CONSERVATIVE POSITIVE CONTRACTIONS IN L^1

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1. Introduction

Let $(X, \mathcal{B}, m)$ be a σ-finite measure space, and let T be a positive contraction defined on $L^1(X, \mathcal{B}, m)$. It was shown by Hopf [9] that the space X can be decomposed into two disjoint subsets with respect to T, the conservative part and the dissipative part. Recently, Neveu [12] remarked that one can single out a particular subset, called the strongly conservative part, from the conservative part. In what follows, we show (theorem 2) that if the positive contraction T is conservative, then the complement of the strongly conservative part can be decomposed further into at most a countable number of disjoint subsets, each of which is characterized by a certain infinite sequence of positive integers. As Neveu has remarked, the strongly conservative part is characterized as the maximal set carrying an element f in $L^1(X, \mathcal{B}, m)$ which is left invariant by T (see theorem 1 below).

The problem of determining the existence of a strictly positive element which is left invariant by T has received considerable attention in recent years in connection with the invariant measure problem for measurable transformations and Markov processes. Various necessary and sufficient conditions for the existence of such an element f (though stated in terms of the existence of a finite, invariant, and equivalent measure) have been obtained by Hopf [8], Dowker [3], [4], Calderón [1], Hajian and Kakutani [6], and Sucheston [13] for the case of an operator T which arises from a measurable transformation; and by Ito [10] and Hajian and Ito [5] for the case of a T which arises from a Markov process. The methods and results of the last two papers cited above can be generalized further without any modification. In fact, Neveu [12] proves some of these assertions for the general case of a positive contraction T operating on $L^1(X, \mathcal{B}, m)$ using much simpler and more elegant arguments.

In Hajian and Ito [5] it was shown by means of a trivial counter-example that the case of Markov processes in general is not quite the same as for the case of invertible measurable transformations. However, assuming that the operator is conservative, most of the annoying minor difficulties disappear, and the theory generalizes smoothly. In the second part of this paper we show

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(theorem 3) that all of the necessary and sufficient conditions previously obtained in [5] for the case of invertible measurable transformations generalize to positive contractions which are assumed to be conservative. We also show (theorem 4) that these conditions are violated in a stronger way, just as in the case of invertible measurable transformations, when a positive contraction is conservative and ergodic and admits no strictly positive invariant element.

In the last part of the paper we restrict to operators which arise from Markov processes and measurable transformations. In each case we prove a theorem which seems to be of some interest. It would be interesting to generalize them to the case of a positive contraction if possible.

2. Definitions and basic concepts

We shall use the following basic notations and definitions: $(X, \mathcal{B}, m)$ represents a σ-finite measure space; ϕ a measurable nonsingular (not necessarily invertible) transformation on $(X, \mathcal{B}, m)$; $P(x, A)$ a nonsingular Markov process; $L^1(X, \mathcal{B}, m)$ the Banach space of real-valued m-integrable functions; and $L^\infty(X, \mathcal{B}, m)$ the Banach space of real-valued m-essentially bounded functions with their respective usual norms. For detailed definitions of these and other concepts used throughout this paper see [5].

Most of the statements involving an element f in $L^1(X, \mathcal{B}, m)$ or $L^\infty(X, \mathcal{B}, m)$ are meant to be qualified by the words a.e., and all the subsets considered are assumed to be measurable even though not explicitly stated.

A linear operator T defined on $L^1(X, \mathcal{B}, m)$ will be called a positive contraction if it satisfies the following two conditions:

(a) T is positive, that is, $Tf \geq 0$ if $f \geq 0$;

(b) $\|T\| \leq 1$.

Let us denote by U the adjoint operator of T; then U is a linear operator on $L^\infty(X, \mathcal{B}, m)$ satisfying the following two conditions:

(a′) U is positive;

(b′) $\|U\| \leq 1$.

We say that an element $f \in L^1(X, \mathcal{B}, m)$ is invariant under T (or simply T-invariant) if $Tf = f$. An element $g \in L^\infty(X, \mathcal{B}, m)$ is invariant under U (or simply U-invariant) if $Ug = g$, and g is U-subinvariant if $Ug \leq g$. A set $B \in \mathcal{B}$ will be called a U-invariant set, or a U-subinvariant set, if the characteristic function $\mathcal{X}_B$ of B is a U-invariant function, or a U-subinvariant function, respectively.

We shall be concerned with the existence of a nonnegative T-invariant $f \not\equiv 0$. We remark that for the existence of such an element and for related problems we may assume without loss of generality that the given measure m is finite. This is because if m is not finite but is σ-finite, then we can consider a finite measure m^* equivalent to m (see [5]). If we write $w = dm^*/dm$, the Radon-Nikodym derivative of m^* with respect to m, then the formula

$$\hat{T}f = (1/w)T(fw) \qquad \text{for} \quad f \in L^1(X, \mathcal{B}, m^*) \tag{1}$$

defines a positive contraction $\hat{T}$ on the space $L^1(X, \mathcal{B}, m^*)$. (Note that $f \in L^1(X, \mathcal{B}, m^*)$ if and only if $fw \in L^1(X, \mathcal{B}, m)$ and that w is positive almost everywhere.) We see easily that $\hat{T}$ has an invariant element f_0 if and only if T leaves $g_0 = f_0 w$ invariant. Furthermore, one can easily show that $\hat{T}^n f = (1/w)T^n(fw)$ holds for each n; therefore, the infinite sum $\sum_{j=1}^{\infty} \hat{T}^{n_j}|f|(x)$ converges or diverges at a point x if and only if the corresponding sum $\sum_{j=1}^{\infty} T^{n_j}|fw|(x)$ converges or diverges at x, respectively. Therefore, in the sequel we assume that $m(X) = 1$.

