SOME LOCAL PROPERTIES OF MARKOV PROCESSES

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1. Introduction

In 1953, Lévy [3] proved that for almost all Brownian motion paths $X(t)$ in the Euclidean space R^N of dimension $N \geq 3$,

$$\Lambda_\rho(\{X(\tau)\colon 0 \leq \tau \leq t\}) \leq Kt, \tag{1}$$

where Λ_ρ is the Hausdorff measure in R^N formed with the function $\rho(a) = a^2 \log \log a^{-1}$, and conjectured that

$$\Lambda_\rho(\{X(\tau)\colon 0 \leq \tau \leq t\}) \geq kt \tag{2}$$

with probability one.

Lévy's conjecture was proved in 1961 by Ciesielski and Taylor [2]. The use of a density theorem of Rogers and Taylor [5] enabled them to obtain (2) by proving that with probability one,

$$\limsup_{a \to 0} T(a, t)/\rho(a) = c_N, \tag{3}$$

where

$$T(a, t) = \int_0^t V(X(\tau); a)\, d\tau, \tag{4}$$

$$V(x; a) = 1, \qquad |x| \leq a, \tag{5}$$
$$= 0, \qquad |x| > a,$$

is the sojourn time up to time t of the path inside a sphere of radius a about the initial point $X(0) = 0$. (Actually, the proof of (2) used only the fact that the lim sup in (3) is bounded below with probability one.) The constant c_N is expressed in terms of the zeros of Bessel functions through an eigenvalue problem for Laplace's equation.

In [2], Ciesielski and Taylor conjectured in turn that the result (3) holds also for $N = 2$ if the function ρ is chosen to be $\rho(a) = a^2 \log \log \log a^{-1}$. This was proved in [4], with the implication, as in [2], that the lower bound (2) holds with probability one for planar Brownian motion, with the above choice of ρ. The proper constant for (3) in this case turned out to be $c_2 = \frac{1}{2}$. Finally, Taylor [6] used (3) and related results to extend (1) to the planar case.

The point is that Taylor's work showed that properties (1) and (2) of the

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Hausdorff measure of the sample path could be obtained for homogeneous processes from the asymptotic behavior of the sojourn times. The point of this paper is that, in turn, asymptotic properties such as (3) depend to a large extent only on the local behavior of the potential kernel of the process.

We will try to emphasize this by proving a form of (3) for a class of processes, including the transient symmetric stable processes, which can be described by conditions on the singularity of the potential kernel.

A general form of the result can be stated as follows. One is given a Markov process $X(t)$, $0 \leq t < S$, in R^N, with stationary transitions, with the strong Markov property, and with almost all path functions right continuous and having limits from the left. The terminal time S may be identically infinite or may be an exponentially distributed variable independent of the paths, introduced to make the potential operator finite.

The potential operator of the process is

$$Hf(x) = \mathcal{E}_x\left\{\int_0^S f(X(t))\,dt\right\}. \tag{6}$$

We will suppose that $Hf(x)$ is finite for each x and for each continuous function f with compact support, and is given by a continuous symmetric kernel relative to Lebesgue measure; that is,

$$Hf(x) = \int H(x, y) f(y)\,dy, \quad H(x, y) = H(y, x). \tag{7}$$

THEOREM. *If the potential kernel of the process satisfies conditions* A, B, *and* C *below, then*

$$\limsup_{a \to 0} T(a; S)/\lambda(a) \log\log a^{-N}\lambda(a) = 1 \tag{8}$$

for almost all paths with initial point at the origin, where $\lambda(a)$ *is the largest eigenvalue of the potential kernel on the sphere of radius* a*:*

$$\lambda(a) = \max\left\{\iint_{|x|, |y| \leq a} H(x, y)\phi(x)\phi(y)\,dx\,dy; \int_{|x| \leq a} \phi^2(x)\,dx = 1\right\}. \tag{9}$$

For the transient stable processes, taking $S \equiv \infty$,

$$\begin{aligned} H(x, y) &= C_{\alpha,N}\,|x - y|^{\alpha - N}, \\ C_{\alpha,N} &= \Gamma\left(\frac{N - \alpha}{2}\right)/2^{\alpha}\pi^{N/2}\Gamma\left(\frac{\alpha}{2}\right), \end{aligned} \tag{10}$$

where $0 < \alpha \leq 2$ if $N \geq 3$, $\alpha < N$ if $N \leq 2$. Because of the homogeneity of the kernel, $\lambda(a) = a^{\alpha}\lambda(1)$, and the result of the theorem becomes

$$\limsup T(a; S)/a^{\alpha} \log\log a^{-1} = \lambda(1). \tag{11}$$

In this case, also, the result is invariant under translation and can be applied toward proving (1) and (2).

