ON LOCAL AND RATIO LIMIT THEOREMS

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1. Introduction

In this paper we obtain local limit theorems, local limit theorems for large deviations, and ratio limit theorems for multi-dimensional probability measures which may be lattice, nonlattice, or a combination of the two.

2. Statements of results

Let R^d denote the set of d-tuples of real numbers $x = (x^1, \cdots, x^d)$. Let μ denote a probability measure on the Borel subsets of R^d with characteristic function f defined by

$$f(\theta) = \int_{R^d} e^{ix\cdot\theta}\mu(dx), \qquad \theta = (\theta_1, \cdots, \theta_d) \in R^d, \tag{2.1}$$

where $x\cdot\theta = x^1\theta_1 + \cdots + x^d\theta_d$.

We assume that μ is nondegenerate in that it is not supported by any $(d-1)$-dimensional affine subspace of R^d. Then by making a suitable linear transformation on R^d, we can assume that μ is normalized in the following sense (see Spitzer [10], pp. 64–75): there is an integer d_1, $0 \leq d_1 \leq d$, and there are real numbers $\alpha^1, \cdots, \alpha^{d_1}$ such that

$$f(2\pi n_1, \cdots, 2\pi n_{d_1}, 0, \cdots, 0) = \exp(2\pi i(n_1\alpha^1 + \cdots + n_{d_1}\alpha^{d_1})) \tag{2.2}$$

for integral $n_1, \cdots, n_{d_1}$, and $|f(\theta)| < 1$ for all other values of θ. If $d_1 = d$, then μ is lattice and if $d_1 = 0$, then μ is nonlattice.

Let $\mu^{(n)}$ denote the n-fold convolution of μ with itself. It is clear that $\mu^{(n)}$ is supported by

$$D_n = \{x \in R^d | x^k - n\alpha^k \text{ is an integer for } 1 \leq k \leq d_1\}. \tag{2.3}$$

Note that D_n is independent of n if and only if we can take $\alpha^1 = \cdots = \alpha^{d_1} = 0$, and in particular, that $D_n = R^d$ if $d_1 = 0$. The statements below can be simplified somewhat in these cases.

For the $0 \leq h < \infty$ set

$$I_h = \{x \in R^d \mid |x^k| \leq h/2 \text{ for } 1 \leq k \leq d\}, \tag{2.4}$$

and

$$\bar{I}_h = \{x \in R^d | x^k = 0 \text{ for } 1 \leq k \leq d_1 \text{ and } |x^k| \leq h/2 \text{ for } d_1 < k \leq d\}. \tag{2.5}$$

Also set $x + I_h = \{y | y - x \in I_h\}$ and $x + \bar{I}_h = \{y | y - x \in \bar{I}_h\}$.

THEOREM 1. *Let ν be a stable probability measure in R^d having a density p. Let μ be normalized and suppose that for some constants B_n and A_n*

$$\lim_{n\to\infty} \mu^{(n)}(B_n x + A_n + I_{B_n h}) = \nu(x + I_h), \qquad x \in R^d \text{ and } 0 \le h < \infty. \tag{2.6}$$

Then

$$\mu^{(n)}(x + \bar{I}_h) = \frac{h^{d-d_1}}{B_n^d}\, p\left(\frac{x - A_n}{B_n}\right) + o(B_n^{-d}), \qquad x \in D_n, \tag{2.7}$$

where $B_n^d o(B_n^{-d}) \to 0$ as $n \to \infty$ uniformly for $x \in R^d$ and h in bounded sets.

COROLLARY 1. *Let μ be normalized and have mean m and covariance Σ. Then*

$$\mu^{(n)}(x + \bar{I}_h) = \frac{h^{d-d_1}}{(2\pi n)^{d/2}|\Sigma|^{1/2}} \exp\left(-\frac{(x - nm)\cdot\Sigma^{-1}(x - nm)}{2n}\right) + o(n^{-d/2}), \qquad x \in D_n, \tag{2.8}$$

where $n^{d/2} o(n^{-d/2}) \to 0$ as $n \to \infty$ uniformly for $x \in R^d$ and h in bounded sets.

Theorem 1 was obtained in the lattice case $d_1 = d$ by Rvačeva [9] and in the nonlattice case $d_1 = 0$ by Stone [12]. It shows that in the present context local limit theorems hold no less generally than integral limit theorems. In fact, the integral form of the central limit theorem in the finite covariance case could be proven in general by first proving corollary 1 and then using Riemann approximating sums. After [12] appeared the author was informed that closely related results were announced by Bretagnolle and Dacunha-Castelle [1] (see also Stone [11]).

