APPLICATION OF ADDITIVE FUNCTIONALS TO THE BOUNDARY PROBLEM OF MARKOV PROCESSES (LÉVY'S SYSTEM OF U-PROCESSES)

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1. Introduction

Under the name of boundary problem for Markov processes, we shall consider the problem of finding all Markov processes whose behavior, before they reach the boundary, is the same as that of given minimal processes (cf. Feller [2]). In this paper, we shall characterize those processes by their U-processes (on the boundary) and certain auxiliary factors (cf. Sato [8]). The precise formulations and the summary of the paper are given in sections 2 and 3. The author wishes to express his gratitude to Professors M. Nagasawa, K. Sato, and T. Ueno for their kind discussions and advice, and especially to Professor K. Sato who read the paper and suggested many improvements.

2. Assumptions and notations

The space S is a compact space with metric r and $S^* = S \cup \{\partial\}$ where ∂ is an isolated (extra) point; D is a fixed open set in S such that $S = \bar{D}$ and V is the boundary of D. As sample paths on the space $S^* = S \cup \{\partial\}$ or $V^* = V \cup \{\partial\}$, we consider paths which are right continuous, have left limits, and stay at ∂ after they reach ∂. The path is denoted by w and $x_t(w) \in S^*$ (or $\xi_t(w) \in V^*$) is the value of w at t. We shall set $x_\infty(w) = \partial$, $\sigma_E = \inf \{t > 0: x_t(w) \in E\}$, and $\zeta = \inf \{t > 0: x_t(w) \in D\}$ with $\inf \phi = \infty$. A Markov process defined on the space of the above sample paths is called a Hunt process if it satisfies the conditions (P.1), (P.2), (P.3), and (P.4) in [5]. Roughly speaking, a Hunt process is a right-continuous and quasi-left-continuous strong Markov process. When referring to "subsets of S," we shall mean only topological Borel subsets of S. For $E \subset S$, $B(E)$ is the set of all bounded measurable functions on E, and $B^+(E)$ (resp. $C(E)$) means the subset of $B(S)$ consisting of the functions which are nonnegative (resp. continuous). Sometimes, we consider f in $B(E)$ as a function on δ^*, setting $f(x) = 0$, $x \notin E$. For $E, F \subset S$, $K(x, A)$, $(x \in E, A \subset F)$ is called a kernel on $E \times F$ if $K(\cdot, A)$ is Borel measurable on E and $K(x, \cdot)$ is a measure on F. If $K(x, A)$ is a kernel, we write $Kf(x) = \int K(x, dy)f(y)$.

2.1. *Minimal process.* A process $\mathbf{M}_0$ is called a *minimal process* on D if the following conditions $(\mathbf{M}_0.1) \sim (\mathbf{M}_0.6)$ are satisfied (throughout this paper $\gamma > 0$ is fixed and $\sigma = \sigma_V$ is the hitting time to V):

$(\mathbf{M}_0.1)$ $\mathbf{M}_0$ is a Hunt process on S;

$(\mathbf{M}_0.2)$ $P^0_\xi(x_t = \xi, 0 \leq t < \infty) = 1, \qquad \xi \in V,$

where $P^0_\cdot$ and $E^0_\cdot$ are the probabilities and the expectations for $\mathbf{M}_0$.

For $f \in B(S)$, let

$$(1) \qquad \begin{aligned} G^0_\alpha f(x) &= E^0_x\left(\int_0^\sigma e^{-\alpha t} f(x_t)\, dt\right), && (\alpha > 0), \\ H_\alpha f(x) &= E^0_x(e^{-\alpha\sigma} f(x_\sigma)\colon \sigma < \infty), && (\alpha \geq 0); \end{aligned}$$

then the Green kernel of $\mathbf{M}_0$ is $G^0_\alpha + (1/\alpha)\, H_\alpha$; that is,

[2.1] $E^0_x(\int_0^\infty e^{-\alpha t} f(x_t)\, dt) = G^0_\alpha f(x) + (1/\alpha)\, H_\alpha f(x)$, and $\mathbf{M}_0$ is uniquely determined by G^0_α and H_α.

We can easily see that

$$(2) \qquad G^0_\alpha - G^0_\beta + (\alpha - \beta) G^0_\alpha G^0_\beta = 0,$$

$$(3) \qquad H_\alpha - H_\beta + (\alpha - \beta) G^0_\alpha H_\beta = 0,$$

$$(4) \qquad G^0_\alpha f(\xi) = 0 \quad \text{and} \quad H_\alpha f(\xi) = f(\xi) \quad \text{for} \quad \xi \in V.$$

$(\mathbf{M}_0.3)$ There exists a measure m_0 on D such that $m_0(E) = 0$ is equivalent to $G^0_\alpha(x, E) = 0$ for every $x \in S$;

$(\mathbf{M}_0.4)$ $G^0_\alpha f \in C(S)$ if $f \in C(S)$, and $H_\alpha f \in C(S)$ if $f \in C(V)$;

$(\mathbf{M}_0.5)$ $\hat{H}_\alpha f = (G^0_\alpha f / g_\gamma) \in C(S)$ if $\alpha > 0$ and $f \in C(S)$, where $g_\gamma(x) = G^0_\gamma 1(x)$ and γ is a fixed positive constant and 1 denotes the function which is 1 on S and 0 at ∂.

Since $G^0_\alpha f(x)/g_\gamma(x)$ is in $C(D)$, $(\mathbf{M}_0.5)$ means that $G^0_\alpha f/g_\gamma$ can be extended to S continuously. By (2) we can easily see that if $(\mathbf{M}_0.5)$ holds for some $\alpha_0 > 0$, then it also holds for every $\alpha > 0$. As functional on $C(S)$, $\hat{H}_\alpha$ can be considered as a kernel on $S \times S$, and $\hat{H}_\alpha f$ is well defined for $f \in B(S)$:

$(\mathbf{M}_0.6)$ $\{\hat{H}_\alpha f\colon f \in C(S)\}$ is dense in $C(S)$.

By (2), we have

$$(5) \qquad \hat{H}_\alpha - \hat{H}_\beta + (\alpha - \beta)\hat{H}_\alpha G^0_\beta = 0.$$

Throughout this paper a minimal process $\mathbf{M}_0$ and $\gamma > 0$ are fixed. The conditions need some explanations. We did not hesitate to impose conditions on $\mathbf{M}_0$, if they are convenient for the following argument and if they are satisfied by ordinary regular processes. Condition $(\mathbf{M}_0.4)$ implies that every point of V is an exit point, and quasi-left continuity on $\mathbf{M}_0$ near V assures that it is not too wide as an exit boundary, although it may be too small. Condition $(\mathbf{M}_0.5)$ assures (except for the smoothness part) that for every point of V there corresponds at most one entrance point, and therefore V is not too small as an entrance boundary. For example, let $S = [-1, 1]$, $V = \{-1, 0, 1\}$, and let $\mathbf{M}_0$ be the Brownian motion stopped by V. Then ($\mathbf{M}_0$.5) is not satisfied near 0, since 0 consists of two entrance points (see [8]). There exist more essential examples which do not satisfy ($\mathbf{M}_0$.5). In these the point of V cannot be divided as an exit boundary, but it consists of many entrance boundary points; that is, G^0_α/g_γ has many limiting values at V.