The conditions of the theorem do not cover the recurrent stable processes. For planar Brownian motion, however, the result proved in [4] agrees with (8), since $\lambda(a) \sim \frac{1}{2}a^2 \log a^{-1}$. For the Cauchy process stopped at an exponentially distributed time independent of the paths,

$$
\tag{12}
\begin{aligned}
H(x, y) &= \frac{1}{\pi} \log \frac{1}{|x-y|} + 0(1), \\
\lambda(a) &= \frac{2}{\pi} a \log \frac{1}{a} + 0(1).
\end{aligned}
$$

Part of the proof below applies, with the result that for the Cauchy process starting at the origin, with probability one,

$$
\tag{13}
\limsup T(a; S)/\lambda(a) \log \log a^{-1}\lambda(a) \geq 1.
$$

As in [4], however, quite different techniques from those of this paper are needed to prove the opposite inequality.

The distinction between the cases, rather than transience and recurrence, is that the singularity of the potential kernel grows slower than a power for the Cauchy process. A condition that restricts processes to those resembling the transient stable processes is the following.

CONDITION A. *As a tends to zero,*

$$
\tag{14}
\mu(a) = \min \{H(x, y): |x| \leq a, |y| \leq a\} \sim Ca^{-\beta}\ell(a),
$$

with $0 < \beta < N$, *where* $\ell(a)$ *is a slowly varying function; that is,*

$$
\tag{15}
\lim_{a \to 0} \ell(ba)/\ell(a) = 1,
$$

uniformly for b bounded above and below.

The proof in outline follows that given in [2]. The first step is to estimate the distribution of the sojourn time in terms of the eigenvalue $\lambda(a)$. Sufficient to do this is condition B.

CONDITION B. *For some integer p and some constant* K_1,

$$
\tag{16}
H_a^{(p)}(x, x) \leq K_1 a^{N(p-1)}(\mu(a))^p, \qquad |x| \leq a,
$$

when a is small.

Here $H_a^{(p)}$ is the p-th iterate of the potential kernel on the sphere of radius a; that is,

$$
\tag{17}
\begin{aligned}
H_a(x, y) &= H_a^{(1)}(x, y) = H(x, y), \qquad |x|, |y| \leq a, \\
H_a^{(p+1)}(x, y) &= \int_{|u| \leq a} H_a^{(p)}(x, u) H(u, y)\, du.
\end{aligned}
$$

The estimate which we will obtain in section 2 under condition B easily provides a sure upper bound for the sojourn times when A also holds. That this upper bound is the best possible is proved by breaking the path into segments at the passage times out of a sequence of spheres. By the strong Markov property, these segments are independent, conditional on the places of passage, and the sojourn times during the segments together with the places of passage form a Markov chain. The estimates of section 2 can be applied to these segments if the spheres are sparse enough that a return to one of them after passage out of the next larger is a rare event, since then the sojourn time during the segment is almost the entire sojourn time.

We therefore need estimates of the distribution of the places of passage out of spheres and of the probability of returning to spheres. These probabilities were found for the symmetric stable processes explicitly in [1], and we can use the same method to get appropriate estimates if the potential kernel has a fairly regular growth at the origin, and if the singularity at the origin predominates.

CONDITION C. *There are constants* K_2 *and* K_3 *such that*

$$h_a(x) \equiv \int_{|x| \le a} H(x, y)\, dy \le K_2 a^N \mu(a \vee |x|) \tag{18}$$

if a *and* $|x|$ *are small;*

$$h_a(x') \le h_a(x) + K_3 a^N \tag{19}$$

if $|x| \le |x'|$, *again with* a *and* $|x'|$ *small. Here* $a \vee |x|$ *is of course the larger of the numbers* a *and* $|x|$.