Next we consider probability measures μ which satisfy Cramér's condition: for some constant $c > 0$

$$\int_{R^d} e^{c|x|}\mu(dx) < \infty. \tag{2.9}$$

Let g denote the moment generating function of μ, defined for all $s \in R^d$ by

$$g(s) = \int_{R^d} e^{x\cdot s}\mu(dx), \qquad x \in R^d. \tag{2.10}$$

Under Cramér's condition, g is continuously differentiable any number of times for $|s| < c$, and in particular

$$g'(0) = \int_{R^d} x\mu(dx) = m. \tag{2.11}$$

Let μ_s, $|s| \le c$, denote the probability measure on the Borel subsets of R^d defined by $d\mu_s/d\mu = (g(s))^{-1}e^{x\cdot s}$, or equivalently for all Borel sets A,

$$\mu_s(A) = \int_A (g(s))^{-1}e^{x\cdot s}\mu(dx). \tag{2.12}$$

If μ is normalized, then so is each μ_s. Let $\mu_s^{(n)}$ denote the n-fold convolution of μ_s with itself. Then for all Borel sets A,

$$\mu_s^{(n)}(A) = \int_A (g(s))^{-n}e^{x\cdot s}\mu^{(n)}(dx) \tag{2.13}$$

and

$$\mu^{(n)}(A) = \int_A (g(s))^n e^{-x\cdot s} \mu_s^{(n)}(dx). \tag{2.14}$$

Let m_s, Σ_s, and f_s denote the mean, covariance, and characteristic functions of μ_s.

Using Fourier analysis we will prove the following theorem.

THEOREM 2. *Suppose μ is normalized and satisfies Cramér's condition. Then*

$$\mu_s^{(n)}(x + \bar{I}_h) = \frac{h^{d-d_1}}{(2\pi n)^{d/2}|\Sigma_s|^{1/2}} \exp\left(-\frac{(x - nm_s)\cdot\Sigma_s^{-1}(x - nm_s)}{2n}\right) + o(n^{-d/2}), \qquad x \in D_n, \tag{2.15}$$

where $n^{d/2}o(n^{-d/2}) \to 0$ as $n \to \infty$ uniformly for $x \in R^d$, h in bounded sets, and s in compact subsets of $\{s \mid |s| < c\}$.

Before stating theorem 3, we need to comment further on the functions m_s and Σ_s, $|s| < c$. These functions are continuously differentiable any number of times. Also if μ has mean 0, then as $s \to 0$, $n_s = \Sigma s + 0(|s|^2)$ and $\Sigma_s = \Sigma + 0(|s|)$. If μ is normalized and in particular nondegenerate, then Σ is nonsingular and m_s has a continuous inverse s_m for s sufficiently small. The function s_m is continuously differentiable any number of times for m sufficiently small.

Recalling the relationship between $\mu_s^{(n)}$ and $\mu^{(n)}$, we obtain immediately from theorem 2 the following theorem.

THEOREM 3. *Suppose μ is normalized, has mean 0, and satisfies Cramér's condition. Then for some constant $c_2 > 0$, and for $x \in D_n$,*

(2.16)

$$\mu^{(n)}(x + \bar{I}_h) = (g(s_{x/n}))^n e^{-x\cdot s_{x/n}} \frac{1}{(2\pi n)^{d/2}|\Sigma_{s_{x/n}}|^{1/2}} \left(\int_{\|\bar{y}\| \le h} e^{-\bar{y}\cdot \bar{s}_{x/n}}\, d\bar{y} + o_n(1)\right),$$

where $o_n(1) \to 0$ as $n \to \infty$ uniformly for $|x| \le c_2 n$ and h in bounded sets (the integral is set equal to 1 if $d_1 = d$).

Theorem 3, a local limit theorem for large deviations, is closely related to work of Cramér [3], Petrov [7], Richter [8], and others. The author was motivated to prove theorem 3 by the realization that it led to an easy proof of theorem 4.