The following processes, which are stopped by the boundaries, satisfy condition ($\mathbf{M}_0$.1) $\sim$ ($\mathbf{M}_0$.6):

(a) one-dimensional diffusion in an interval with exit boundaries;
(b) Brownian motion in a unit sphere or half plane;
(c) space time Brownian motion in a band-like domain parallel to the time axis;
(d) stable processes with exponent $\alpha > 1$ in a finite interval.

In the cases (b) and (c), suitable compactifications are needed.

2.2. *Processes with minimal process* $\mathbf{M}_0$ (*extension of* $\mathbf{M}_0$). The process $\mathbf{M}$ is called an *extension of* $\mathbf{M}_0$ if the following conditions ($\mathbf{M}$.1), ($\mathbf{M}$.2), and ($\mathbf{M}$.3) are satisfied:

($\mathbf{M}$.1) $\mathbf{M}$ is a Hunt process on S.

($\mathbf{M}$.2) Let G_α be the Green kernel of $\mathbf{M}$. There exists measure m on S such that $m(E) = 0$ is equivalent to $G_\alpha(x, E) = 0$ for every $x \in S$.

Under ($\mathbf{M}$.1), ($\mathbf{M}$.2) is equivalent to the condition (L) in [3] or the condition (P.5) in [5]. (See also [4].)

($\mathbf{M}$.3) The process $\mathbf{M}$ stopped by V is $\mathbf{M}_0$; that is,

$$E_x\left(\int_0^\sigma e^{-\alpha t} f(x_t)\,dt\right) = G^0_\alpha f(x), \tag{6}$$

$$E_x(e^{-\alpha\sigma} f(x_\sigma)) = H_\alpha f(x), \tag{7}$$

where the expectations and the probabilities for $\mathbf{M}$ are denoted by $E_.$ and $P_.$. By (4) and ($\mathbf{M}$.3),

[2.2] every point of V is regular to V with respect to $\mathbf{M}$.

Sometimes we shall use the following conditions.

($\mathbf{M}$.4) The sample path of $\mathbf{M}$ has no sojourn on V. a.e.; that is, $G_\alpha \chi_V(x) = 0$ for every $x \in S$. (We say that an assertion $\mathbf{A}$ holds a.e., if $P_x(\mathbf{A}) = 1$ for all x.)

($\mathbf{M}$.5) The sample path of $\mathbf{M}$ has no jump from V to D, a.e.; that is, $P_x(\sum_s \chi_V(x_{s-})\chi_D(x_s) = 0) = 1$ for every $x \in S$, where χ_V and χ_D are the characteristic functions on V and D respectively.

($\mathbf{M}$.6) The sample path of $\mathbf{M}$ has no jump from V to V, a.e.; that is, $P_x(\sum_s \chi_V(x_{s-})\chi_V(x_s) = 0) = 1$ for every $x \in S$.

($\mathbf{M}$.C) If $f \in C(S)$, then $G_\alpha f \in C(S)$.

Under ($\mathbf{M}$.1) and ($\mathbf{M}$.C), G_α, restricted on $C(S)$, becomes the resolvent of a strongly continuous semigroup on $C(S)$.

2.3. *Process on the boundary* (U-*process*). In this paper by "additive functionals" we mean nonnegative additive functionals only, unless otherwise stated. The definition is given in ([4], (A.1) $\sim$ (A.6)). For two additive functionals A and B, $A \approx B$ if and only if $P_x(A(t, w) = B(t, w)$ for all $t) = 1$ for all x, and $A \ll B$ if and only if $P_x(A(t, w) \leq B(t, w)$ for all $t) = 1$ for all x. We set $f \cdot A(t) = \int_0^t f(x_t)\, dA$ for $f \in B(S)$.

For given **M**, the *U-process* of **M** (introduced by T. Ueno) is obtained as follows (see [9] and [5]). Let **M** be a process satisfying (**M**.1), (**M**.2), and [2.2]. Then there exists a unique continuous additive functional Φ such that

$$E_x\Big(\int_0^\infty e^{-\gamma t}\, d\Phi\Big) = E_x\Big(\int_\sigma^\infty e^{-\gamma t}\, dt\Big) = H_\gamma G_\gamma 1(x); \tag{8}$$

that is, $\Phi = \tilde{T}_\gamma$ where T is the additive functional $T(t) = t \wedge \zeta$, and in general, for a continuous additive functional A, such that $E_x(\int_0^\infty e^{-\alpha t}\, dA) < \infty$, we define a continuous additive functional $\tilde{A}_\alpha$ by $E_x(\int_0^\infty e^{-\alpha t}\, d\tilde{A}_\alpha) = E_x(\int_\sigma^\infty e^{-\alpha t}\, dA)$ (see [4]). The functional Φ satisfies

$$\chi_V \cdot \Phi \approx \int_0^t \chi_V \cdot d\Phi \approx \Phi. \tag{9}$$

Now set $\tau(s) = \sup \{t\colon \Phi(t) \leq s\}$; then the process $\tilde{\mathbf{M}}$ on V defined by $(x_{\tau(s)}, P_\xi)$, $(\xi \in V)$ is called the *U-process* of **M** on V. The Green kernel $\tilde{G}_\lambda$ of $\tilde{\mathbf{M}}$ is given by

$$\begin{aligned} \tilde{G}_\lambda f(\xi) &= E_\xi\Big(\int_0^\infty e^{-\lambda s} f(x_{\tau(s)})\, ds\Big) \\ &= E_\xi\Big(\int_0^\infty e^{-\lambda\Phi(t)} f(x_t)\, d\Phi\Big), \end{aligned} \tag{10}$$

for every $\lambda > 0$, $\xi \in V$ and $f \in B(V)$.

It can also be shown that

($\tilde{\text{M}}$.1) $\tilde{\mathbf{M}}$ is a Hunt process on V;

($\tilde{\text{M}}$.2) there exists a measure ν on V such that $\nu(E) = 0$ is equivalent to $\tilde{G}_\lambda(\xi, E) = 0$ for every $\xi \in S$.

The results stated above are proved in [5]. Further properties of **M** are discussed in section 6. In the following, the expectations and the probabilities by $\tilde{\mathbf{M}}$ are denoted by $\tilde{E}_.$ and $\tilde{P}_.$.

2.4. *Definition of a Lévy system.* For any process **M** satisfying (**M**.1) and (**M**.2), a Lévy system was introduced by S. Watanabe in [11]. Here, we give the definition in a slightly different form. (It is easy to show that the following definition is equivalent to that in [11]).

Let L be a continuous additive functional and $P(x, dy)$ be a kernel on $S \times S$.

DEFINITION 2.1. *A pair (P, L) is called a* Lévy system *of* **M** *if and only if $P(x, \{x\}) = 0$ and*

$$E_x\Big(\sum_{s \leq t} f(x_{s-}, x_s)\Big) = E_x\Big(\int_s^t Pf(x_s)\, dL\Big) \tag{11}$$

for any $t > 0$ and $f \in B_0^+(S \times S)$, where

$$B_0^+(S \times S) = \{f\colon f \in B(S \times S) \quad \text{and} \quad f(x, x) = 0\} \tag{12}$$

and $Pf(x) = \int P(x, dy) f(y)$.