Let us denote by $\mathbb{1}$ the function taking constant value one a.e. Let us write $C = \{x | \sum_{n=0}^{\infty} T^n \mathbb{1}(x) = \infty\}$ and $D = X - C$. Hopf [9] showed that for any function $f \in L^1(X, \mathcal{B}, m)$ with $f > 0$ there exists a subset N_f of C such that $m(N_f) = 0$ and such that $\sum_{n=0}^{\infty} T^n f(x) = \infty$ for all $x \in C - N_f$. Furthermore, for any function $g \in L^1(X, \mathcal{B}, m)$ there exists a subset N_g of D such that $m(N_g) = 0$ and such that $\sum_{n=0}^{\infty} T^n|g|(x) < \infty$ for all $x \in D - N_g$. The set C thus defined is called the conservative part of the operator T and D the dissipative part.

We say that a positive contraction T is ergodic (or, equivalently, U is ergodic) if the only U-invariant functions are constants. Note that in general the ergodicity of T defined in this way does not necessarily imply that it is conservative.

For a positive contraction T defined on $L^1(X, \mathcal{B}, m)$ and for any set $A \in \mathcal{B}$ with $m(A) > 0$, we define $T_A f = \mathcal{X}_A T(\mathcal{X}_A f)$. Clearly, T_A defines a positive contraction on $L^1(A, \mathcal{B}_A, m_A)$ where $\mathcal{B}_A$ denotes the σ-algebra of those sets in $\mathcal{B}$ which are contained in A, and m_A denotes the restriction of m to $\mathcal{B}_A$. If we denote by U_A the adjoint of T_A defined on $L^\infty(A, \mathcal{B}_A, m_A)$, then U_A can be shown to be induced by U in the following manner: $U_A g = \mathcal{X}_A U(\mathcal{X}_A g)$.

We shall use the following two lemmas quite often in the sequel.

LEMMA 1. *The following four conditions are equivalent:*

(i) *$A \in \mathcal{B}$ is U-subinvariant;*

(ii) *for any $f \in L^1(X, \mathcal{B}, m)$, the inclusion $\{x | f(x) \neq 0\} \subset X - A$ implies that $\{x | Tf(x) \neq 0\} \subset X - A$;*

(iii) *for any $f \in L^1(X, \mathcal{B}, m)$, $T_A^k f = \mathcal{X}_A T^k f$ holds for every positive integer k;*

(iv) *for any $g \in L^\infty(X, \mathcal{B}, m)$, $U_A^k g = U^k(\mathcal{X}_A g)$ holds for every positive integer k.*

PROOF. We shall prove that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i). For the first implication, suppose (ii) does not hold. Then there exists an element $f \in L^1(X, \mathcal{B}, m)$ such that $\{x | f(x) \neq 0\} \subset X - A$ and $Tf(x) \neq 0$ for all x in some set $B \subset A$ with $m(B) > 0$. We may assume, taking the absolute value of f if necessary, that $f \geq 0$ on X, and hence $Tf > 0$ on B. Then,

$$0 < \int_B Tf \, dm = \int_X (U\mathcal{X}_B) f \, dm \leq \int_X (U\mathcal{X}_A) f \, dm \leq \int_X \mathcal{X}_A f \, dm = 0, \tag{2}$$

which is a contradiction.

For the second implication, (ii) implies that the support of $T(\mathcal{X}_{X-A} f)$ must be contained in $X - A$. Therefore, $\mathcal{X}_A T(\mathcal{X}_{X-A} f) = 0$, which is equivalent to the statement $T_A f = \mathcal{X}_A Tf$. Now suppose that $T_A^n f = \mathcal{X}_A T^n f$ for every positive integer $n \leq k - 1$, then

$$(3)\qquad T_A^k f = T_A(T_A^{k-1}f) = T_A(\mathfrak{X}_A T^{k-1}f) = T_A(T^{k-1}f) = \mathfrak{X}_A T(T^{k-1}f) = \mathfrak{X}_A T^k f.$$

By induction we conclude (iii).

To prove that (iii) $\Rightarrow$ (iv), we take $g \in L^\infty(X, \mathfrak{B}, m)$ and $f \in L^1(X, \mathfrak{B}, m)$, then

$$(4)\qquad \int_X (U_A^k g) f\, dm = \int_A (U_A^k g) f\, dm = \int_A g(T_A^k f)\, dm = \int_A g\mathfrak{X}_A(T^k f)\, dm$$
$$= \int_X g\mathfrak{X}_A(T^k f)\, dm = \int_X U^k(g\mathfrak{X}_A) f\, dm.$$

Since this is true for any $f \in L^1(X, \mathfrak{B}, m)$, $U_A^k g = U^k(g\mathfrak{X}_A)$.

Finally, in the fourth implication, (iv) implies that $U_A \mathbb{1} = U\mathfrak{X}_A$. But from the way U_A was induced from U, it follows that $U_A \mathbb{1} = \mathfrak{X}_A(U\mathfrak{X}_A)$. Therefore, $\mathfrak{X}_A(U\mathfrak{X}_A) = U\mathfrak{X}_A$. Since $\|U\| \leq 1$ implies $U\mathfrak{X}_A \leq \mathbb{1}$, it follows that $U\mathfrak{X}_A \leq \mathfrak{X}_A$.

REMARK. There seems to be some confusion in the literature on the definition on invariant sets. In [2], Chacon defines a set B to be invariant if for any $f \in L^1(X, \mathfrak{B}, m)$, $\{x|f(x) \neq 0\} \subset B$ implies $\{x|Tf(x) \neq 0\} \subset B$. According to Tsurumi [14], however, a set B is T-invariant if $T_B f = \mathfrak{X}_B Tf$ holds for every $f \in L^1(X, \mathfrak{B}, m)$. We shall not use either of these definitions. Lemma 1 above shows that a set A is U-subinvariant in our sense if and only if $X - A$ is invariant in Chacon's sense and if and only if A is T-invariant in Tsurumi's sense.