It is easy to verify that the transient symmetric stable processes satisfy A, B, and C. Note also that the Cauchy process satisfies B and C, and also A if $\beta = 0$ were allowed. One can construct other processes for which the conditions hold, for example diffusions, but obviously the necessary estimates can be obtained more efficiently in any special case. The form of the conditions is intended not to provide generality but rather to indicate that the singularity of the potential kernel determines the path's local behavior in a relatively detailed way.

2. Estimates of the sojourn time distribution

In this section we need assume only property B of the potential kernel, to prove lemma 1.

LEMMA 1. *As* $t/\lambda(a)$ *tends to infinity,*

$$\log \mathcal{P}_x\{T(a; S) > t\} = -t/\lambda(a) + 0(1), \tag{20}$$

uniformly for $|x| \le a$, *if* a *is small.*

We begin with some facts about the spectrum of the operator H_a. First, a simple semigroup argument shows that H, and hence also its restriction H_a to the sphere of radius a, is positive definite. Let R_s be the resolvent operator of the process

$$R_s f(x) = \mathcal{E}_x \left\{ \int_0^S e^{-st} f(x(t))\, dt \right\}. \tag{21}$$

Putting $f = H(\cdot, y)$, it is just a matter of interchanging the order of integration to prove a special case of the resolvent equation $sR_sH = sHR_s = H - R_s$, $s > 0$. This and the symmetry of H imply that R_s is symmetric. It is also standard that $H = \int_0^\infty R_s^2\, ds$, which with the symmetry of R_s implies that H is positive definite.

Now B states that for some integer p, Mercer's theorem applies to the p-th iterate of H_a, namely H_a has an orthonormal sequence of continuous eigenfunctions $\phi_n(x) = \phi_n(x; a)$, $n = 1, 2, \cdots$, with corresponding eigenvalues $\lambda_n = \lambda_n(a)$, satisfying

$$H_a^{(p)}(x, y) = \sum \lambda_n^p \phi_n(x)\phi_n(y) \le K_1 a^{N(p-1)}(\mu(a))^p, \tag{22}$$

the series converging absolutely and uniformly for $|x|$, $|y| \le a$. In particular, if we let $\lambda_1 = \lambda = \lambda(a)$ be the largest eigenvalue, then

$$\lambda^p \phi_1^2(x) \le K_1 a^{N(p-1)}(\mu(a))^p. \tag{23}$$

Integrating over the sphere,

$$\lambda(a) \le K_4 a^N \mu(a). \tag{24}$$

On the other hand, with the usual notation for the inner product of square integrable functions on the sphere, $\lambda = (H_a\phi_1, \phi_1) \ge (H_a\psi, \psi)$ whenever $(\psi, \psi) = 1$. This implies that λ is simple and $\phi_1 \ge 0$, at least if a is small, for since $H(x, y) \ge \mu(a) > 0$, $(H_a|\phi_1|, |\phi_1|)$ exceeds $(H_a\phi_1, \phi_1)$ unless $\phi_1 = |\phi_1|$. It follows from $\phi_1 \ge 0$ that

$$\begin{aligned} \lambda\phi_1(x) &= H_a\phi_1(x) \\ &= \int_{|y| \le a} H(x, y)\phi_1(y)\, dy \\ &\ge \mu(a)(\phi_1, 1), \end{aligned} \tag{25}$$

so that, upon integration over the sphere,

$$\lambda(a) \ge k_1 a^N \mu(a). \tag{26}$$

By (23) and (26), $\phi_1(x) = 0(a^{-N/2})$, $|x| \le a$, which implies

$$(\phi_1, 1) \ge k_2 a^{N/2}(\phi_1, \phi_1) = k_2 a^{N/2}. \tag{27}$$

Turning now to the proof of lemma 1, we write

$$\begin{aligned} F(x, t) &= \mathcal{P}_x\{T(a; S) > t\}, \\ G(x) &= \mathcal{E}_x\{e^{-uT(a;S)}\}. \end{aligned} \tag{28}$$

Using the Markov property,

$$\begin{aligned} G(x) &= 1 - \mathcal{E}_x\left\{1 - \exp\left[-u \int_0^S V(X(\tau); a)\, d\tau\right]\right\} \\ &= 1 - \mathcal{E}_x\left\{\int_0^S dt\, uV(X(t); a) \exp\left[-u \int_t^S V(X(\tau); a)\, d\tau\right]\right\} \\ &= 1 - \mathcal{E}_x\left\{\int_0^S dt\, uV(X(t); a)G(X(t))\right\} \\ &= 1 - uH_aG(x). \end{aligned} \tag{29}$$