THEOREM 4. *Suppose μ is normalized. Then for every integer n_0, $h > 0$, and $\epsilon > 0$, there is a $\delta > 0$ such that if $n \ge \delta^{-1}$, $x \in D_n$, $y \in D_{n+n_0}$, $|x - y| \le \epsilon^{-1}$ and $\mu^{(n)}(x + \bar{I}_h) \ge e^{-\delta n}$, then*

$$\left|\frac{\mu^{(n+n_0)}(y + \bar{I}_h)}{\mu^{(n)}(x + \bar{I}_h)} - 1\right| \le \epsilon. \tag{2.17}$$

Let us say that μ is not one-sided if $\mu\{x | x\cdot\theta > 0\} > 0$ for all non-zero $\theta \in R^d$. It is exactly in this case that there is an (necessarily unique) $s_0 \in R^d$ such that $\inf_{s\in R^d} g(s) = g(s_0)$. Sufficient conditions to guarantee that $s_0 = 0$ are that μ has mean 0 or, more trivially, that $g(s) = \infty$ for $s \ne 0$.

THEOREM 5. *Suppose μ is normalized and not one-sided, and let s_0 be defined* *as above. Then for every integer* n_0, $h > 0$, *and* $\epsilon > 0$, *there is a* $\delta > 0$ *such that if* $n \geq \delta^{-1}$, $x \in D_n$, $y \in D_{n+n_0}$, $|x - y| \leq \epsilon^{-1}$ *and* $|x| \leq \delta n$, *then*

$$\left|\frac{\mu^{(n+n_0)}(y + \bar{I}_h)}{\mu^{(n)}(x + \bar{I}_h)} - (g(s_0))^{n_0} e^{(x-y)\cdot s_0}\right| \leq \epsilon. \tag{2.18}$$

Note that (2.18) reduces to (2.17) if and only if $s_0 = 0$.

If we suppose further that $D_n = D$ is independent of n (see discussion of this above) and ignore the uniformity in x, then the statement of theorem 5 simplifies as follows.

COROLLARY 2. *Suppose* μ *is normalized,* $D_n = D$ *is independent of* n, *and* μ *is not one-sided. Let* s_0 *be as defined above. Then for every integer* n_0, $x \in D$, $y \in D$, *and* $h > 0$,

$$\lim_{n\to\infty} \frac{\mu^{(n+n_0)}(y + \bar{I}_h)}{\mu^{(n)}(x + \bar{I}_h)} = (g(s_0))^{n_0} e^{(x-y)\cdot s_0}. \tag{2.19}$$

Again note that $s_0 = 0$ is necessary and sufficient for (2.19) to reduce to

$$\lim_{n\to\infty} \frac{\mu^{(n+n_0)}(y + \bar{I}_h)}{\mu^{(n)}(x + \bar{I}_h)} = 1. \tag{2.20}$$

Corollary 2 includes theorems of Chung and Erdös [2] and Kemeny [4] in the lattice case and an unpublished theorem of Ornstein [6] in the nonlattice case. Ornstein obtained the result that (in our notation) if $d = 1$, $d_1 = 0$, and either μ has mean 0, or the positive and negative tails of μ have infinite means (both conditions guarantee that $s_0 = 0$), then (2.20) holds. Ornstein's method differs considerably from the one used here. It seems also to be capable of yielding corollary 2 and possibly much more

3. Proofs

The proof of theorem 1 and its corollary are omitted since the necessary modifications of the proofs in [12] are presented in the proof of theorem 2.

To begin the proof of theorem 2 write $x = (\tilde{x}, \bar{x})$, where $\tilde{x} = (x^1, \cdots, x^{d_1}) \in R^{d_1}$ and $\bar{x} = (x^{d_1+1}, \cdots, x^d) \in R^{d-d_1}$. If $d_1 = 0$ or $d_1 = d$, then $\tilde{x}$ or $\bar{x}$ is undefined. Similarly, write $\theta = (\tilde{\theta}, \bar{\theta})$, where $\tilde{\theta} \in R^{d_1}$ and $\bar{\theta} \in R^{d-d_1}$.

Define $K(\bar{x})$, $\bar{x} \in R^{d-d_1}$, and $k(\bar{\theta})$, $\bar{\theta} \in R^{d-d_1}$, by

$$K(\bar{x}) = \frac{1}{(2\pi)^{d-d_1}} \left(\prod_{j=d_1+1}^{d} \frac{\sin x^j/2}{x^j/2}\right)^2 \tag{3.1}$$

and

$$k(\bar{\theta}) = \begin{cases} \prod_{j=d_1+1}^{d} (1 - |\theta_k|), & \|\bar{\theta}\| < 1, \\ 0, & \|\bar{\theta}\| \geq 1. \end{cases} \tag{3.2}$$

For the $a > 0$ set

$$K_a(\bar{x}) = a^{-(d-d_1)} K(a^{-1}\bar{x}), \qquad \bar{x} \in R^{d-d_1}, \tag{3.3}$$

and

(3.4) $$k_a(\bar{\theta}) = k(a\bar{\theta}), \qquad \bar{\theta} \in R^{d-d_1}.$$

Then K_a is a probability density on R^{d-d_1} with characteristic function (Fourier transform) k_a. The essential point here is that k_a has compact support.

