26)\qquad m_n \leqq n m_{n-1} m_1$$

is valid for any n. To do this, let us consider the difference

$$(27)\qquad n m_{n-1} m_1 - m_n = \int_0^\infty \varphi^{n-1}(x)[n m_1 - \varphi(x)] e^{-x}\, dx.$$

If $\varphi(x) < n m_1$ for all x, then inequality (26) is satisfied. Otherwise, a point x_1 is found for which $\varphi(x_1) = n m_1$.

Since $\varphi(x) \geqq (\varphi(x_1)/x_1)x$ for $x < x_1$ and $\varphi(x) \leqq (\varphi(x_1)/x_1)x$ for $x > x_1$, then

$$(28)\quad \int_0^\infty \varphi^{n-1}(x)[n m_1 - \varphi(x)] e^{-x}\, dx \geqq \left|\frac{\varphi(x_1)}{x_1}\right|^{n-1} \int_0^\infty x^{n-1}[n m_1 - \varphi(x)] e^{-x}\, dx$$
$$= \left|\frac{\varphi(x_1)}{x_1}\right|^{n-1} \int_0^\infty (n! - x^{n-1})\, \varphi(x)\, e^{-x}\, dx.$$

Let x_2 be the point at which $n! = x_2^{n-1}$. Then, as above, we have

$$(29)\qquad n m_{n-1} m_1 - m_n \geqq \left|\frac{\varphi(x_1)}{x_1}\right|^{n-1} \int_0^\infty (n! - x^{n-1}) \varphi(x)\, e^{-x}\, dx$$
$$\geqq \left|\frac{\varphi(x_1)}{x_1}\right|^{n-1} \frac{\varphi(x_2)}{x_2} \int_0^\infty (n! - x^{n-1}) x\, e^{-x}\, dx = 0.$$

Inequality (26) is proved. It may also be written as

$$(30)\qquad 1 \geqq \frac{m_2}{m_1^2 \cdot 2!} \geqq \frac{m_3}{m_1^3 \cdot 3!} \geqq \cdots \geqq \frac{m_n}{m_1^n\, n!} \geqq \cdots.$$

Hence, in particular, we obtain

$$(31)\qquad D\xi = \sigma^2 \leqq T_0^2 = (M\xi)^2.$$

This latter inequality may be used to estimate the mean time of faultless operation of an aging element.

Suppose that we test n identical elements until they all fail. Let $t_1, t_2, \cdots, t_n$ denote the lifetimes of these elements, and let $\bar{t}$ be their arithmetic mean:

$$(32)\qquad \bar{t} = \frac{1}{n} \sum_{i=1}^{n} t_i.$$

For large n the quantity

$$(33)\qquad \frac{\bar{t} - T_0}{\sigma} \sqrt{n}$$

is distributed according to a normal law, with sufficient accuracy. Hence, with probability $1 - \alpha$ it will be included within the limits

$$-x_\alpha < \frac{\bar{t} - T_0}{\sigma}\sqrt{n} < x_\alpha \tag{34}$$

where x_α is determined from normal law tables by the condition

$$\frac{1}{\sqrt{2\pi}}\int_{-x_\alpha}^{x_\alpha} e^{-\frac{1}{2}x^2}\,dx = 1 - \alpha. \tag{35}$$

But then, by virtue of inequality (31), the inequality

$$-x_\alpha < \frac{\bar{t} - T_0}{T_0}\sqrt{n} < x_\alpha \tag{36}$$

will be satisfied with even greater probability. Hence,

$$\frac{\bar{t}}{1 + \frac{x_\alpha}{\sqrt{n}}} < T_0 < \frac{\bar{t}}{1 - \frac{x_\alpha}{\sqrt{n}}}. \tag{37}$$

We have obtained a confidence interval for the mean time T_0.

5. Aging elements and the theory of standbys

(a) *Hot standby.* All the elements in a hot standby operate in the same regime and, hence, have the same reliability. If $p(t)$ denotes the probability of faultless operation of a single element in the time t, and $P_n(t)$ the probability of faultless operation of the standby group, then, as is known,

$$P_n(t) = 1 - [1 - p(t)]^n. \tag{38}$$

The mean time of faultless operation of the standby group is expressed by the integral

$$T_n = \int_0^\infty P_n(t)\,dt. \tag{39}$$

This formula is not convenient for practical use. When the elements forming the standby group are aging elements, we may obtain a very simple estimate for the mean time T_n. Let us write the quantity T_n as

$$T_n = \int_0^\infty [1 - (1 - e^{-\Lambda(t)})^n]\,dt = \int_0^\infty [1 - (1 - e^{-x})^n]d\,\varphi(x) \tag{40}$$

where $\varphi(x)$ is the function inverse to $\Lambda(t)$. Integrating by parts we obtain

$$T_n = n\int_0^\infty e^{-x}(1 - e^{-x})^{n-1}\,\varphi(x)\,dx. \tag{41}$$

Since $\varphi(x)$ is convex upward, then $\varphi(x) \leqq \varphi(a) + \varphi'(a)(x - a)$ for any $a > 0$. Therefore,

$$T_n \leqq n\int_0^\infty e^{-x}(1 - e^{-x})^{n-1}[\varphi(a) + \varphi'(a)(x - a)]\,dx. \tag{42}$$