This equation allows us to compute the expansion of G in terms of the eigenfunctions of H_a: $(G, \phi_n) = (1 + u\lambda_n(a))^{-1}(\phi_n, 1)$. We can invert the Laplace transform to get $(F, \phi_n) = \exp\{-t/\lambda_n\}(\phi_n, 1)$. Note now that the expansion for F converges uniformly for $|x| \le a$:

$$F(x, t) = \sum \exp \{-t/\lambda_n\} (\phi_n, 1)\phi_n(x), \tag{30}$$

and can be differentiated term by term in t. Write

$$\begin{aligned} F_1(x, t) &= -\frac{\partial}{\partial t} F(x, t) \\ &= \sum \frac{1}{\lambda_n} e^{-t/\lambda_n}(\phi_n, 1)\phi_n(x); \end{aligned} \tag{31}$$

then $H_a F_1 = F$. Since $F_1 \geq 0$,

$$\begin{aligned} F(x, t) &= \int_{|y| \leq a} H(x, y) F_1(y, t)\, dy \\ &\geq \mu(a) \int_{|y| \leq a} F_1(y, t)\, dy \\ &= \mu(a) \sum \frac{1}{\lambda_n} e^{-t/\lambda_n}(\phi_n, 1)^2 \\ &\geq \frac{\mu(a)}{\lambda(a)} e^{-t/\lambda}(\phi_1, 1)^2 \\ &\geq (k_2^2/K_4) e^{-t/\lambda(a)} \end{aligned} \tag{32}$$

by (24) and (27), if $|x| \leq a$.

To get an upper bound for $F(x, t)$, use the Schwartz inequality

$$F(x, t) \leq \{\sum (\phi_n, 1)^2\}^{1/2} \{\sum e^{-2t/\lambda_n} \phi_n^2(x)\}^{1/2}. \tag{33}$$

Using condition B and the bound (26), if $2t/\lambda(a) \geq p$,

$$\begin{aligned} e^{2t/\lambda(a)} \sum e^{-2t/\lambda_n} \phi_n^2(x) &= \sum \exp \left[-\frac{2t}{\lambda} \left(\frac{\lambda}{\lambda_n} - 1 \right) \right] \phi_n^2(x) \\ &\leq \sum (\lambda_n/\lambda)^p \phi_n^2(x) \\ &= (\lambda(a))^{-p} H_a^{(p)}(x, x) \\ &\leq K_1 k_1^{-p} a^{-N}. \end{aligned} \tag{34}$$

Since also $\sum (\phi_n, 1)^2 = (1, 1)^2 = 0(a^N)$, it follows that when $|x| \leq a$ and $t/\lambda(a)$ is large, $F(x, t) = 0(e^{-t/\lambda(a)})$, and this completes the proof of lemma 1.

3. Estimates of hitting probabilities

Next we obtain from condition C estimates of probabilities relating to the passage of the process out of a sphere and its subsequent return to the sphere. The strong Markov property is the basic tool in what follows.

Let E be an open subset of R^N, and let $P = P(E)$ be the passage time of a path of the process into E, with the convention that $P(E) = S$ if the path never enters E:

$$\begin{aligned} P = P(E) &= \inf \{t < S;\ X(t) \in E\}, \\ &= S \quad \text{if} \quad X(t) \notin E, \qquad 0 \leq t < S. \end{aligned} \tag{35}$$

According to the strong Markov property, the stopped process $X(t)$, $0 \leq t < P$, and the renewed process $X(P + t)$, $0 \leq t < S - P$, are independent, conditional on the value of $X(P)$ and the event $P < S$; and with the same conditions, the renewed process has the same transition function as the original process.

In particular, the sojourn time $T(E'; S)$ in a subset E' of E depends only on the renewed process, so that, as in [1],

$$\text{(36)} \qquad \int_{E'} H(x, y)\, dy = \mathcal{E}_x\{T(E', S)\} = \mathcal{E}_x\left\{\int_{E'} H(X(P), y)\, dy;\ P < S\right\}.$$

Let us write $P = P(a)$ in case E is the exterior of the sphere of radius a about the origin; that is, $P(a) = \inf\{t\colon |X(t)| > a\}$.

