A NOTE ON MARKOV SEMIGROUPS WHICH ARE COMPACT FOR SOME BUT NOT ALL $t > 0$

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1. Introduction

In this note $\{P_t: t \geq 0\}$ will be a strongly continuous semigroup of transition operators on ℓ and R_λ will be the corresponding resolvent operator for $\lambda > 0$.

The following three statements are correct.

(i) If, for some $t > 0$, P_t is quasi-compact, then it is quasi-compact for all $t > 0$.

(ii) [(iii)] If, for some $\lambda > 0$, λR_λ is compact [quasi-compact], then it is compact [quasi-compact] for all $\lambda > 0$.

Statements (i) and (ii) are very easy to prove, and (iii) is established in each of the accompanying papers by D. Williams and J. G. Basterfield. This note shows that the fourth similar assertion is false by giving an example of a Markov semigroup for which P_t is compact if $t > 1$, but not compact if $0 < t < 1$.

2. The example

The states are labelled 0 and (m, n) for $n = 1, 2, \cdots$ and $m = 1, 2, \cdots, n$. Here 0 is an absorbing state and the (m, n)-th state feeds into the $(m - 1, n)$-th state at rate $n (m > 1)$, $(1, n)$ feeds into 0 also at rate n.

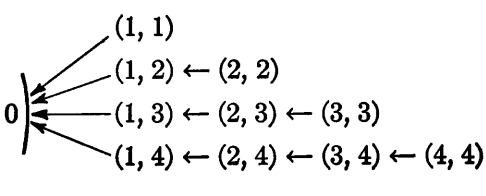

FIGURE 1

Thus $p_{(n,n),0}(t)$ is the distribution function of the sum of n independent negative exponential random variables, each with mean $1/n$; hence, if $t < 1$, then $p_{(n,n),0}(t) \rightarrow 0$ as $n \rightarrow \infty$ and if $t > 1$, then $p_{(n,n),0}(t) \rightarrow 1$. It is also clear that $p_{(m,n),0}(t) \geq p_{(n,n),0}(t)$ for $1 \leq m \leq n$ and for all t, and that for $j \geq N$,

$$(1)\qquad \sum_{n=N}^{\infty}\sum_{m=1}^{n} p_{(i,j),(m,n)}(t) = \sum_{m=1}^{i} p_{(i,j),(m,j)}(t) = 1 - p_{(i,j),0}(t).$$

When $t < 1$, if $0 < \epsilon < 1$, then for every given N we can choose $j \geq N$ such that $1 - p_{(j,j),0}(t) > \epsilon$, that is

$$(2)\qquad \sum_{n=N}^{\infty}\sum_{m=1}^{n} p_{(j,j),(m,n)}(t) > \epsilon,$$

and so P_t is not compact. On the other hand, if $t > 1$, then given $\epsilon > 0$ there exists N such that $1 - p_{(j,j),0}(t) < \epsilon$ for all $j \geq N$. Then

$$(3)\qquad 0 \leq \sum_{n=N}^{\infty}\sum_{m=1}^{n} p_{(i,j),(m,n)}(t) = 1 - p_{(i,j),0}(t) \leq 1 - p_{(j,j),0}(t) < \epsilon$$

for all $j \geq N$ and all $i \leq j$, and so P_t is compact.

3. The spectrum of the semigroup

For such a semigroup, R_λ is not compact for any λ, by Williams' theorem, but the assertions (i), (ii), and (iii) of his theorem remain true.

For when P_t is compact for some $t > 0$, it must be compact for all $s \geq t$ because $P_s = P_t P_{s-t}$. Thus for any $s > 0$, there exists an integer n such that $(P_s)^n = P_{ns}$ is compact. This implies that each nonzero point λ of the spectrum of P_s is a pole of order k, that the null-space of $(\lambda - P_s)^k$ is of nonzero finite dimension, and that the spectrum has no nonzero accumulation points ([2], VII.4.6). The proof of (i), (ii), and (iii) now follows as in section 4 of Williams' paper. Again, each P_t and R_λ will be quasi-compact so that assertion 1 is approachable.

4. Compactness at the critical value

It is of interest to note that, in general, P_t may be compact or not compact at the critical value of t. To construct an example of the first case we modify the semigroup of section 2 by taking the rate of departure from the (m, n)-th state to be $n/(1 - a_n)$ instead of n, where $0 < a_n < 1$ and $a_n \to 0$ as $n \to \infty$. In this case $p_{(n,n),0}(t)$ is the distribution function of a random variable with mean $1 - a_n$ and variance $(1 - a_n)^2/n$. From the Chebychev inequality we obtain

$$(4)\qquad 0 < 1 - p_{(n,n),0}(1) \leq (1 - a_n)^2/na_n^2$$

so that if also $na_n^2 \to \infty$ as $n \to \infty$ (for example, if $a_n = n^{-1/4}$), then

$$(5)\qquad 1 - p_{(n,n),0}(1) \to 0$$

and P_1 is compact. As before, P_t is not compact for $0 < t < 1$.

For an example in which P_1 is not compact, we can use a similar modification with $n/(1 + a_n)$ for the rate of departure from (m, n). We impose the same conditions on a_n.

I wish to thank Professor D. G. Kendall for suggesting this problem and Dr. Williams for allowing me to see his unpublished work. The present work was supported by the Science Research Council.

REFERENCES

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[2] N. DUNFORD and J. T. SCHWARZ, *Linear Operators, Part I*, New York, Interscience, 1958.

[3] D. WILLIAMS, "Uniform ergodicity in Markov chains," *Proceedings of the Fifth Berkeley Symposium on Mathematical Statistics and Probability*, Berkeley and Los Angeles, University of California Press, 1966, Vol. II, Part II, pp. 187–191.