The Lévy system is unique in the following sense: if (P, L) and (P', L') are Lévy systems of the same process $\mathbf{M}$, then

$$Pf\cdot L \approx P'f\cdot L' \quad \text{for any} \quad f \in B_0^+(S\times S). \tag{13}$$

Condition (11) is equivalent to

$$E_x\Big(\sum_{t>0} e^{-\alpha t} f(x_{t-}, x_t)\Big) = E_x\Big(\int_0^\infty e^{-\alpha t} Pf(x_t)\, dL\Big) \tag{14}$$

for some $\alpha > 0$ and every $f \in B_0^+(S\times S)$.

The existence of the Lévy system is proved in [11]. It is also shown that [2.3] there exists an increasing sequence $\{E_n\}$ such that $E \subset S \times S - D$, $E_n \uparrow S\times S - D$, $(n \to \infty)$ and $E_x(\int_0^\infty e^{-\alpha t} P\chi_{E_n}\cdot dL)$ is bounded in x, for a fixed $\alpha > 0$. Here $D = \{(x, x): x \in S\}$.

Similarly, a continuous additive functional A_∞ is called the *killing functional* of $\mathbf{M}$ if

$$E_x(A_\infty(\infty)) = P_x(\zeta < \infty), \tag{15}$$

which is equivalent to

$$E_x(A_\infty(t)) = P_x(\zeta < t), \qquad t < \infty, \tag{16}$$

or

$$E_x\Big(\int_0^\infty e^{-\alpha t}\, dA_\infty\Big) = E_x(e^{-\alpha\zeta}), \qquad \alpha > 0. \tag{17}$$

(See [11] and [7].)

3. A problem of Sato and related topics

The purpose of this paper is to characterize $\mathbf{M}$ by its U-process $\tilde{\mathbf{M}}$ for a given minimal process $\mathbf{M}_0$. The problem is divided into the following three questions, where V, $\gamma > 0$, and $\mathbf{M}_0$ are fixed.

3.1. *Is* $\mathbf{M}$ *uniquely determined by* $\tilde{\mathbf{M}}$? K. Sato proposed this problem and solved it under a different formulation (see [8]). The answer is also affirmative in our case when $\mathbf{M}$ has no jump from V to D and no sojourn on V.

Let $\mathbf{M}$ satisfy $(\mathbf{M}.1)$–$(\mathbf{M}.3)$ and let $\tilde{\mathbf{M}}$ be the U-process of $\mathbf{M}$. Let

$$T_c = \{s;\, \tau(s-) < \tau(s),\, \tau(s-) < \zeta,\, x_{\tau(s-)-} = x_{\tau(s)}\}. \tag{1}$$

Let ℓ and m be elements of $B^+(V)$ and Q be a kernel on $V \times D$ such that

$$\ell\cdot\Phi \approx \chi_V\cdot T, \tag{2}$$

$$E_x\Big(\int_0^\infty e^{-\gamma t} m\, d\,\Phi\Big) = E_x\Big(\sum_{s\in T_c}\int_{\tau(s-)}^{\tau(s)} e^{-\gamma t}\, dt\Big), \tag{3}$$

$$Qf\cdot\Phi \approx \chi_V P_D(fg_\gamma)\cdot L \tag{4}$$

for any $f \in B(S)$, where (P, L) is the Lévy system of $\mathbf{M}$ and $P_D f(x) = \int_D P(x, dy) f(y)$. It will be shown that (ℓ, m, Q), satisfying (2), (3), and (4), exist. Let ν be a measure which satisfies $(\tilde{\mathbf{M}}.2)$; then (ℓ, m, Q) is uniquely determined up to equivalence with respect to ν. We shall call the system $(\tilde{\mathbf{M}}, \ell, m, Q)$ the *boundary system* of $\mathbf{M}$ (the system $(\tilde{\mathbf{M}}, \ell, m, Q)$ changes if γ changes, and the dependence of $(\tilde{\mathbf{M}}, \ell, m, Q)$ on γ will be discussed in appendix II). We shall also see that

($\tilde{\mathbf{M}}$.3) $\ell(\xi) + m(\xi) + Q(\xi, D) = 1$, a.e. ν.

Roughly speaking, $\ell(\xi)$, $m(\xi)$, and $Q(\xi, D)$ represent the (suitably weighted) proportion of sojourn on V, of reflection, and of jump from ξ to D $(\xi \in V)$, and the measure $Q(\xi, \cdot)$ denotes the mode of jump from V to D averaged by g_γ. In fact, we shall see in section 5 that the following proposition holds.

PROPOSITION 2. (i) *The path of* $\mathbf{M}$ *has no sojourn on* V, *a.e., if and only if* $\ell = 0$, *a.e.*, ν.

(ii) *The path of* $\mathbf{M}$ *has no excursion which begins from* V, *a.e., if and only if* $m = 0$, *a.e.*, ν.

(iii) *The path of* $\mathbf{M}$ *has no jump from* V *to* D, *a.e., if and only if* $Q = 0$, *a.e.*, ν.

Now the problem stated is answered as follows in section 5.

THEOREM 4. *The process* $\mathbf{M}$ *is uniquely determined by the boundary system* $(\tilde{\mathbf{M}}, \ell, m, Q)$.

3.2. *What process can* $\tilde{\mathbf{M}}$ *be?* Let $\mathbf{M}$ satisfy ($\mathbf{M}$.1) $\sim$ ($\mathbf{M}$.3). We shall investigate the properties which characterize the system $(\tilde{\mathbf{M}}, \ell, m, Q)$ as a boundary system. We have already seen that $(\tilde{\mathbf{M}}, \ell, m, Q)$ satisfies ($\tilde{\mathbf{M}}$.1) $\sim$ ($\tilde{\mathbf{M}}$.3). Moreover, we shall see that

($\tilde{\mathbf{M}}$.4) $m\hat{H}_\alpha\chi_V = 0$, a.e. ν for every $\alpha > 0$;

($\tilde{\mathbf{M}}$.5) set $E = \{\xi \colon \ell(\xi) + m(\xi) > 0\}$; then

$$\int_0^t \left(\frac{1}{1-\chi_E} + Q\left(\frac{1}{g_\gamma}\right)\right) dt = \infty \qquad \text{for every} \quad t > 0, \quad \text{a.e.} \tag{5}$$

Roughly speaking, ($\tilde{\mathbf{M}}$.4) implies that the path of $\mathbf{M}$ has no reflection at the purely exit points (exit but nonentrance points). In fact, in one-dimensional diffusion, $\hat{H}_\alpha\chi_V(\xi) > 0$ when ξ is a purely exit point. Condition ($\tilde{\mathbf{M}}$.5) implies that the path of $\mathbf{M}$ should have sufficiently many infinitesimal jumps near the point at which the path has no sojourn and no reflection. For example, if $\ell = m = 0$ and the carrier of $Q(\xi, \cdot)$ is contained in some compact set independent of ξ, then $\mathbf{M}$ should be an instantaneous return process which we exclude in our formulation.