LEMMA 2. *There is a positive number* ϵ *so that for small* a *and* b, *if* $\mu(b)/\mu(a)$ *is also small and* $|x| \leq a$,

$$\text{(37)} \qquad \mathcal{P}_x\{|X(P(a))| \leq b,\ P < S\} \geq 1 - K_5\left(\left(\frac{\mu(b)}{\mu(a)}\right)^{\epsilon} + \frac{1}{\mu(b)}\right).$$

The form of the result arises from the asymptotic behavior assumed in condition A, but without using the assumption $\beta > 0$.

For the proof we use (36) with E' a spherical shell $\{y\colon a < |y| < c\}$. Using the notation of C,

$$\text{(38)} \qquad \begin{aligned} h_c(x) - h_a(x) &= \int_{a<|y|<c} H(x, y)\, dy \\ &= \mathcal{E}_x\{h_c(X(P)) - h_a(X(P));\ P < S\} \\ &\leq (h_c(x) + K_3 c^N)\mathcal{P}_x\{|X(P)| \leq b,\ P < S\} \\ &\quad + K_2 c^N \mu(b)\mathcal{P}_x\{|X(P)| > b,\ P < S\}, \end{aligned}$$

since necessarily $|X(P)| \geq a$. This implies

$$\text{(39)} \qquad \mathcal{P}_x\{|X(P)| \leq b,\ P < S\} \geq 1 - \frac{h_a(x) + K_3 c^N}{h_c(x) - K_2 c^N \mu(b)}.$$

Since the minimum of H on the sphere of radius c is $\mu(c)$, $h_c(x)$ is bounded below by a multiple of $c^N\mu(c)$. If $\mu(b)/\mu(a)$ is small, we can find $c > a$ with $\mu(c) = M\mu(b)$, M being chosen so large that $K_2 c^N \mu(b) < \frac{1}{2}h_c(x)$. This, together with the bound for $h_a(x)$ in condition C, gives

$$\text{(40)} \qquad \frac{h_a(x) + K_3 c^N}{h_c(x) - K_2 c^N \mu(b)} = 0\left(\frac{a^N \mu(a)}{c^N \mu(c)} + \frac{1}{\mu(c)}\right).$$

Now since a slowly varying function must increase more slowly than any power, condition A implies that $\mu(a)/\mu(b)$ increases as a power less than N of c/a as c and a/c tend to zero,

$$\text{(41)} \qquad \frac{\mu(a)}{\mu(c)} = 0\left(\left(\frac{c}{a}\right)^{N/(1+\epsilon)}\right)$$

for some $\epsilon > 0$. But then

$$\frac{a^N \mu(a)}{c^N \mu(c)} = 0\left(\left(\frac{\mu(c)}{\mu(a)}\right)^{\epsilon}\right); \tag{42}$$

since $\mu(c)$ was chosen equal to a constant multiple of $\mu(b)$, we have the bound stated in lemma 2.

The other estimates we need concern the return time to the sphere, given by

$$R(a) = \inf \{t > P(a): |X(t)| < a\}. \tag{43}$$

Again we set $R(a) = S$ for paths which do not return in the interval $(P(a), S)$. If the process states at a point x with $|x| > a$, then of course $P(a) = 0$, and the return time coincides with the first passage time into the sphere.

LEMMA 3. *Suppose a, b and $\mu(b)/\mu(a)$ are all small. If $|x| \geq b$, then*

$$\mathcal{P}_x\{R(a) < S\} \leq K_6 \mu(b)/u(a); \tag{44}$$

whereas if $|x| \leq a$,

$$\mathcal{P}_x\{R(a) < S\} \geq k_3. \tag{45}$$

To prove the first estimate, use (36) taking for E the interior of the sphere of radius a. Since H has the lower bound $\mu(a)$ on the sphere, and since necessarily $|X(R)| \leq a$,

$$\begin{aligned} \int_{|y|<a} H(x, y)\, dy &= \mathcal{E}_x\left\{\int_{|y|<a} H(X(R), y)\, dy;\ R < S\right\} \\ &\geq k_1 a^N \mu(a) \mathcal{P}_x\{R(a) < S\}. \end{aligned} \tag{46}$$

If $|x| \geq b$, the bound assumed for the left side in condition C gives (44).