For any $f \in L^1(X, \mathfrak{B}, m)$, we define $f^+ = \max(f, 0)$ and $f^- = -\min(f, 0)$. Then $f^+ \geq 0$, $f^- \geq 0$, $f = f^+ - f^-$ and $|f| = f^+ + f^-$. Furthermore, if $f = f_1 - f_2$ with $f_1 \geq 0$ and $f_2 \geq 0$, then we must have $f_1 \geq f^+$ and $f_2 \geq f^-$.

LEMMA 2. *Suppose an element $f \in L^1(X, \mathfrak{B}, m)$ is T-invariant, then so are f^+, f^-, and $|f|$.*

PROOF. The T-invariance of f implies that $f^+ - f^- = T(f^+ - f^-) = Tf^+ - Tf^-$. The positivity of T then implies $f^+ \leq Tf^+$ and $f^- \leq Tf^-$. But since T is a contraction, we must have $f^+ = Tf^+$ and $f^- = Tf^-$. Therefore, f^+ and f^- are both T-invariant, and since $|f| = f^+ + f^-$, so is $|f|$.

We also need the following lemma which is due to Neveu [12].

LEMMA 3. *Let T be a positive contraction defined on $L^1(X, \mathfrak{B}, m)$. Suppose T does not admit an invariant element which is strictly positive. Then there exist a set B and an infinite sequence of positive integers $\{n_i|i = 1, 2, \cdots\}$ such that $m(B) > 0$ and*

$$(5)\qquad \sum_{i=1}^{\infty} U^{n_i}\mathfrak{X}_B(x) < 2 \qquad \text{a.e. on} \quad X.$$

3. Decomposition of the conservative part

The following theorem elaborates the remark of Neveu made in [12]. We have also learned that Krengel has discussed this decomposition recently in an as yet unpublished work of his.

THEOREM 1. *There exists a subset S of the conservative part, called the strongly conservative part, satisfying the following conditions:*

(i) *every T-invariant element in $L^1(X, \mathfrak{B}, m)$ has support in S;*

(ii) *for every nonnegative function* $f \in L^1(X, \mathcal{B}, m)$ *with* $f > 0$ *on* S *and for every infinite sequence of positive integers* $\{n_i | i = 1, 2, \cdots\}$,

$$\sum_{i=1}^{\infty} T^{n_i} f(x) = \infty \qquad \text{a.e. on} \quad S. \tag{6}$$

(iii) $X - S$ *is a* U*-subinvariant set.*

PROOF. Let us denote by $\mathfrak{A}$ the collection of all sets in $\mathcal{B}$ of the form $\{x | g(x) > 0\}$ where g is a nonnegative T-invariant element in $L^1(X, \mathcal{B}, m)$. Let $\delta = \sup_{A \in \mathfrak{A}} m(A)$. Since it is clear that $\mathfrak{A}$ is closed under the formation of countable unions, there exists a set $S \in \mathfrak{A}$ such that $m(S) = \delta$. Let $f_0 \geq 0$ be an element in $L^1(X, \mathcal{B}, m)$ such that $Tf_0 = f_0$ and $S = \{x | f_0(x) > 0\}$. To prove (i) we suppose that g is a T-invariant function such that $g \neq 0$ on some set $B \subset X - S$ with $m(B) > 0$. Then by lemma 2, $f_0 + |g|$ is also T-invariant, and therefore, $\{x | f_0 + |g| > 0\} \in \mathfrak{A}$. We also have $m\{x | f_0 + |g| > 0\} \geq m(S) + m(B) > \delta$, which is a contradiction.

To prove (ii), let us consider the operator T_S which is a positive contraction on $L^1(S, \mathcal{B}_S, m_S)$. The function f_0 is a strictly positive invariant element for T_S since $T_S f_0 = \mathfrak{X}_S T(\mathfrak{X}_S f_0) = \mathfrak{X}_S T f_0 = \mathfrak{X}_S f_0 = f_0$. Therefore, by theorem 4 of [5], for every function $\tilde{f} \in L^1(S, \mathcal{B}_S, m_S)$ with $\tilde{f} > 0$ on S, and for every infinite sequence of positive integers $\{n_i | i = 1, 2, \cdots\}$,

$$\sum_{i=1}^{\infty} T_S^{n_i} \tilde{f}(x) = \infty \qquad \text{a.e. on} \quad S. \tag{7}$$

If we now take any function $f \in L^1(X, \mathcal{B}, m)$ which is nonnegative on X and is strictly positive on S, then for each integer k $T_S^k f \leq \mathfrak{X}_S T^k f$. Therefore, for any such f and for every infinite sequence of positive integers $\{n_i | i = 1, 2, \cdots\}$,

$$\sum_{i=1}^{\infty} T^{n_i} f(x) = \infty \qquad \text{a.e. on} \quad S. \tag{8}$$

Finally, to show (iii), suppose $X - S$ is not U-subinvariant. Then there exists a subset B of S such that $m(B) > 0$ and for all $x \in B$, $U\mathfrak{X}_{X-S}(x) > 0$. Then

$$0 < \int_B f_0 U\mathfrak{X}_{X-S}\, dm \leq \int_X f_0\, U\mathfrak{X}_{X-S}\, dm = \int_{X-S} Tf_0\, dm = \int_{X-S} f_0\, dm = 0, \tag{9}$$

which is a contradiction. This completes the proof of the theorem.