Integrating by parts in the reverse order, we find

$$(43)\qquad T_n < \varphi(a) - a\varphi'(a) + \varphi'(a)\int_0^\infty [1 - (1 - e^{-x})^n]\,dx$$
$$= \varphi(a) + \varphi'(a)\left(1 + \tfrac{1}{2} + \cdots + \frac{1}{n} - a\right).$$

Let $a = 1 + \frac{1}{2} + \cdots + 1/n$. Then $T_n < \varphi(1 + \frac{1}{2} + \cdots + 1/n)$ from which one obtains $\Lambda(T_n) \leqq 1 + \frac{1}{2} + \cdots + 1/n$ and

$$(44)\qquad P(T_n) = e^{-\Lambda(T_n)} > \exp\left\{-\left(1 + \tfrac{1}{2} + \cdots + \frac{1}{n}\right)\right\}.$$

Furthermore, it may be shown that $1 + \frac{1}{2} + \cdots + (1/n) < \ln(n + 1) + c$ where $c = 0.57712\ldots$ is the Euler constant. Hence it follows that

$$(45)\qquad P(T_n) > \frac{e^{-c}}{n+1} \approx \frac{0.56}{n+1}.$$

In order to appreciate how much more accurate this estimate is, let us try to estimate T_n from the other side. To do this, let us assume in addition that the tangent to the graph of $P(t)$ at the point where $P(t) = 1/n + 1$, lies below the graph. This assumption is natural enough. Let $q(t) = 1 - p(t)$ and let $\psi(x)$ be the inverse function to $q(t)$. According to this condition, the tangent to the graph of $\psi(x)$ at the point $a = n/n + 1$ lies below the graph. Thus,

$$(46)\qquad T_n = \int_0^\infty [1 - q^n(t)]\,dt = \int_0^1 (1 - x^n)d\psi(x) = n\int_0^1 x^{n-1}\psi(x)\,dx$$
$$\geqq n\int_0^1 x^{n-1}[\psi(a) + \psi'(a)(x - a)]\,dx$$
$$= \psi(a) - a\psi'(a) + \frac{n}{n+1}\psi'(a) = \psi\left(\frac{n}{n+1}\right);$$

hence, $q(T_n) > (n/n + 1)$ or $P(T_n) < (1/n + 1)$. Combining this inequality with inequality (45) we obtain

$$(47)\qquad \frac{0.56}{n+1} < P(T_n) < \frac{1}{n+1}.$$

This double inequality shows that the simple approximate formula

$$(48)\qquad P(T_n) = \frac{1}{n+1}$$

may be used to estimate the mean time T_n. Its accuracy is perfectly adequate if it is taken into account that the mean time is determined with great error, even in the exact integral formula, since values of the function $p(t)$ for large times t are ordinarily known very roughly. However, these values for large t's are the primary contributors to the determination of T_n.

(b) *Cold standby.* In the case of a cold standby, when the standby elements do not age and may not fail prior to the time of their connection, the lifetime of the whole standby group is the sum of the independent lifetimes of all the elements. Hence, the mean time T_n is determined trivially as $T_n = nT_0$, where T_0 is the mean lifetime of a single element.

The probability of faultless operation of the standby group is expressed in a much more complicated way. If $q(t)$ denotes the failure probability of a single element in the time t, and $Q_n(t)$ the failure probability of the standby group, this latter probability is then determined from the recursion relation

$$Q_n(t) = \int_0^t Q_{n-1}(t - \tau)dq(\tau), \qquad Q_1(t) = q(t). \tag{49}$$

As is seen, in order to determine the probability $Q_n(t)$, it is necessary to perform a sequence of several integrations. Moreover, we should know the function $q(t)$ in the whole range $(0, t)$. It turns out that for aging elements it is possible to obtain a convenient estimate of $Q_n(t)$, without these disadvantages. To do this let us use the almost obvious result [3]: Let $\bar{q}(\tau)$ be a monotonely increasing function such that $q(\tau) \leqq \bar{q}(\tau)$ for $\tau \leqq t$. If $\overline{Q}_n(\tau)$ is a function determined by formula (49) with $\bar{q}(\tau)$ replacing $q(\tau)$, then $Q_n(\tau) \leqq \overline{Q}_n(\tau)$ for $\tau \leqq t$. Since $\Lambda(\tau)$ is convex downward for aging elements, then