Let $(\tilde{P}, \tilde{L})$ and $\tilde{A}_\infty$ be the Lévy system and the killing functional of $\tilde{\mathbf{M}}$. Then we shall see that

($\tilde{\mathbf{M}}$.6) $(\tilde{P}, \tilde{L}) \gg ((m + Q)\Theta, T)$, that is, $\tilde{P}f\cdot\tilde{L} \gg \{(m + Q)\Theta\}f\cdot\tilde{T}$ for every $f \in B^+(V \times V)$.

Here $T = t \wedge \zeta$ in $\tilde{\mathbf{M}}$ and $\Theta(x\cdot d\eta) = \lim_{\alpha\to\infty} \alpha\hat{H}_\alpha H(x, d\eta)$, $x \notin d\eta$, and $\Theta(\xi, \{\xi\}) = 0$, (Θ is a kernel on $S \times V$).

($\tilde{\mathbf{M}}$.7) $A_\infty \gg (m + Q)\theta\cdot T$

where $\theta(x) = \lim \alpha\hat{H}_\alpha(I - H)1(x)$ and I is the identity operator.

Roughly speaking, ($\tilde{\mathbf{M}}$.6) implies that the path of $\tilde{\mathbf{M}}$ has at least the jumps due to the excursions of $\mathbf{M}$, and ($\tilde{\mathbf{M}}$.7) means that the path of $\tilde{\mathbf{M}}$ has at least the killing due to that of the minimal process $\mathbf{M}_0$. In fact, we shall prove in section 6 the following propositions.

PROPOSITION 8. *The paths of* $\mathbf{M}$ *have no jumps from* V *to* V, *a.e.* $(\mathbf{M})$, *if and only if* $(\tilde{P}, \tilde{L}) \approx ((m + Q)\Theta, T)$.

PROPOSITION 9. *The paths of* $\mathbf{M}$ *are continuous for* $t \in [0, \zeta)$ *if and only if* $Q = 0$, *a.e.* ν, $(\tilde{P}, \tilde{L}) \approx (m\Theta, T)$ *and the paths of* $\mathbf{M}_0$ *are continuous for* $t \in [0, \zeta)$.

PROPOSITION 11. *The left limit* $x_{\zeta-} \notin V$ *when* $\zeta < \infty$ *a.e.* $(\mathbf{M})$ *if and only if* $\tilde{A}_\infty \approx (m + Q)\theta \cdot T$, $(\tilde{\mathbf{M}})$.

PROPOSITION 12. *The process* $\mathbf{M}$ *is conservative if and only if* $\mathbf{M}_0$ *is conservative and* $\tilde{A}_\infty \approx (m + Q)\theta \cdot T$, $(\tilde{\mathbf{M}})$.

3.3. *Does there exist an* $\mathbf{M}$ *for given* $(\tilde{\mathbf{M}}, \ell, m, Q)$? The conditions $(\tilde{\mathbf{M}}.1) \sim (\tilde{\mathbf{M}}.7)$ for the boundary system are almost sufficient. In fact, we need the following smoothness condition which seems to depend on our method of proof:

$(\tilde{\mathbf{M}}.C)$ $\tilde{G}_\lambda f$, $\tilde{G}_\lambda(\ell + (m + Q)\hat{H}_\alpha)f \in C(V)$ if $f \in C(S)$.

Then the problem is solved in the following way. Let $\tilde{\mathbf{M}}$ be a Markov process on V, let ℓ and m be in $B^+(V)$, and let Q be a kernel on $V \times D$.

THEOREM 16 (see section 7). *To a system* $(\tilde{\mathbf{M}}, \ell, m, Q)$ *satisfying* $(\tilde{\mathbf{M}}.1)$–$(\tilde{\mathbf{M}}.7)$ *and* $(\tilde{\mathbf{M}}.C)$, *there corresponds one and only one process* $\mathbf{M}$ *on* S *satisfying* $(\mathbf{M}.1)$–$(\mathbf{M}.3)$ *and* $(\mathbf{M}.C)$, *whose boundary system is* $(\tilde{\mathbf{M}}, \ell, m, Q)$.

Under $(\mathbf{M}.C)$, the necessity of the condition $(\tilde{\mathbf{M}}.C)$ will be discussed in section 6.4. From the results of that section, we shall obtain the following theorem when $\ell = 0$ and $Q = 0$.

THEOREM 17 (see section 7). *Let* $\tilde{\mathbf{M}}$ *be a process on* V. *Then* $\tilde{\mathbf{M}}$ *is a U-process of a certain process* $\mathbf{M}$ *on* S *which satisfies* $(\mathbf{M}.1)$–$(\mathbf{M}.5)$ *and* $(\mathbf{M}.C)$ *if and only if* $\tilde{\mathbf{M}}$ *satisfies* $(\tilde{\mathbf{M}}.1)$, $(\tilde{\mathbf{M}}.2)$, *and*

$(\tilde{\mathbf{M}}.4)$ $\tilde{G}_\lambda \hat{H}_\alpha \chi_V = 0$ *for some* $\lambda > 0$ *and* $\alpha > 0$,

$(\mathbf{M}.6)$ $(\tilde{P}, \tilde{L}) \gg (\Theta, T)$,

$(\mathbf{M}.7)$ $\tilde{A}_\infty \gg \theta \cdot T$,

$(\mathbf{M}.C)$ $\tilde{G}_\lambda f \in C(V)$ *if* $f \in C(V)$.

The process $\mathbf{M}$ *is uniquely determined by* $\tilde{\mathbf{M}}$.

THEOREM 19 (see section 7). Let $\mathbf{M}$ satisfy $(\mathbf{M}.1) \sim (\mathbf{M}.5)$.

(i) *The paths of* $\mathbf{M}$ *are continuous for* $t \in [0, \zeta)$ *if and only if the paths of* $\mathbf{M}_0$ *are continuous for* $t \in [0, \zeta)$ *and* $(\tilde{P}, \tilde{L}) \approx (\Theta, T)$;

(ii) $\mathbf{M}$ *is conservative if and only if* $\mathbf{M}_0$ *is conservative and* $\tilde{A}_\infty \approx \theta \cdot T$.

The relation between our formulation and the lateral condition will be discussed in appendix III.

3.4. *Remaining problems.* The above arguments shift the problem (determining every $\mathbf{M}$ which is an extension of $\mathbf{M}_0$) to the existence of a process on V whose Lévy system has the assigned property. However, the general theorem for constructing such an $\tilde{\mathbf{M}}$ remains unproved.

(When the number of boundary points is finite, all $\tilde{\mathbf{M}}$ are easily obtained. As another example, in the trivial case $\ell = 1$ and $m = Q = 0$, $\tilde{\mathbf{M}}$ is arbitrary except for the smoothness conditions $(\tilde{\mathbf{M}}.1)$, $(\tilde{\mathbf{M}}.2)$, and $(\tilde{\mathbf{M}}.C)$.)

Further, the conditions ($\mathbf{M}_0$.5) and ($\mathbf{M}_0$.6) should be replaced by deeper (probabilistic) and more general ones. In the general case, to every point of V there corresponds a set of entrance boundary points, and it seems that we can assign the beginning point of excursion with any probability law.