We next state a few more lemmas which we need for the further decomposition of $C - S$.

LEMMA 4. *Let* A *be any set in* $\mathcal{B}$ *with* $m(A) > 0$. *If the contraction* T_A *has a nontrivial invariant element in* $L^1(A, \mathcal{B}_A, m_A)$, *then* T *also has a nontrivial invariant element in* $L^1(X, \mathcal{B}, m)$.

PROOF. Let $\tilde{f}$ be a nontrivial function in $L^1(A, \mathcal{B}_A, m_A)$, which is invariant under T_A. By lemma 2, we can suppose that $\tilde{f} \geq 0$ on A. We define an element $f \in L^1(X, \mathcal{B}, m)$ by

$$f = \begin{cases} \tilde{f} & \text{on} \quad A \\ 0 & \text{on} \quad X - A. \end{cases} \tag{10}$$

From the definition of T_A it follows that $T_A f = T_A \tilde{f}$. The invariance of $\tilde{f}$ under T_A implies $\tilde{f} = T_A \tilde{f} = T_A f = \mathfrak{X}_A T(\mathfrak{X}_A f) = \mathfrak{X}_A Tf$. Therefore, on A we have $f = \tilde{f} = Tf$. On the other hand, since T is positive and $f \geq 0$,

$$\int_X f\, dm = \int_A f\, dm = \int_A Tf\, dm \leq \int_X Tf\, dm. \tag{11}$$

But since T is a contraction, we must have $\int_X Tf\, dm \leq \int_X f\, dm$, which implies $\int_A Tf\, dm = \int_X Tf\, dm$. Therefore, $Tf = 0$ a.e. on $X - A$. Thus, $Tf = f$ a.e. on X, and this proves the lemma.

LEMMA 5. *Let B be any set in $\mathfrak{B}$. Then the set $\hat{B} = B \cup \bigcup_{n=1}^{\infty} \{x | U^n \mathfrak{X}_B(x) > 0\}$ is the smallest U-subinvariant set containing B.*

PROOF. We write $B_0 = B$ and $B_n = \{x | U^n \mathfrak{X}_B(x) > 0\}$ for $n = 1, 2, \cdots$. We first show that $U\mathfrak{X}_{B_n}(x) \leq \mathfrak{X}_{B_{n+1}}(x)$ a.e. on X for $n = 0, 1, \cdots$. Since $\|U\| \leq 1$ implies that $U\mathfrak{X}_{B_n}(x) \leq 1$ a.e., it suffices to show that $U\mathfrak{X}_{B_n}(x) = 0$ a.e. on $X - B_{n+1} = \{x | U^{n+1}\mathfrak{X}_B(x) = 0\}$. The chain of equalities

$$\begin{aligned} 0 &= \int_{X - B_{n+1}} U^{n+1}\mathfrak{X}_B\, dm = \int_X (T\mathfrak{X}_{X - B_{n+1}})(U^n \mathfrak{X}_B)\, dm \\ &= \int_{B_n} (T\mathfrak{X}_{X - B_{n+1}})(U^n \mathfrak{X}_B)\, dm \end{aligned} \tag{12}$$

implies that $T\mathfrak{X}_{X - B_{n+1}} = 0$ a.e. on B_n. Therefore,

$$0 = \int_{B_n} T\mathfrak{X}_{X - B_{n+1}}\, dm = \int_{X - B_{n+1}} U\mathfrak{X}_{B_n}\, dm, \tag{13}$$

which proves that $U\mathfrak{X}_{B_n} = 0$ a.e. on $X - B_{n+1}$.

We now show that $\hat{B}$ is U-subinvariant. Suppose not, then there exists a set $E \subset X - \hat{B}$ such that $\int_E U\mathfrak{X}_{\hat{B}}\, dm > 0$. But since $E \cap \hat{B} = \emptyset$, it follows that

$$\int_E U\mathfrak{X}_{\hat{B}}\, dm \leq \int_E U\left(\sum_{n=0}^{\infty} \mathfrak{X}_{B_n}\right) dm \leq \int_E \sum_{n=0}^{\infty} \mathfrak{X}_{B_{n+1}}\, dm = 0, \tag{14}$$

which is a contradiction.

Suppose now that F is a U-subinvariant set containing B. The positivity of U then implies $U^n\mathfrak{X}_B \leq U^n\mathfrak{X}_F \leq \mathfrak{X}_F$. Therefore, $B_n \subset F$ for each n. Consequently, $\hat{B} \subset F$, which implies that $\hat{B}$ is the smallest U-subinvariant set containing B.

LEMMA 6. *The contraction T is conservative if and only if every U-subinvariant function is U-invariant.*

PROOF. The proof follows immediately from lemma 7 of [10] once we show that T being conservative, is equivalent to the following condition:

(a) for every set E with $m(E) > 0$, $\sum_{n=0}^{\infty} \int_E T^n 1\, dm = \infty$.

If T is conservative the condition (a) is obviously satisfied. Conversely, suppose that T is not conservative. For each positive integer k let us put

$$A_k = \left\{x \middle| \sum_{n=0}^{\infty} T^n 1(x) \leq k\right\}. \tag{15}$$

Then $\bigcup_{k=1}^{\infty} A_k = D$, so that $m(D) > 0$ implies the existence of a positive integer k such that $m(A_k) > 0$. We then have

$$\sum_{n=0}^{\infty} \int_{A_k} T^n \mathbb{1}(x)\, dm = \int_{A_k} \sum_{n=0}^{\infty} T^n \mathbb{1}(x)\, dm \leq km(A_k) < \infty. \tag{16}$$

Therefore, condition (a) is not satisfied.