$$\Lambda(\tau) \leqq \frac{\tau}{t}\Lambda(t) \qquad \text{for all} \quad \tau \leqq t, \tag{50}$$

and therefore,

$$q(\tau) = 1 - e^{-\Lambda(\tau)} \leqq 1 - e^{-(\tau/t)\Lambda(t)} = \bar{q}(\tau). \tag{51}$$

However,

$$\overline{Q}_n(\tau) = 1 - \left\{1 + \left[\frac{\tau}{t}\Lambda(t)\right] + \frac{1}{2!}\left[\frac{\tau}{t}\Lambda(t)\right]^2 + \cdots + \frac{1}{(n-1)!}\left[\frac{\tau}{t}\Lambda(t)\right]^{n-1}\right\}e^{-(\tau/t)\Lambda(t)}. \tag{52}$$

Hence for $\tau = t$ we obtain

$$Q_n(t) \leqq 1 - \left[1 + \Lambda(t) + \frac{1}{2!}\Lambda^2(t) + \cdots + \frac{1}{(n-1)!}\Lambda^{n-1}(t)\right]e^{-\Lambda(t)}. \tag{53}$$

This estimate is simple and convenient. Moreover, it depends only on the value of the failure probability $q(t)$ at the terminal time t.

6. Estimates for aging elements in renewal theory

Let us consider the renewal process in which the spacing between adjacent recovery times is an aging random variable distributed according to the law $F(t)$. As is known [1], the renewal function is expressed by the series

$$H(t) = F(t) + F_2(t) + \cdots + F_n(t) + \cdots \tag{54}$$

where $F_n(t)$ is the convolution of n identical laws $F(t)$. Then, as has already been shown above, $F(\tau) \leqq 1 - \exp\{-(\tau/t)\Lambda(t)\}$ for $\tau \leqq t$, and therefore, $H(\tau) \leqq (\tau/t)\Lambda(t)$ and

$$H(t) \leqq \Lambda(t). \tag{55}$$

It is convenient to use this estimate for small t. For large t it is possible to use the estimate

$$\frac{t}{T_0} - 1 \leqq H(t) \leqq \frac{t}{T_0} \tag{56}$$

which is easily obtained from the following argument.

If $B(t) \geqq F(t)$ and $A(t)$ is the solution of the equation

$$A(t) = B(t) + \int_0^t A(t-x)dF(x), \tag{57}$$

then $H(t) \leqq A(t)$. In order to use this to derive inequality (56), let us use the fact (section 3), that the hazard rate for the residual time $\lambda_1(t)$ increases monotonely. Therefore,

$$\lambda_1(t) = \frac{1 - F(t)}{\int_t^\infty [1 - F(x)]\, dx} \geqq \lambda_1(0) = \frac{1}{T_0} \tag{58}$$

from which follows

$$F(t) \leqq \frac{1}{T_0}\int_0^t [1 - F(x)]\, dx = B(t). \tag{59}$$

However, as is easy to verify, $A(t) = t/T_0$ will be the solution of equation (57), and hence, $H(t) \leqq t/T_0$. As follows from Wald's identity, the left side of inequality (56) is always true.

In conclusion, we obtain an estimate for the distribution law of the residual time

$$\varphi(t) = \frac{1}{T_0}\int_0^t [1 - F(x)]\, dx. \tag{60}$$

Since the function $\varphi(t)$ is convex upward, then $\varphi(t) \leqq t/T_0$. If the law $F(t)$ is an aging law, then $\varphi(t)$ is also an aging law, and

$$\varphi(t) = 1 - \exp\left\{-\int_0^t \lambda_1(x)\, dx\right\} \geqq 1 - \exp\{-t\lambda_1(0)\} = 1 - \exp\{-t/T_0\}. \tag{61}$$

Hence, for aging elements the two-sided inequality

$$1 - \exp\{-t/T_0\} \leqq \varphi(t) \leqq t/T_0 \tag{62}$$

is valid, from which follows, in particular, that the error of the approximate formula $\varphi(t) \approx t/T_0$ does not exceed $\frac{1}{2}(t/T_0)^2$.

The estimates presented in the paper for aging elements certainly do not exhaust all estimates possible here. The purpose of the present work is to show how useful the consideration of aging elements is in reliability theory.

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