4. Properties of the excursion at V

Let $\Phi = \tilde{T}_\gamma$ and let its right continuous inverse $\tau(s)$ be defined as in 2.3. In this section, with the exception of the last four lemmas, we assume only (**M**.1), (**M**.2), and [2.2] on **M**. The following are proved in [5]:

[4.1] $\tau(0) = \sigma$, a.e. (lemma 5.6 in [5]),

[4.2] $x_{\tau(s)} \notin D$ for any s, a.e. (theorem 5.7 in [5]),

[4.3] for any s, $x_t \notin V$ if $t \in (\tau(s-), \tau(s))$, a.e. (theorem 5.9 in [4]), where $\tau(0-) = 0$.

From [4.2], we can easily see that

(1) for every $s > 0$ and $\epsilon > 0$, there exists a t such that $x_t \in V$ and $t \in (\tau(s-) - \epsilon, \tau(s-))$, a.e. Especially, $x_{\tau(s-)-} \in V$ for any $s > 0$, a.e.

Let $\rho = \rho(k)$ be a Markov time such that

(2) (i) $\rho \leq (1/k) \wedge \sigma_{D_k} \wedge \inf \{t\colon r(x_0, x_t) \geq 1/k\}$;

(ii) put $\sigma_1 = \sigma_1(k) = \sigma$, $\rho_n = \rho_n(k) = \sigma_n + \rho(w_{\sigma_n}^+)$, and $\sigma_{n+1} = \sigma_{n+1}(k) = \rho_n + \sigma(w_{\rho_n}^+)$,

and $\lim \rho_n = \lim \sigma_n = \infty$, where $D_k = \{x\colon r(x, V) \geq 1/k\}$ and $x_t(w_s^+) = x_{t+s}(w)$.

(If we set $\rho(k) = (1/k) \wedge \sigma_{D_k} \wedge \inf \{t\colon r(x_0, x_t) \geq (1/k)\}$, then $\rho(k)$ satisfies these conditions.)

It is also easily seen that

(3) for every n and k, $\rho_n(k) = \sigma_{n+1}(k)$ if $x_{\rho_n(k)} \in V$ and $\rho_n(k) < \sigma_{n+1}(k)$ if $x_{\rho_n(k)} \in D$, a.e.

Noting that $\chi_V \cdot \Phi \approx \Phi$ and $P_\xi(\tau(0) = \sigma = 0) = 1$ for $\xi \in V$, we have

$$\Phi(\rho_n) = \Phi(\sigma_{n+1}) \qquad \text{for every } n \text{ and } k, \text{ a.e.,} \tag{4}$$

$$\tau(\Phi(\sigma_n)) = \sigma_n \qquad \text{for every } n \text{ and } k, \text{ a.e.} \tag{5}$$

DEFINITION 4.1. *Let* $T(w) = \{s > 0\colon \tau(s-) < \tau(s), \tau(s-) < \zeta\}$ *and* $N_k(w) = \{n\colon x_{\rho_n(k)} \in D\}$.

[4.5] For a fixed k, the mapping $n \to s = \Phi(\sigma_{n+1})$ is one-to-one from N_k to T, and

$$\sigma_n < \tau(s-) \leq \rho_n < \tau(s) = \sigma_{n+1}, \qquad \text{a.e.} \tag{6}$$

PROOF. For $n \in N_k$, $\rho_n < \sigma_{n+1} \wedge \zeta$, and $s = \Phi(\rho_n) = \Phi(\sigma_{n+1})$ by (4). So $\tau(s-) \leq \rho_n < \sigma_{n+1} = \tau(s)$ by (5) and $s \in T$. If $n' < n$, $\tau(\Phi(\sigma_{n'+1})) = \sigma_{n'+1} \leq \rho_n < \sigma_{n+1}$ and $\Phi(\sigma_{n'+1}) < \Phi(\sigma_{n+1}) = s$, which proves that the mapping is one-to-one. Setting $n' = n - 1$, $\Phi(\sigma_n) < s$ and $\sigma_n < \tau(s-)$, we have (6).

DEFINITION 4.2. *Let* $T_k(w) = \{s\colon s = \Phi(\sigma_{n+1}), n \in N_k\}$, *and for* $s \in T_k$, $n(k, s) = \{n\colon n \in N_k,\ s = \Phi(\sigma_{n+1})\}$, $\hat{\sigma}(k, s) = \sigma_{n(k,s)}$, $\rho(k, s) = \rho_{n(k,s)}$, *and* $\sigma(k, s) = \sigma_{n(k,s)+1}$.

By (6), we have

[4.6] for $s \in T_k$, $x_{\rho(k,s)} \in D$, and $\hat{\sigma}(k, s) < \tau(s-) \leq \rho(k, s) < \sigma(k, s) = \tau(s)$, a.e.;

[4.7] $T_k \subset T$ and $\liminf_{k\to\infty} T_k = T$, a.e.

PROOF. For any $s \in T$, since $x_t \in D$ if $t \in (\tau(s-), \tau(s) \wedge \zeta)$, there exist $t_0(w)$ and $k_0(w)$ such that $x_{t_0}(w) \in D_{k_0}$ and $t_0 \in (\tau(s-), \tau(s))$. For every $k \geq k_0$, since $\tau(s) > \tau(0) = \sigma_1(k)$ taking n such that $\sigma_n(k) < \tau(s) \leq \sigma_{n+1}(k)$, $\sigma_n(k) \leq \rho_n \leq t_0 < \tau(s) = \sigma_{n+1}(k)$, for $\sigma_n(k) \leq \tau(s-)$ and $\sigma_{n+1}(k) = \tau(s)$ by [4.2] and [4.3]. Therefore $s = \Phi(\sigma_{n+1})$ and $x_{\rho_n} \in D$ by (3). Thus, $s \in T_k$ for $k \geq k_0$.

[4.8] For $s \in T$, $\lim_{k\to\infty} \hat{\sigma}(k, s) = \lim_{k\to\infty} \rho(k, s) = \tau(s-)$, a.e.

[4.9] For $s \in T$, $\lim_{k\to\infty} x_{\hat{\sigma}(k,s)} = \lim_{k\to\infty} x_{\rho(k,s)-} = x_{\tau(s-)-}$ and $\lim_{k\to\infty} x_{\rho(k,s)} = x_{\tau(s)}$, a.e.

PROOF. By [4.7], the limits in [4.8] and [4.9] have a meaning. Noting that $\rho(k, s) - \hat{\sigma}(k, s) \leq (1/k)$ and $r(x_{\hat{\sigma}(k,s)}, x_{\rho(k,s)-}) \leq (1/k)$, we have [4.8] and [4.9] by [4.6].

Let $\mu_k = \sigma_{D_k^c}$; then $\mu_k \leq \sigma$ and $x_{\lim \mu_k} = \lim x_{\mu_k} \in V$ if $\lim \mu_k < \infty$, a.e. Therefore, $\sigma = \lim \mu_k$ and $x_\sigma = \lim x_{\mu_k}$ if $\sigma < \infty$, a.e. Therefore, we have the following:

$$\text{for any } t < \sigma,\ x_{t-} \notin V, \text{ a.e.}; \tag{7}$$

$$\text{if } \sigma < \infty \text{ and } x_{\sigma-} \in V,\ x_{\sigma-} = x_\sigma, \text{ a.e.} \tag{8}$$

[4.10] For any s, $x_{t-} \notin V$ if $t \in (\tau(s-), \tau(s))$, a.e.