COROLLARY. *If T is conservative, the strongly conservative part S and its complement $X - S = C - S$ are both U-invariant sets.*

PROOF. By theorem 1, $X - S$ is U-subinvariant. Since T is conservative, lemma 6 implies that $X - S$ is U-invariant. Clearly the complement of any U-invariant set is also U-invariant.

When T is conservative, we decompose $X - S$ further.

THEOREM 2. *Assume that T is conservative. Then there exists at most a countable number of disjoint subsets $\{A_j | j = 1, 2, \cdots\}$ of $X - S$ satisfying the following conditions:*

(i) *each set A_j is U-invariant;*

(ii) *$\bigcup_{j=1}^{\infty} A_j = X - S$ modulo a set of measure zero;*

(iii) *for each $j = 1, 2, \cdots$, there exists an infinite sequence of positive integers $\{n_i^j | i = 1, 2, \cdots\}$ such that for every $f \in L^1(X, \mathcal{B}, m)$,*

$$\sum_{i=1}^{\infty} T^{n_i^j} |f|(x) < \infty \qquad \text{a.e. on } A_j. \tag{17}$$

PROOF. Consider the contraction T_{X-S} on $L^1(X - S, \mathcal{B}_{X-S}, m_{X-S})$. By lemma 4, T_{X-S} does not have any nontrivial invariant element in $L^1(X - S, \mathcal{B}_{X-S}, m_{X-S})$. Therefore, lemma 3 applies to T_{X-S}, and we get a set $B \subset X - S$ and an infinite sequence of positive integers $\{n_i^1 | i = 1, 2, \cdots\}$ such that $m(B) > 0$ and

$$\sum_{i=1}^{\infty} U_{X-S}^{n_i^1} \chi_B(x) < 2 \qquad \text{a.e. on } X - S. \tag{18}$$

Since $X - S$ is U-invariant, lemma 1 implies $U_{X-S}^k \chi_B = U^k \chi_B$ for each k. Therefore,

$$\sum_{i=1}^{\infty} U^{n_i^1} \chi_B(x) < 2 \qquad \text{a.e. on } X. \tag{19}$$

Let

$$A_1 = B \cup \bigcup_{n=1}^{\infty} \{x | U^n \chi_B(x) > 0\}. \tag{20}$$

By lemma 5 and lemma 6, A_1 is the U-invariant subset of X. Lemma 5 also implies that $A_1 \cap S = \emptyset$, since $X - S$ is a U-invariant set containing B. Let us define an element $h \in L^\infty(X, \mathcal{B}, m)$ by

$$h(x) = \begin{cases} \sum_{k=1}^{\infty} (1/2^{k+1}) U^k \chi_B(x) & \text{for } x \in A_1, \\ 0 & \text{for } x \in X - A_1. \end{cases} \tag{21}$$

Then $h \geq 0$, and

$$\sum_{i=1}^{\infty} U^{n_i^1} h(x) \leq \sum_{i=1}^{\infty} U^{n_i^1} \left(\sum_{k=1}^{\infty} (1/2^{k+1}) U^k \chi_B\right)(x) \tag{22}$$
$$= \sum_{k=1}^{\infty} (1/2^{k+1}) U^k \left(\sum_{i=1}^{\infty} U^{n_i^1} \chi_B\right) \leq 1 \qquad \text{a.e. on} \quad X.$$

Therefore, for any $f \in L^1(X, \mathfrak{B}, m)$,

$$\infty > \|f\| \geq \int_X |f| \sum_{i=1}^{\infty} U^{n_i^1} h\, dm = \int_X \left(\sum_{i=1}^{\infty} T^{n_i^1} |f|\right) h\, dm \tag{23}$$
$$\geq \int_{A_1} \left(\sum_{i=1}^{\infty} T^{n_i^1} |f|\right) h\, dm.$$

Since $h > 0$ on A_1, we must have

$$\sum_{i=1}^{\infty} T^{n_i^1} |f|(x) < \infty \qquad \text{a.e. on} \quad A_1. \tag{24}$$

Since the union of two disjoint U-invariant sets is again U-invariant, $S \cup A_1$ is U-invariant, and therefore, $X - (S \cup A_1)$ is also U-invariant. Consider the contraction $T_{X-(S \cup A_1)}$ and repeat the same argument as above. We then obtain a U-invariant subset A_2 of $X - (S \cup A_1)$ and an infinite sequence of positive integers $\{n_i^2 | i = 1, 2, \cdots\}$ such that

$$\sum_{i=1}^{\infty} T^{n_i^2} |f|(x) < \infty \qquad \text{a.e. on } A_2 \text{ for every} \quad f \in L^1(X, \mathfrak{B}, m). \tag{25}$$

We keep repeating the above process. After at most a countable number of steps we will exhaust the set $X - S$, and in this way obtain the decomposition $\{A_j | j = 1, 2, \cdots\}$ of $X - S$ together with the associated sequences of positive integers $\{n_i^j | i = 1, 2, \cdots\}$ with the desired property. This completes the proof of the theorem.

REMARK. The sequences $\{n_i^j | i = 1, 2, \cdots\}$ are a natural generalization of weakly wandering sequences defined for invertible measurable transformations in [5]. In fact, if T arises from such a transformation, then for each $j = 1, 2, \cdots$ there exists a set B_j of positive measure which is weakly wandering under $\{n_i^j | i = 1, 2, \cdots\}$.

4. Existence of strictly positive invariant elements for a conservative T

In [5] it was shown that the following two conditions are necessary but in general not sufficient for the existence of a strictly positive invariant element for an operator T which arises from a Markov process.