Next suppose $a < |x| \leq c$, and apply the bounds in the opposite direction:

$$\begin{aligned} k_1 a^N \mu(c) &\leq \int_{|y|<a} H(x, y)\, dy \\ &= \mathcal{E}_x\left\{\int_{|y|<a} H(X(R), y)\, dy;\ R < S\right\} \\ &\leq K_2 a^N \mu(a) \mathcal{P}_x\{R(a) < S\}. \end{aligned} \tag{47}$$

Finally, suppose $|x| \leq a$. The time $R(a)$ depends only on the path after the passage time $P = P(a)$, so that by the strong Markov property,

$$\begin{aligned} \mathcal{P}_x\{R(a) < S\} &= \mathcal{E}_x\{\mathcal{P}_{X(P)}\{R(a) < S\};\ P < S\} \\ &\geq \mathcal{E}_x\{\mathcal{P}_{X(P)}\{R(a) < S\};\ |X(P)| \leq c,\ P < S\} \\ &\geq (k_1\mu(c))/(K_2\mu(a)) \mathcal{P}_x\{|X(P)| \leq c,\ P < S\}. \end{aligned} \tag{48}$$

If we choose c so that $\mu(c)$ is a fixed but sufficiently small multiple of $\mu(a)$, then the last factor on the right has a positive lower bound by lemma 2, so that (45) follows.

4. Asymptotic behavior of the sojourn times

Turning to the main result, we find that a sure upper bound for the sojourn times in small spheres is easily obtained from the estimates of section 2, assuming $\beta > 0$ in condition A. This is the sole use of that part of A.

Fix a positive number δ. Since $\lambda(a)$ is a continuous increasing function of a, there is a sequence of numbers a_n decreasing to zero with $\lambda(a_{n+1})/\lambda(a_n) = 1 - \delta$. For this sequence we have certainly that $\log \lambda(a_n) = n \log (1 - \delta) + 0(1)$. On the other hand, $\lambda(a)$ is bounded below by a multiple of $a^N\mu(a)$, by (26), and μ has the asymptotic behavior assumed in A, so that

$$\begin{aligned}(49)\qquad \log \lambda(a_n) &= \log a_n^N \mu(a_n) + 0(1)\\ &= (N - \beta) \log a_n + 0(1).\end{aligned}$$

With the above this gives

$$(50)\qquad \log a_n^{-N}\lambda(a_n) = -\frac{\beta n}{N - \beta} \log (1 - \delta) + 0(1);$$

writing

$$(51)\qquad \sigma(a) = \log \log a^{-N}\lambda(a),$$

we have $\sigma(a_n) = \log n + 0(1)$, assuming $\beta > 0$.

By lemma 1,

$$\begin{aligned}(52)\qquad \log \mathcal{P}_0\{T(a_n;\ S) \geq (1 + \delta)\lambda(a_n)\sigma(a_n)\} &= -(1 + \delta)\sigma(a_n) + 0(1)\\ &= -(1 + \delta) \log n + 0(1).\end{aligned}$$

Thus the series $\sum \mathcal{P}_0\{T(a_n;\ S) \geq (1 + \delta)\lambda(a_n)\sigma(a_n)\}$ converges, and with probability one, $T(a_n;\ S) < (1 + \delta)\lambda(a_n)\sigma(a_n)$ for all but finitely many n.

Fix a path for which this holds. If a is small enough, then $a_{n+1} < a \leq a_n$, with $T(a;\ S) \leq T(a_n;\ S) < (1 + \delta)\lambda(a_n)\sigma(a_n)$. On the other hand, by the choice of a_n, $\lambda(a) \geq \lambda(a_{n+1}) = (1 - \delta)\lambda(a_n)$, while obviously $\sigma(a) = \sigma(a_n) + 0(1)$. The result is that with probability one, for a small,

$$(53)\qquad T(a;\ S) \leq \frac{1 + \delta}{1 - \delta} \lambda(a)\sigma(a) + 0(1).$$

Since δ can be arbitrarily small, $\limsup_{a\to 0} T(a;\ S)/\lambda(a)\sigma(a) \leq 1$.