PROOF. For $s \notin T$ ($s = 0$, $\tau(s-) \geq \zeta$ or $\tau(s-) = \tau(s)$), [4.10] is obvious. By (7),

$$P_x(x_{t-} \in V,\ \exists\, t \in (\rho_n(k), \sigma_n(k))) = E_x(P_{x_{\rho_n}}(x_{t-} \in V,\ \exists\, t \in (0, \sigma))) = 0. \tag{9}$$

Therefore $x_{t-} \notin V$ if $t \in (\rho_n(k), \sigma_n(k))$ for any n and k, a.e., and $x_{t-} \notin V$ if $t \in (\rho(k, s), \tau(k, s))$ for any k and $s \in T_k$. The statement [4.10] follows from [4.6], [4.8], and [4.7].

[4.11] For any $s \in T$, $x_{\tau(s-)} \in V$ implies $x_{\tau(s-)-} = x_{\tau(s-)}$, a.e.

PROOF. Since $x_{\rho(k,s)} \in D$, $\rho(k, s) > \tau(s-)$ and $\lim x_{\rho(k,s)-} = x_{\tau(s-)}$ by [4.8]. On the other hand, $\lim x_{\rho(k,s)-} = x_{\tau(s-)-}$ by [4.9] and [4.11] follows.

[4.12] For every $s \in T$, $\tau(s) < \infty$ and $x_{\tau(s)-} \in V$ implies $x_{\tau(s)-} = x_{\tau(s)}$, a.e.

PROOF. By (8),

$$\begin{aligned}
P_x(x_{\sigma_{n+1}-} \in V,\ x_{\sigma_{n+1}-} \neq x_{\sigma_{n+1}},\ \sigma_{n+1} < \infty) \qquad & \\
= E_x(P_{x_{\rho_n}}(x_{\sigma-} \in V,\ x_{\sigma-} \neq x_\sigma,\ \sigma < \infty);\ \rho_n < \infty) & \\
= 0 &
\end{aligned} \tag{10}$$

for any n and k, and so [4.12] follows from [4.7].

DEFINITION 4.3. *Let*

$$\begin{aligned}
T^* &= \{s > 0\colon x_{\tau(s-)-} \neq x_{\tau(s)};\ \tau(s) < \zeta\},\\
T_d &= \{s\colon x_{\tau(s-)-} \neq x_{\tau(s-)};\ s \in T\},\\
T_c &= \{s\colon x_{\tau(s-)-} = x_{\tau(s-)},\ s \in T\}.
\end{aligned} \tag{11}$$

With this notation, the following relation holds:

[4.13] $\{t > 0: x_{t-} \in V, x_t \in D\} = \{\tau(s-): s \in T_d\}$, a.e.

PROOF. If $x_{t-} \in V$ and $x_t \in D$, then by [4.10], $t = \tau(s-)$ or $\tau(s)$ for some s. Since $x_{\tau(s)} \notin D$ by [4.2], $t = \tau(s-) < \tau(s) = t + \sigma(w_t^+)$ and $s \in T_d$. If $s \in T_d$, $\tau(s-) < \zeta$, $x_{\tau(s-)-} \in V$ by (1) and $x_{\tau(s-)} \notin V$ by [4.11], so $x_{\tau(s-)} \in D$.

[4.14] If $s \in T^* - T$, then $\tau(s-) = \tau(s)$, a.e. And

(12) $$\{t > 0: x_{t-}, x_t \in V, \quad x_{t-} \neq x_t\} = \{\tau(s-) = \tau(s): s \in T^* - T\}, \text{ a.e.}$$

PROOF. The first assertion is obvious from the definition. If x_{t-}, $x_t \in V$, and $x_{t-} \neq x_t$, then by [4.3] (or [4.10]), $t = \tau(s-)$ or $\tau(s)$ for some s. In either case, s cannot be contained in T. Moreover, $t < \zeta$ and $t \neq \tau(0) = \sigma$ (by (8)) imply that $t = \tau(s-) = \tau(s)$ for some $s > 0$. Therefore, $s \in T^* - T$. If $s \in T^* - T$, setting $t = \tau(s-) = \tau(s)$, $x_t = x_{\tau(s)} \in V$ and $x_{t-} = x_{\tau(s-)-} \in V$, and $x_{t-} \neq x_t$ by definition of T^*.

DEFINITION 4.4. *Let* $T_{d,k}(w) = \{s: s \in T_k, \quad x_{\rho(k,s)-} \in V\}$, a.e.

With this definition we have

[4.15] $T_{d,k} \subset T_d$ and $\liminf_{k\to\infty} T_{d,k} = T_d$, a.e.

PROOF. By [4.6] and [4.10], if $s \in T_{d,k}$, then $\rho(k, s) = \tau(s-)$ and $x_{\rho(k,s)} \in D$ so that $s \in T_d$. If $s \in T_d$, by [4.13] $x_{\tau(s-)} \in D$, and there exists $k_0(w)$ such that $s \in T_k$ and $x_{\tau(s-)} \in D_k$ for $k \geq k_0$. Therefore by [4.6], $\hat{\sigma}(k, s) < \tau(s-) \leq \rho(k, s)$ implies that $\rho(k, s) = \tau(s-)$ and $x_{\rho(k,s)-} = x_{\tau(s-)-} \in V$. Hence $s \in T_{d,k}$ for all $k \geq k_0(w)$.

In the proof of [4.15], we see that

[4.16] for any $s \in T_{d,k}$, $\rho(k, s) = \tau(s-)$.

For later use we shall give some remarks. Let

(13) $$\xi = \inf \{s: x_{\tau(s)} = \partial\}.$$

Then it is easily seen that

(14) $$\tau(\xi-) \leq \zeta \leq \tau(\xi) = \infty.$$

And by (1) and [4.3],

(15) $$\tau(\xi-) = \sup \{t: x_t \in V\}, \text{ a.e.}$$

Therefore by theorem 5.2 in [5], we have

$$[4.17] \quad \begin{array}{lll} \tau(\xi-) = \infty & \text{if and only if} & \xi = \infty, \text{ a.e.,} \\ \tau(\xi-) < \zeta & \text{if and only if} & \xi \in T, \text{ a.e.,} \\ \tau(\xi-) = \zeta < \infty & \text{if and only if} & \zeta < \infty \text{ and } x_{\zeta-} \in V, \text{ a.e.} \end{array}$$

(If $\tau(\xi-) < \zeta < \tau(\xi)$, then $x_{\zeta-} \notin V$ by [4.10].) Let $\mathbf{M}$ be an extension of $\mathbf{M}_0$. Noting that

(16) $$P_x(\zeta \in (0, \sigma], \zeta < \infty) = P_x^0(\zeta \leq \sigma, \zeta < \infty)$$

and

(17) $$P_x(x_{t-} \neq x_t, t < \zeta \text{ and } t \leq \sigma) = P_x^0(x_{t-} \neq x_t, t < \zeta),$$

we can prove the following lemmas by the same method as in the proof of [4.10].

[4.18] Suppose that the paths of $\mathbf{M}_0$ are continuous for $t \in [0, \zeta)$, a.e. If $t < \zeta$ and $t \in (\tau(s-), \tau(s)]$, $x_{t-} = x_t$, a.e.