(A) for every measurable function f with $f > 0$, and for every infinite sequence of positive integers $\{n_i | i = 1, 2, \cdots\}$,

$$\sum_{i=1}^{\infty} U^{n_i} f(x) = \infty \qquad \text{a.e.} \tag{26}$$

(B) for every decomposition $\{E_k | k = 1, 2, \cdots\}$ of the space X, for every measurable set B with $m(B) > 0$, and for every infinite sequence of positive integers $\{n_i | i = 1, 2, \cdots\}$ there exists an index k (depending on the set B and the sequence $\{n_i\}$) such that

$$\sum_{i=1}^{\infty} \int_B U^{n_i} \mathfrak{X}_{E_k} \, dm = \infty. \tag{27}$$

We also consider the following condition:

(C) for every decomposition $\{E_k | k = 1, 2, \cdots\}$ of the space X, and for every infinite sequence of positive integers $\{n_i | i = 1, 2, \cdots\}$,

$$m\left\{\bigcap_{k=1}^{\infty} \left\{x \middle| \sum_{i=1}^{\infty} U^{n_i} \mathfrak{X}_{E_k}(x) < \infty\right\}\right\} = 0. \tag{28}$$

We now prove the following theorem.

THEOREM 3. *For any positive contraction T defined on $L^1(X, \mathfrak{B}, m)$ the conditions* (A), (B), *and* (C) *are equivalent (regardless of T being conservative). Any (and hence all) of these conditions are necessary and sufficient for the existence of a strictly positive invariant element for a conservative T.*

PROOF. We first show the equivalence of the conditions (A), (B), and (C).

To prove (A) $\to$ (C), suppose (C) does not hold. Then there exist a decomposition $\{E_k | k = 1, 2, \cdots\}$ and an infinite sequence of positive integers $\{n_i | i = 1, 2, \cdots\}$ such that

$$m\left\{\bigcap_{k=1}^{\infty} \left\{x \middle| \sum_{i=1}^{\infty} U^{n_i} \mathfrak{X}_{E_k}(x) < \infty\right\}\right\} > 0. \tag{29}$$

We denote the above set of positive measure by Λ. Next choose an $\varepsilon > 0$ with $\varepsilon < m(\Lambda)$, and then choose a sequence of positive integers $\{N_k | k = 1, 2, \cdots\}$ such that for each k, $m(\Lambda_k) < \varepsilon/2^k$, where

$$\Lambda_k = \Lambda \cap \left\{x \middle| \sum_{i=1}^{\infty} U^{n_i} \mathfrak{X}_{E_k}(x) > N_k\right\}. \tag{30}$$

Let $\Lambda^* = \bigcup_{k=1}^{\infty} \Lambda_k$; then $m(\Lambda - \Lambda^*) \geq m(\Lambda) - \sum_{k=1}^{\infty} \varepsilon/2^k > 0$. If $x \in \Lambda - \Lambda^*$, then

$$\sum_{i=1}^{\infty} U^{n_i} \mathfrak{X}_{E_k}(x) \leq N_k \qquad \text{for all} \quad k. \tag{31}$$

We define

$$f = \sum_{k=1}^{\infty} (1/2^k N_k) \, \mathfrak{X}_{E_k} \tag{32}$$

Then $f > 0$ on X and

$$\sum_{i=1}^{\infty} U^{n_i} f(x) \leq 1 \qquad \text{on} \quad \Lambda - \Lambda^*, \tag{33}$$

contradicting (A).

That (C) $\to$ (B), follows immediately from the monotone convergence theorem.

To show that (B) $\rightarrow$ (A), suppose (A) does not hold. Then there exist a strictly positive function $f \in L^{\infty}(X, \mathcal{B}, m)$, an infinite sequence of positive integers $\{n_i | i = 1, 2, \cdots\}$, and a set B with $m(B) > 0$ such that $\lim_{i\to\infty} U^{n_i}f(x) = 0$ boundedly on B. Therefore, there exists an infinite subsequence $\{n_j | j = 1, 2, \cdots\}$ of the sequence $\{n_i\}$ for which

$$\sum_{j=1}^{\infty} \int_B U^{n_j} f \, dm < \infty. \tag{34}$$

We define $E_1 = \{x | f(x) > 1\}$ and $E_k = \{x | 1/(k-1) \geq f(x) > 1/k\}$ for $k = 2, 3, \cdots$. Then $\{E_k | k = 1, 2, \cdots\}$ is a countable decomposition of X, and for each k, $f > (1/k)\mathfrak{X}_{E_k}$. Therefore, for each $k = 1, 2, \cdots$,

$$\sum_{j=1}^{\infty} \int_B U^{n_j} \mathfrak{X}_{E_k} \, dm \leq k \sum_{j=1}^{\infty} \int_B U^{n_j} f \, dm < \infty, \tag{35}$$

which contradicts (B).

Finally, we show that the condition (A) is sufficient for the existence of a strictly positive invariant element for a conservative T. This will complete the proof of the theorem since the necessity of conditions (A) and (B) was already shown in [5]. We suppose that T has no strictly positive invariant element. Then by lemma 3, there exist a set B with $m(B) > 0$ and an infinite sequence of positive integers $\{n_i | i = 1, 2, \cdots\}$ such that

$$\sum_{i=1}^{\infty} U^{n_i} \mathfrak{X}_B(x) < 2 \qquad \text{a.e. on} \quad X. \tag{36}$$

Let

$$\hat{B} = B \cup \bigcup_{n=1}^{\infty} \{x | U^n \mathfrak{X}_B(x) > 0\}; \tag{37}$$

since T is conservative, $\hat{B}$ is a U-invariant set. Let

$$f = \begin{cases} \sum_{k=1}^{\infty} (1/2^{k+1}) U^k \mathfrak{X}_B & \text{on} \quad \hat{B}, \\ 1 & \text{on} \quad X - \hat{B}. \end{cases} \tag{38}$$

Then $f > 0$ on X, and the invariance of $\hat{B}$ implies

$$\sum_{i=1}^{\infty} U^{n_i} f = \sum_{i=1}^{\infty} U^{n_i} \left(\sum_{k=1}^{\infty} (1/2^{k+1}) U^k \mathfrak{X}_B \right) \qquad \text{on} \quad \hat{B}. \tag{39}$$

Therefore,

$$\sum_{i=1}^{\infty} U^{n_i} f \leq 1 \qquad \text{on} \quad \hat{B}; \tag{40}$$

this contradicts (A).