The proof of the opposite inequality was sketched in the introduction. Fix a positive number δ. Since $\mu(a)$ is continuous, monotonic, and becomes infinite as $a \to 0$, we can find a sequence of numbers a_n decreasing to zero with

$$(54)\qquad \log \mu(a_n) = n^{1+\delta},$$

at least for n sufficiently large.

As in section 3, let $P_n = P(a_n) = \inf \{t\colon |X(t)| > a_n\}$ be the first passage time out of the sphere of radius a_n, again with $P_n = S$ if the passage does not occur before the terminal time S. Let $Y_n = X(P_n)$ for those paths for which $P_n < S$, and let

$$T_n = \int_{P_n}^{P_{n-1}} V(X(t);\ a_n)\, dt \tag{55}$$

be the sojourn time in the sphere during the segment (P_n, P_{n-1}).

Fix n momentarily. For $m > n$, T_m and Y_{m-1} depend only on the stopped path $X(t)$, $0 \le t < P_n$; for $m \le n$, T_m and Y_{m-1} depend only on the renewed path $X(P_n + t)$. By the strong Markov property, these two sets of variables are conditionally independent on the event $P_n < S$ and on the value of Y_n. In other words, (T_n, Y_{n-1}), $P_n < S$, form a Markov chain.

We proceed to estimate the transition function of this chain. Since, conditional on $P_n < S$, the renewed process is a copy of the original one with initial point Y_n,

$$\mathcal{P}\{T_n \le (1 - \delta)\lambda(a_n)\sigma(a_n),\ P_{n-1} < S | X(t),\ t < P_n\} = Q_n(Y_n) \tag{56}$$

for almost all paths with $P_n < S$, where

$$Q_n(x) = \mathcal{P}_x\{T_n \le (1 - \delta)\lambda(a_n)\sigma(a_n),\ P_{n-1} < S\}. \tag{57}$$

Now $Q_n(x)$ is defined for all x by the above, although it applies in (56) only for $|x| \ge a_n$. For $|x| \le a_n$, since the event on the right involves the path only after time P_n,

$$Q_n(x) = \mathcal{E}_x\{Q_n(Y_n);\ P_n < S\}. \tag{58}$$

Also for such x we can find a suitable estimate for $Q_n(x)$.

LEMMA 4. *If $|x| \le a_n$, for n large, then $Q_n(x) \le 1 - k_4 n^{-1}$.*

To prove the lemma, suppose first that a path starts at a point x with $|x| \ge a_n$. Then $P_n = 0$, and

$$T_n = T(a_n;\ S) - \int_{P_{n-1}}^{S} V(X(t);\ a_n)\, dt. \tag{59}$$

The integral on the right has the value zero unless the path returns to the sphere of radius a_n after time P_{n-1}. The probability of this event is

$$\begin{aligned} \mathcal{E}_x\{\mathcal{P}_{Y_{n-1}}\{R(a_n) < S\};\ P_{n-1} < S\} &\le K_c \mu(a_{n-1})/\mu(a_n) \\ &= K_c \exp\left[-n^{1+\delta} + (n-1)^{1+\delta}\right] \\ &= o(n^{-1}) \end{aligned} \tag{60}$$

by (44) and (54). Thus except with a probability small compared with $1/n$, $T_n = T(a_n;\ S)$, and

$$Q_n(x) \le Q_n^*(x) + o(n^{-1}), \qquad |x| \ge a_n, \tag{61}$$

where

$$Q_n^*(x) = \mathcal{P}_x\{T(a_n;\ S) \le (1 - \delta)\lambda(a_n)\sigma(a_n)\}. \tag{62}$$

In turn, $Q_n^*(s)$ is defined for all values of x, and lemma 1 gives an estimate when $|x| \le a_n$. In fact, since by (54)

$$(63) \qquad \sigma(a_n) = \log\log a_n^{-N}\lambda(a_n) = \log\log \mu(a_n) + 0(1) = (1+\delta)\log n + 0(1),$$

we have

$$(64) \qquad Q_n^*(x) \le 1 - \exp[-(1-\delta)\sigma(a_n) + 0(1)] \le 1 - k_5 n^{-(1-\delta^2)} \le 1 - k_5 n^{-1}, \qquad |x| \le a_n.$$