For $|s| < c_1$, $x \in D_n$, $h > 0$, and $a > 0$ set,

(3.5) $$V_n(s, x, h, a) = \int_{R^{d-d_1}} \mu_s^{(n)}((\tilde{x}, \bar{x} - \bar{y}) + \bar{I}_h)K_a(y)\, dy.$$

A form of the Fourier inversion theorem yields that

(3.6) $$V_n(s, x, h, a) = \frac{h^{d-d_1}}{(2\pi)^d} \int_{\|\tilde{\theta}\| \le \pi} d\tilde{\theta} \int_{\|\bar{\theta}\| \le a^{-1}} d\bar{\theta}\, e^{-ix\cdot\theta} k_a(\theta) f_s^n(\theta) \prod_{j=d_1+1}^{d} \frac{\sin h\theta_j/2}{h\theta_j/2}.$$

Choose c_1 such that $0 < c_1 < c$. From the relation $f_s(\theta) = f(\theta - is)$, it follows that

(3.7) $$e^{-im_s\cdot\theta} f_s(\theta) = e^{-\theta\cdot\Sigma_s\theta/2} + o(|\theta|^2),$$

where $|\theta|^{-2}o(|\theta|^2) \to 0$ as $\theta \to 0$ uniformly for $|s| \le c_1$. Also for any $N > 0$ there is an $\epsilon > 0$ such that $|f_s(\theta)| \le 1 - \epsilon$ for $N^{-1} \le \tilde{\theta} \le \pi$, $N^{-1} \le \bar{\theta} \le N$, and $|s| \le c_1$. It now follows by a standard computation such as in the proof of lemma 1 of [12] that for any $N > 0$ and for all $x \in D_n$,

(3.8)
$$V_n(s, x, h, a) = \frac{h^{d-d_1}}{(2\pi n)^{d/2}|\Sigma_s|^{1/2}} \exp\left(-(x - nm_s)\cdot\Sigma_s^{-1}(x - nm_s)/2n\right) + o(n^{-d/2}),$$

where $n^{d/2}o(n^{-d/2}) \to 0$ as $n \to \infty$ uniformly for $|s| \le c_1$, $x \in R^d$, $0 \le h \le N$, and $N^{-1} \le a \le N$.

Theorem 2 now follows by a proof similar to that of lemma 2 of [2]. This proof is unnecessarily complicated, however, by the fact that in [12] 0 was chosen to be a corner instead of the center of the cube I_h. A simpler proof can be based on the fact that if $|s| < c$, $x \in R^d$, $h > 0$ and $0 < \delta < 1$, then

(3.9) $$(\tilde{x}, \bar{x} - \bar{y}) + \bar{I}_{h(1-\delta)} \subseteq x + \bar{I}_h \subseteq (\tilde{x}, \bar{x} - \bar{y}) + \bar{I}_{h(1+\delta)}, \qquad \|\bar{y}\| \le \delta/2,$$

and hence

(3.10) $$\mu_s^{(n)}((\tilde{x}, \bar{x} - \bar{y}) + \bar{I}_{h(1-\delta)}) \le \mu_s^{(n)}(x + \bar{I}_h) \le \mu_s^{(n)}((\tilde{x}, \bar{x} - \bar{y}) + \bar{I}_{h(1+\delta)}), \qquad \|\bar{y}\| \le \delta/2.$$

As discussed in section 2, theorem 3 follows immediately from theorem 2. Theorem 4 is clearly equivalent to lemma 4.

LEMMA 1. *Suppose μ is normalized. Then for every integer n_0, $h > 0$, and $\epsilon > 0$, there is a $\delta > 0$ such that if $n \ge \delta^{-1}$, $x \in D_n$, $y \in D_{n+n_0}$, and $|x - y| \le \epsilon^{-1}$ then*

(3.11) $$\mu^{(n+n_0)}(y + \bar{I}_h) \le e^{-\delta n} + (1 + \epsilon)\mu^{(n)}(x + \bar{I}_h).$$

Lemma 1 will be proven first under the additional assumption that Cramér's condition is satisfied.