REMARK. The following three conditions, which were shown in [10] to be necessary but not sufficient for the existence of a strictly positive invariant element, are also sufficient under the added assumption that T is conservative:

(D) for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $m(B) < \delta$ implies $\sup_n \int_B T^n \mathbb{1} \, dm < \varepsilon$;

(D)′ for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $m(B) < \delta$ implies $\sup_n (1/n) \sum_{k=0}^{n-1} \int_B T^k \mathbb{1}\, dm < \varepsilon$;

(D)″ the mean ergodic theorem holds for T in $L^1(X, \mathcal{B}, m)$.

PROOF. It is clear that (D) implies (D)′. It was shown in [10] that (D)′ and (D)″ are equivalent, and that (D) is necessary for the existence of a strictly positive invariant element for T. We therefore show that (D)″ is sufficient assuming that T is conservative. To this end, consider $\bar{T}\mathbb{1} = s\text{-}\lim_{n\to\infty} (1/n) \sum_{k=0}^{n-1} T^k \mathbb{1}$; where s-lim means the limit in the sense of the norm in $L^1(X, \mathcal{B}, m)$. Clearly $\bar{T}\mathbb{1}$ is T-invariant. Let $A = \{x \mid \bar{T}\mathbb{1}(x) > 0\}$; then as in the proof of theorem 1, we can show that $X - A$ is U-subinvariant, and hence U-invariant. But on U-invariant sets the integrals of $\bar{T}\mathbb{1}(x)$ and $\mathbb{1}(x)$ have the same value. Therefore,

$$m(X) = \int_X \bar{T}\mathbb{1}(x)\, dm = \int_A \bar{T}\mathbb{1}(x)\, dm = m(A). \tag{41}$$

This means $X = A$; therefore, $\bar{T}\mathbb{1}$ is a strictly positive T-invariant function.

Let us now suppose that T is ergodic as well as conservative. Then the only U-invariant sets are the null set and the whole space X. Therefore, in the decomposition $X = \bigcup_{j=1}^{\infty} A_j \cup S$ obtained in theorem 2, only one of these sets is of positive measure. Thus, we can state the following theorem.

THEOREM 4. *Let T be a positive contraction acting on $L^1(X, \mathcal{B}, m)$. Assume that T is conservative and ergodic. Then one and only one of the following conditions holds:*

(i) *for every $f \in L^1(X, \mathcal{B}, m)$ with $f > 0$, and for every infinite sequence of positive integers $\{n_i \mid i = 1, 2, \cdots\}$,*

$$\sum_{i=1}^{\infty} T^{n_i} f(x) = \infty \qquad \text{a.e. on } X. \tag{42}$$

(ii) *There exists an infinite sequence of positive integers $\{n_i \mid i = 1, 2, \cdots\}$ such that for every $f \in L^1(X, \mathcal{B}, m)$,*

$$\sum_{i=1}^{\infty} T^{n_i} |f|(x) < \infty \qquad \text{a.e. on } X. \tag{43}$$

REMARK. It is clear that condition (i) of the above theorem holds if and only if T has a strictly positive invariant element, and condition (ii) holds if and only if T has no nontrivial invariant element.

5. Some special cases

Let us consider a nonsingular Markov process $P(x, B)$ defined on the measure space $(X, \mathcal{B}, m)$. Such a process gives rise in a natural way to a positive contraction T in $L^1(X, \mathcal{B}, m)$ (see [5]). We say that a nonsingular Markov process is conservative, or is ergodic, whenever the induced contraction T is conservative or ergodic, respectively. When a positive contraction T arises from a nonsingular

Markov process $P(x, B)$, then the adjoint U on $L^\infty(X, \mathcal{B}, m)$ is given by the following simple formula,

$$Uf(x) = \int_X f(y)P(x, dy) \qquad \text{for} \quad f \in L^\infty(X, \mathcal{B}, m). \tag{44}$$

A finite invariant measure for $P(x, B)$ which is absolutely continuous with respect to m corresponds in a one-to-one way to a nonnegative invariant element for the induced contraction T. It is possible for $P(x, B)$ to admit infinite but σ-finite invariant measures. Existence of such measures cannot be characterized by the existence of invariant elements for the induced contraction T. However, we have the following characterization which generalizes theorem 3 of [5].