Supposing once more that $|x| \ge a_n$, the entire sojourn time $T(a_n;\, S)$ involved in $Q_n^*(x)$ occurs after the return time $R(a_n)$, vanishing if no return occurs. Hence, for $|x| \ge a_n$,

$$(65) \qquad Q_n^*(x) = \mathcal{E}_x\{Q_n^*(X(R(a_n)));\ R(a_n) < S\} + \mathcal{P}_x\{R(a_n) = S\} \le 1 - k_5 n^{-1}\mathcal{P}_x\{R(a_n) < S\}.$$

Finally, suppose $|x| \le a_n$ again. Using (58), (61), and the above,

$$(66) \qquad Q_n(x) \le \mathcal{E}_x\{Q_n^*(Y_n);\ P_n < S\} + o(n^{-1}) \le 1 - k_5 n^{-1}\mathcal{E}_x\{\mathcal{P}_{Y_n}\{R(a_n) < S\};\ P_n < S\} + o(n^{-1}) \le 1 - k_3 k_5 n^{-1} + o(n^{-1})$$

by (45), and this completes the proof of lemma 4.

With the preliminaries now over, fix m, and for $n \ge m$ let

$$(67) \qquad F_n(x) = \mathcal{P}_x\{T_j \le (1-\delta)\lambda(a_j)\sigma(a_j),\ m \le j \le n,\ P_{m-1} < S\}.$$

Since (T_n, Y_{n-1}) form a Markov chain,

$$(68) \qquad F_n(x) = \mathcal{E}_x\{F_{n-1}(Y_{n-1});\ T_n \le (1-\delta)\lambda(a_n)\sigma(a_n),\ P_{n-1} < S\};$$

and using the Schwartz inequality,

$$(69) \qquad (F_n(x))^2 \le \mathcal{E}_x\{(F_{n-1}(Y_{n-1}))^2;\ P_{n-1} < S\}Q_n(x).$$

We will establish, at least for large m,

$$(70) \qquad (F_n(x))^2 \le A_{m,n}Q_n(x), \qquad |x| \le a_{n-1},$$

where $A_{m,n} = \Pi_m^{n-1}(1 - k_4 j^{-1}) + 1/m - 1/n$. This is certainly true if $n = m$. Suppose it holds for given n. Then since $0 \le F_n \le 1$,

$$(71) \qquad (F_{n+1}(x))^2 \le Q_{n+1}(x)[\mathcal{E}_x\{(F_n(Y_n))^2;\ |Y_n| \le a_{n-1},\ P_n < S\} + \mathcal{P}_x\{|Y_n| > a_{n-1},\ P_n < S\}].$$

If n is large enough, lemma 2 applies, and

$$(72) \qquad \mathcal{P}_x\{|Y_n| > a_{n-1},\ P_n < S\} \le K_5\left(\left(\frac{\mu(a_{n-1})}{\mu(a_n)}\right)^\epsilon + \frac{1}{\mu(a_{n-1})}\right) = K_5 \exp[-\epsilon(n^{1+\delta} - (n-1)^{1+\delta})] + K_5\exp[-n^{1+\delta}] = o(n^{-2}).$$

On the other hand, using the induction hypothesis, (58), and lemma 4,

$$\begin{aligned}(73)\qquad \mathcal{E}_x\{(F_n(Y_n))^2;\ |Y_n| \le a_{n-1}, P_n < S\} &\le A_{m,n}\mathcal{E}_x\{Q_n(Y_n);\ P_n < S\}\\ &\le A_{m,n}Q_n(x)\\ &\le A_{m,n}(1 - k_4n^{-1})\end{aligned}$$

if $|x| \le a_n$. Thus for such x,

$$\begin{aligned}(74)\qquad (F_{n+1}(x))^2 &\le Q_{n+1}(x)\left[A_{m,n}(1 - k_4n^{-1}) + \frac{1}{2n^2}\right]\\ &\le Q_{n+1}(x)A_{m,n+1}\end{aligned}$$

if n is large, completing the proof of (18).

Putting $x = 0$, we have from (18) that $\lim_{m\to\infty}\lim_{n\to\infty} F_n(0) = 0$; but this implies that with probability one, $T(a_n;\ S) > (1 - \delta)\lambda(a_n)\sigma(a_n)$ occurs infinitely many times, or that $\limsup_{a\to 0} T(a;\ S)/\lambda(a)\sigma(a) \ge (1 - \delta)$. Since δ can be taken as small as desired, this completes the proof of the main result.

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