LEMMA 2. *If μ has mean m and satisfies Cramér's condition, then for every $\tau > 0$ there is a $\delta > 0$ such that* $\mu^{(n)}\{x \mid |x - nm| \geq \tau n\} \leq e^{-\delta n}$, $n \geq \delta^{-1}$.

It suffices to prove this known lemma for $d = 1$ and $m = 0$. In this case we can find positive numbers s and δ such that $|s| < c$ and $g(\pm s)e^{-\tau s} \leq e^{-2\delta}$. Then for $n \geq \delta^{-1}$

$$\mu^{(n)}([\tau n, \infty)) \leq (g(s))^n e^{-\tau n s} \leq e^{-2\delta n} \leq e^{-\delta n}/2, \tag{3.12}$$

and similarly $\mu^{(n)}((-\infty, -\tau n]) \leq e^{-\delta n}/2$. Thus $\mu^{(n)}\{x \mid |x| \geq \tau n\} \leq e^{-\delta n}$ as desired.

Let μ be normalized and satisfy Cramér's condition. In proving lemma 1 we can, without further loss of generality, assume that μ has mean 0. It follows easily from theorem 3 and lemma 2 that for every integer n_0, $h > 0$, and $\epsilon > 0$ we can find $\delta > 0$ and $\tau > 0$ such that if $n \geq \delta^{-1}$, $x \in D_n$, $y \in D_{n+n_0}$, and $|x - y| \leq \epsilon^{-1}$, then

$$\mu^{(n+n_0)}(y + \bar{I}_h) \leq (1 + \epsilon)\mu^{(n)}(x + \bar{I}_h), \qquad |x| \leq \tau n, \tag{3.13}$$

and

$$\mu^{(n+n_0)}(y + I_h) \leq e^{-\delta n}, \qquad |x| > \tau n. \tag{3.14}$$

This yields lemma 1 (hence also theorem 4 under the additional assumption that μ satisfies Cramér's condition).

We can reduce the general case to this special case by means of lemma 3.

LEMMA 3. *There are two probability measures μ_1 and μ_2 such that μ_1 satisfies Cramér's condition, μ is absolutely continuous with respect to μ_1, and $2\mu = \mu_1 + \mu_2$.*

To effect the desired decomposition we can, for example, choose $x_0 \in R^d$ such that $\mu\{x_0\} = 0$, define μ_1 by

$$\mu_1(A) = 2 \int_A e^{-\eta|x - x_0|}\mu(dx), \qquad \text{all Borel sets } A, \tag{3.15}$$

η denoting a positive number such that $\mu_1(R^d) = 1$, and set $\mu_2 = 2\mu - \mu_1$.

We proceed to a proof of lemma 1. Let μ_1 be normalized and decomposed according to lemma 3. Then μ_1 is normalized and satisfies Cramér's condition and, therefore, lemma 1 holds with μ replaced by μ_1. Also

$$\mu^{(n)} = \sum_{j=0}^{n} \binom{n}{j} \frac{1}{2^n} \mu_1^{(n-j)} * \mu_2^{(j)}, \tag{3.16}$$

where * denotes convolution.

Choose integer n_0, $h > 0$, and $\epsilon > 0$. By applying lemma 1 to μ_1 and lemma 2 to the binomial distribution, we can find $\delta > 0$ and $0 < \tau < \frac{1}{2}$ such that if $n \geq \delta^{-1}$, $|j - (n/2)| \leq \tau n$, $x \in D_{n-j}$, $y \in D_{n+n_0-j}$, and $|x - y| \leq \epsilon^{-1}$, then

$$\begin{aligned} &\binom{n + n_0}{j} 2^{-(n+n_0)} \mu_1^{(n+n_0-j)}(y + \bar{I}_h) \\ &\qquad \leq \tfrac{1}{2}\binom{n + n_0}{j} 2^{-(n+n_0)} e^{-\delta n} + (1 + \epsilon)\binom{n}{j} 2^{-n} \mu_1^{(n-j)}(x + \bar{I}_h), \end{aligned} \tag{3.17}$$

and if $n \geq \delta^{-1}$, then

$$\sum_{\left|j - \frac{n}{2}\right| > \tau n} \binom{n + n_0}{j} 2^{-(n+n_0)} \leq \frac{e^{-\delta n}}{2}. \tag{3.18}$$

Lemma 1, and hence also theorem 4, now follow immediately.