THEOREM 5. *Let a nonsingular Markov process $P(x, B)$ defined on $(X, \mathcal{B}, m)$ be conservative and ergodic. Suppose there exists an infinite but σ-finite measure μ invariant under $P(x, B)$ and equivalent to m. Then there exists an infinite sequence of positive integers $\{n_i | i = 1, 2, \cdots\}$ such that for every $f \in L^1(X, \mathcal{B}, \mu)$,*

$$\sum_{i=1}^{\infty} U^{n_i}|f|(x) < \infty \qquad \text{a.e. on} \quad X. \tag{45}$$

PROOF. The invariance of the measure μ under $P(x, B)$ implies that the operator U can be defined by the formula (44) not only for $f \in L^\infty(X, \mathcal{B}, \mu) \equiv L^\infty(X, \mathcal{B}, m)$, but also for $f \in L^1(X, \mathcal{B}, \mu)$. It is easy to see that U considered as an operator on $L^1(X, \mathcal{B}, \mu)$ is a positive contraction. The invariance of μ implies also that the adjoint U^* of this contraction U (U^* is defined also on $L^\infty(X, \mathcal{B}, \mu)$) extends to a positive contraction on $L^1(X, \mathcal{B}, \mu)$. It is easy to see that $P(x, B)$ is conservative and ergodic if and only if the only U-subinvariant functions in $L^\infty(X, \mathcal{B}, \mu)$ are constants. On the other hand, a slight modification of the arguments used in theorem 3 of [11] shows that constants are the only U-subinvariant functions in $L^\infty(X, \mathcal{B}, \mu)$ if and only if they are the only U^*-subinvariant functions in $L^\infty(X, \mathcal{B}, \mu)$. It is easy to show that if there exists a nontrivial U-subinvariant element in $L^1(X, \mathcal{B}, \mu)$, then there must be a nontrivial U-subinvariant element in $L^1(X, \mathcal{B}, \mu) \cap L^\infty(X, \mathcal{B}, \mu)$. Therefore, the fact that $P(x, B)$ is conservative and ergodic implies that there is no U-subinvariant element in $L^1(X, \mathcal{B}, \mu)$, except 0, since the measure μ is infinite.

Let $w = d\mu/dm$, the Radon-Nikodym derivative of μ with respect to m, and define a contraction operator $\hat{U}$ on $L^1(X, \mathcal{B}, m)$ by the formula

$$\hat{U}(fw) = wUf \qquad \text{for every} \quad f \in L^1(X, \mathcal{B}, \mu). \tag{46}$$

(Note that every element of $L^1(X, \mathcal{B}, m)$ is of the form fw with $f \in L^1(X, \mathcal{B}, \mu)$.) The adjoint of $\hat{U}$ is the same U^* regarded as an operator on $L^\infty(X, \mathcal{B}, m)$. The transformation $\hat{U}$ is conservative and ergodic since the only U^*-subinvariant functions in $L^\infty(X, \mathcal{B}, m)$ are constants by comments made in the preceding paragraph. Suppose now that a function $fw \in L^1(X, \mathcal{B}, m)$ is invariant under $\hat{U}$; then we have $fw = \hat{U}(fw) = wUf$, which implies that f is an element in $L^1(X, \mathcal{B}, \mu)$ left invariant by U. Therefore, $f = 0$, and this implies that the contraction $\hat{U}$ on $L^1(X, \mathcal{B}, m)$ has no nontrivial invariant element. Consequently, by theorem 4 (see the remark following theorem 4), there exists an infinite sequence of positive integers $\{n_i|i = 1, 2, \cdots\}$ such that for every $fw \in L^1(X, \mathcal{B}, m)$,

$$\sum_{i=1}^{\infty} \hat{U}^{n_i}|fw|(x) < \infty \qquad \text{a.e. on} \quad X. \tag{47}$$

Since $\hat{U}^k|fw| = wU^k|f|$ for each positive integer k, this completes the proof of the theorem.

COROLLARY. *Let a nonsingular Markov process $P(x, B)$ defined on $(X, \mathcal{B}, m)$ be conservative and ergodic. Suppose there exists an infinite but σ-finite measure μ invariant under $P(x, B)$ and equivalent to m. Then there exists an infinite sequence of positive integers $\{n_i|i = 1, 2, \cdots\}$ such that for any decomposition $\{E_k|k = 1, 2, \cdots\}$ of the space X with $\mu(E_k) < \infty$ for $k = 1, 2, \cdots$,*

$$m\left\{\bigcap_{k=1}^{\infty}\left\{x\middle|\sum_{i=1}^{\infty} U^{n_i}\mathfrak{X}_{E_k}(x) < \infty\right\}\right\} = 1. \tag{48}$$

Let ϕ be a measurable and nonsingular transformation defined on the measure space $(X, \mathcal{B}, m)$. In discussing the problem of invariant measures, Hopf [8] introduced the concept of a bounded set (see also Halmos [7]). In [7] Halmos showed that there exists a σ-finite invariant measure μ equivalent to m if and only if the whole space X is the union of a countable number of bounded sets. Using these concepts we may easily prove the following.

THEOREM 6. *Let ϕ be a measurable and nonsingular transformation defined on the measure space $(X, \mathcal{B}, m)$. Assume ϕ is conservative and let S be the strongly conservative part of X. Then $S' = X - S$ may be decomposed further into two disjoint invariant sets R and $Q = S' - R$ satisfying the following conditions:*

(i) *every σ-finite invariant measure which is absolutely continuous with respect to m has support disjoint from Q;*

(ii) *there exists a σ-finite invariant measure μ which is equivalent to m on R.*

PROOF. Suppose A_1 is a bounded set of positive measure contained in S'. Since ϕ is conservative, by lemma 5,

$$A_1^* = \bigcup_{i=0}^{\infty} \phi^{-i}A_1 \tag{49}$$

is an invariant set, and is the countable union of bounded sets $\phi^{-i}A_1$, $i = 0, 1, \cdots$. If A_2 is a bounded set of positive measure contained in $S' - A_1^*$, then as above form A_2^*. We now use the principle of exhaustion and obtain a maximal set R contained in S', and R consists of the disjoint union of sets of the form $A^* = \bigcup_{i=0}^{\infty} \phi^{-1}A$ where A is a bounded set of positive measure. It is clear that R is the union of a countable number of bounded sets, and $Q = S' - R$ does not contain any bounded sets of positive measure. Parts (i) and (ii) of the theorem now are immediate consequences of the result in [7] mentioned above.

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