[4.19] Suppose that $\mathbf{M}_0$ is conservative. If $\zeta < \infty$, then $\zeta \notin (\tau(s-), \tau(s)]$ for all s.

[4.20] Under the same assumption as in [4.18], the paths of $\mathbf{M}$ have no jump from D to S, a.e.

PROOF. If $x_{t-} \neq x_t$, $x_{t-} \in D$ and $x_t \in S$, then $t = \tau(s-)$ by [4.18], a.e. However, $x_{\tau(s-)-} \in V$ by (1).

[4.21] Under the same assumption as in [4.19], $x_{\zeta-} \in V$ when $\zeta < \infty$, a.e.

PROOF. Since $\zeta = \tau(s-)$ by [4.19], $x_{\zeta-} \in V$ when $\zeta < \infty$.

5. Decomposition of the resolvent. The boundary system

5.1. *Lemmas on continuous additive functionals.* In this section we shall assume (**M**.1) and (**M**.2) for **M**.

[5.1] (Dynkin [1]) Let A, B, and C be continuous additive functionals, such that $E_x(\int_0^\infty e^{-A(t)}\, dC(t))$ and $E_x(\int_0^\infty e^{-B(t)}\, dC(t))$ are bounded in x. Then

$$(1)\quad E_x\Big(\int_0^\infty e^{-A(t)} f(x_t)\, dC(t)\Big) - E_x\Big(\int_0^\infty e^{-B(t)} f(x_t)\, dC(t)\Big) \\ + E_x\Big\{\int_0^\infty e^{-A(t)}(dA(t) - dB(t))\, E_{x_t}\Big(\int_0^\infty e^{-B(s)} f(x_s)\, dC(s)\Big)\Big\} = 0.$$

[5.2] Let X and A be continuous additive functionals, where $E_x(\int_0^\infty e^{-\alpha t}\, dA) < \infty$ for some $\alpha > 0$, and X is not necessarily nonnegative and $X(t) = 0$ for $0 \leq t \leq \sigma$. Let $\tilde{A} = \tilde{A}_\alpha$, that is,

$$(2)\qquad E_x\Big(\int_0^\infty e^{-\alpha t}\, d\tilde{A}\Big) = E_x\Big(\int_\sigma^\infty e^{-\alpha t}\, dA\Big).$$

Then

$$(3)\qquad E_x\Big(\int_\sigma^\infty e^{-\alpha t + X(t)}\, dA\Big) = E_x\Big(\int_0^\infty e^{-\alpha t + X(t)}\, d\tilde{A}\Big).$$

PROOF. Set $\rho = \inf\{t\colon (x(t)) \geq \epsilon\}$, $(\epsilon > 0)$, $\sigma_1 = \sigma$, $\rho_n = \sigma_n + \rho(w_{\sigma_n}^+)$, and $\sigma_{n+1} = \rho_n + \sigma(w_{\rho_n}^+)$. Since $X(t) = X(\rho_n)$ for $t \in [\rho_n, \sigma_{n+1}]$, $(X(t) - X(\sigma_n)) \leq \epsilon$ for $t \in [\sigma_n, \sigma_{n+1}]$. And by lemma 4.1 in [5]

$$(4)\qquad \varphi(\xi) = E_\xi\Big(\int_0^{\sigma_2} e^{-\alpha t}\, dA\Big) = E_\xi\Big(\int_0^{\sigma_2} e^{-\alpha t}\, dA\Big) \quad \text{for any} \quad \xi \in V.$$

Put

$$(5)\qquad I_1 = E_x\Big(\int_\sigma^\infty e^{-\alpha t - X(t)}\, dA\Big)$$

and

$$(6)\qquad I_2 = E_x\Big(\int_\sigma^\infty e^{-\alpha t - X(t)}\, d\tilde{A}\Big) = E_x\Big(\int_0^\infty e^{-\alpha t - X(t)}\, d\tilde{A}\Big).$$

Then,

$$(7)\quad e^{-\epsilon}E_x\Big(\sum_{n=1}^{\infty} e^{-\alpha\sigma_n+X(\sigma_n)}\varphi(x_{\sigma_n})\Big) \leq I_i$$
$$\leq e^{\epsilon}E_x\Big(\sum_{n=1}^{\infty} e^{-\alpha\sigma_n+X(\sigma_n)}\varphi(x_{\sigma_n})\Big), \qquad i = 1, 2.$$

Since $\epsilon > 0$ is arbitrary, we have [5.2].

[5.3] Let A and B be continuous additive functionals. If, for every $f \in B^+(s)$,

$$(8)\qquad E_x\Big(\int_0^\infty e^{-\alpha t}f(x_t)\,dA\Big) \geq E_x\Big(\int_0^\infty e^{-\alpha t}f(x_t)\,dB\Big),$$

then $A \gg B$.

Proof. Since $f\cdot A \approx 0$ implies $f\cdot B \approx 0$ for any $f \in B^+(s)$, there exists a nonnegative Borel measurable function g such that $B \approx g\cdot A$ (see theorem 7.1 in [7]). Hence,

$$(9)\quad 0 \geq E_x\Big(\int_0^\infty e^{-\alpha t}\chi(g > 1)(dB - dA)\Big) = E_x\Big(\int_0^\infty e^{-\alpha t}\chi(g > 1)(g - 1)\,dA\Big),$$

where

$$(10)\qquad \chi(g > 1)(x) = \begin{cases}1, & g(x) > 1,\\ 0, & g(x) \leq 1.\end{cases}$$

So $\chi(g > 1)(g - 1)\cdot A \approx 0$. Therefore,

$$(11)\qquad A - B \approx (1 - g)\cdot A \approx \chi(g \leq 1)(1 - g)\cdot A \gg 0.$$

5.2. *Definition of the boundary system.* The notations are the same as in section 4. In section 5.2 we shall only assume (**M**.1), (**M**.2), and [2.2] on **M**.

[5.4] For any f, $h \in B(S)$ and $\alpha > 0$,

$$(12)\quad \lim_{k\to\infty} E_x\big(\textstyle\sum e^{-\alpha\rho_n(k)}\chi_V(x_{\rho_n(k)}-)f(x_{\rho_n(k)}-)\chi_D(x_{\rho_n(k)})h(x_{\rho_n(k)})\big)$$
$$= E_x\Big(\int_0^\infty e^{-\alpha t}\chi_V(x_t)f(x_t)P_Dh(x_t)\,dL\Big),$$

where (P, L) is a Lévy system of **M** and $P_Dh(x) = \int_D P(x, dy)h(y)$.