LEMMA 4. *Suppose μ is normalized and that $\overline{\lim}_{n\to\infty} (\mu^{(n)}(I_{h_0}))^{1/n} = 1$ for some finite h_0. Then for every integer n_0, $h > 0$, and $\epsilon > 0$, there is a $\delta > 0$ such that if $n \geq \delta^{-1}$, $x \in D_n$, $y \in D_{n+n_0}$, $|x - y| \leq \epsilon^{-1}$, and $|x| \leq \delta n$, then (2.17) holds.*

In proving this lemma, we can assume that $h_0 \geq 1$. It follows from theorem 4 and the present hypotheses that for any $\delta > 0$ there is a positive integer n such that $\mu^{(n)}(-x + I_{h_0}) \geq e^{-\delta n}$ for $x \in I_{h_0}$. Since for $k \geq 1$

$$\mu^{(kn)}(I_{h_0}) \geq \int_{I_{h_0}} \mu^{(kn-n)}(dx)\mu^{(n)}(-x + I_{h_0}), \tag{3.19}$$

we see that $\mu^{(kn)}(I_{h_0}) \geq e^{-\delta kn}$, $k \geq 1$. Theorem 4 now implies a series of consequences. First, $\lim_{n\to\infty} (\mu^{(n)}(I_{h_0}))^{1/n} = 1$. Thus there exists $x_n \in D_n$ with $|x_n| \leq 1$ and $\lim_{n\to\infty} (\mu^{(n)}(x_n + \bar{I}_{h_0}))^{1/n} = 1$. Therefore, for any $h > 0$ and $\delta > 0$ there exist $\tau > 0$ and n_0 such that

$$\mu^{(n)}(x + \bar{I}_h) \geq e^{-\delta n} \qquad \text{for} \quad n \geq n_0, x \in D_n, \quad \text{and} \quad |x| \leq \tau n. \tag{3.20}$$

This result, together with a final application of theorem 4, yields lemma 4.

The next result was suggested by theorem 5 of Kesten [5].

LEMMA 5. *If μ is nondegenerate and $\inf_{s\in R^d} g(s) = g(0) = 1$, then*

$$\lim_{n\to\infty} (\mu^{(n)}(I_{h_0}))^{1/n} = 1 \tag{3.21}$$

for some h_0.

Under the hypothesis of lemma 5 μ is not one-sided and $\lim_{|s|\to\infty} g(s) = \infty$. We assume, without loss of generality, that μ is normalized. For $\epsilon > 0$ set

$$g_\epsilon(s) = \int_{R^d} e^{x\cdot s} e^{-\epsilon|x|^2} \mu(dx). \tag{3.22}$$

Let s_ϵ be the unique minimizing point of $g_\epsilon(\cdot)$. Then as $\epsilon \to 0$, s_ϵ stays bounded and Fatou's lemma implies $g_\epsilon(s_\epsilon) \to g(0) = 1$. Let ν_ϵ be the probability measure defined on all Borel sets $A \subset R^d$ by

$$\nu_\epsilon(A) = \int_A \frac{e^{x\cdot s_\epsilon}}{g_\epsilon(s_\epsilon)} e^{-\epsilon|x|^2} \mu(dx). \tag{3.23}$$

Then ν_ϵ is normalized and has mean 0 and exponentially decreasing tails. Thus $\lim_{n\to\infty} (\nu_\epsilon^{(n)}(I_h))^{1/n}$ for $h \geq 1$. Since

$$\mu^{(n)}(I_h) \geq (g_\epsilon(s_\epsilon))^n \int_{I_h} e^{-x\cdot s_\epsilon} \nu_\epsilon^{(n)}(dx), \tag{3.24}$$

it follows that, for $h \geq 1$, $\liminf_{n\to\infty} (\mu^{(n)}(I_h))^{1/n} \geq g_\epsilon(s_\epsilon)$, and hence that $\lim_{n\to\infty} (\mu^{(n)}(I_h))^{1/n} = 1$, as desired.

Finally, to prove theorem 5, let μ be normalized and not one-sided, and let s_0 be as defined just above the statement of theorem 5. Then lemma 5, and hence lemma 4, apply to the probability measure μ_{s_0}. Using the relation between $\mu^{(n)}$ and $\mu^{(n)}_{s_0}$, we get theorem 5.

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