Proof. Without losing generality, we may assume f, $h \in B^+(S)$. By [4.15] and [4.16],

$$(13)\quad \lim_{k\to\infty} E_x\big(\textstyle\sum e^{-\alpha\rho_n}\chi_V f(x_{\rho_n}-)\chi_D h(x_{\rho_n}\big) = \lim_{k\to\infty} E_x\big(\sum_{T_d(k)} e^{-\alpha\tau(s-)}f(x_{\tau(s-)}-)h(x_{\tau(s-)})\big)$$
$$= E_x\big(\textstyle\sum_{T_d} e^{-\alpha\tau(s-)}f(x_{\tau(s-)}-)h(x_{\tau(s-)})\big).$$

By [4.13],

$$(14)\qquad E_x\big(\textstyle\sum_{T_d} e^{-\alpha\tau(s-)}f(x_{\tau(s-)}-)h(x_{\tau(s-)})\big) = E_x\big(\sum_{t>0} e^{-\alpha t}\chi_V f(x_{t-})\chi_D h(x_t)\big)$$
$$= E_x\Big(\int_0^\infty e^{-\alpha t}\chi_V fP_Dh\,dL\Big).$$

[5.5] For any f, $h \in C(S)$ and $\alpha > 0$,

(15) $$\lim_{k\to\infty} E_x(\sum e^{-\alpha\rho_n(k)}\chi_D(x_{\rho_n(k)}-)f(x_{\rho_n(k)}-)h(x_{\rho_n(k)})g_\gamma(x_{\rho_n(k)}))$$
$$= E_x\Big(\sum_{T_c} e^{-\alpha\tau(s-)}f(x_{\tau(s-)})h(x_{\tau(s-)}) \int_{\tau(s-)}^{\tau(s)} e^{-\gamma(t-\tau(s-))}\, dt\Big).$$

PROOF. We have

(16) $$E_x(\sum e^{-\alpha\rho_n}\chi_D f(x_{\rho_n}-)hg_\gamma(x_{\rho_n}))$$
$$= E_x\Big(\sum e^{-\alpha\rho_n}\chi_D f(x_{\rho_n}-)h(x_{\rho_n}) \int_{\rho_n}^{\sigma_{n+1}} e^{-\gamma(t-\rho_n)}\, dt\Big)$$
$$= E_x\Big(\sum_T \chi(s\in T_k)e^{-\alpha\rho(k,s)}\chi_D f(x_{\rho(k,s)}-)h(x_{\rho(k,s)}) \int_{\rho(k,s)}^{\sigma(k,s)} e^{-\gamma(t-\rho(k,s))}\, dt\Big).$$

Set $\gamma_0 = \alpha \wedge \gamma$; then each term in the last member is dominated by $\|fh\| \int_{\tau(s-)}^{\tau(s)} e^{-\gamma_0 t}dt$, which is independent of k. And

(17) $$\sum_T \|fh\| \int_{\tau(s-)}^{\tau(s)} e^{-\gamma_0 t}dt \le \frac{1}{\gamma_0} \|fh\|.$$

Noting that

(18) $$\lim_{k\to\infty} e^{-\alpha\rho(k,s)}\chi_D f(x_{\rho(k,s)}-)h(x_{\rho(k,s)}) \int_{\rho(k,s)}^{\sigma(k,s)} e^{-\gamma(t-\rho(k,s))}\, dt$$
$$= \begin{cases} 0 & \text{if } s\in T_d \\ e^{-\alpha\tau(s-)}f(x_{\tau(s-)})h(x_{\tau(s-)}) \int_{\tau(s-)}^{\tau(s)} e^{-\gamma(t-\tau(s-))}\, dt & \text{if } s \in T_c \end{cases}$$

by [4.8], [4.9], [4.16], and the definition of T_c, [5.5] follows by the dominated convergence theorem.

Let

(19) $$\Phi_0 = \widetilde{(\chi_D\cdot T)}_\gamma \quad\text{and}\quad \Phi_1 = \chi_V\cdot T;$$

then

(20) $$\Phi = \Phi_1 + \Phi_0.$$

Let f be in $C(S)$ and

(21) $$\bar\rho(k) = (1/k) \wedge \sigma_{D_k} \wedge \inf\{t\colon r(x_t, x_0) \ge (1/k)$$
$$\text{or}\quad |f(x_t) - f(x_0)| \ge (1/k)\}.$$

Then $\{\bar\rho(k)\}$ satisfies the conditions (2) in section 4. Moreover, noting $g_\gamma(x) = E_x(\int_0^\sigma e^{-\gamma t}\, d(\chi_D\cdot T))$, by theorem 4.4 in [5] we have

(22) $$\lim_{k\to\infty} E_x\Big(\sum_{n=1}^\infty e^{-\alpha\bar\rho_n(k)}f(x_{\bar\rho_n(k)}-)g_\gamma(x_{\bar\rho_n(k)})\Big) = E_x\Big(\int_0^\infty e^{-\alpha t}f(x_t)\, d\Phi_0\Big).$$

Setting $h = g_\gamma$ in [5.4] and $h = 1$ in [5.5], we have

(23) $$E_x\Big(\int_0^\infty e^{-\alpha t}f(x_t)\, d\Phi_0\Big) = E_x\Big(\sum_{T_c} e^{-\alpha\tau(s-)}f(x_{\tau(s-)}) \int_{\tau(s-1)}^{\tau(s)} e^{-\gamma(t-\tau(s-)}\, dt\Big)$$
$$+ E_x\Big(\int_0^\infty e^{-\alpha t}\chi_V(x_t)f(x_t)P_D g_\gamma(x_t)\, dL\Big)$$

for any $f \in C(S)$, and hence also for any $f \in B(S)$.

Let

(24) $$\Phi_3 \approx \chi_V P_D g_\gamma \cdot L.$$

Then, from [5.3] and (23), we can see that $\Phi_0 \gg \Phi_3$. Let

(25) $$\Phi_2 \approx \Phi_0 - \Phi_3.$$

Then Φ_2 may be considered as a continuous (nonnegative) additive functional. And by (23),

(26) $$E_x\left(\int_0^\infty e^{-\alpha t} f(x_t)\, d\Phi_2\right) = E_x\left(\sum_{T_c} e^{-\alpha\tau(s-)} f(x_{\tau(s-)}) \int_{\tau(s-)}^{\tau(s)} e^{-\gamma(t-\tau(s-))}\, dt\right).$$

For $\xi \in V$ and $h \in B(D)$ let

(27) $$Q^*h(\xi) = \begin{cases} \dfrac{P_D(hg_\gamma)(\xi)}{P_D g_\gamma(\xi)} & \text{if } P_D g_\gamma(\xi) \neq 0, \\ \displaystyle\int \nu(d\xi) h(\xi) & \text{if } P_D g_\gamma(\xi) = 0, \end{cases}$$

where ν is an arbitrary measure on D such that $\nu(D) = 1$. Then Q^* is a kernel on $V \times D$ and

(28) $$Q^*h(\xi) P_D g_\gamma(\xi) = P_D(hg_\gamma)(\xi) \quad \text{for} \quad \xi \in V \quad \text{and} \quad Q^*h\Phi_3 \approx \chi_V(P_D h g_\gamma)\cdot L.$$

Combining [5.4], [5.5], (26), and (28), we have

[5.6] For any $f, h \in C(s)$, and $\{\rho(k)\}$ satisfying (2) in section 4,

(29) $$\lim_{k\to\infty} E_x\left(\sum e^{-\alpha\rho_n(k)} f(x_{\rho_n(k)-}) h(x_{\rho_n(k)}) g_\gamma(x_{\rho_n(k)})\right)$$
$$= E_x\left(\int_0^\infty e^{-\alpha t} f(x_t) h(x_t)\, d\Phi_2\right) + E_x\left(\int_0^\infty e^{-\alpha t} f(x_t) Q^*h(x_t)\, d\Phi_3\right).$$

(This holds if $f(X_t)$ and $g(x_t)$ are right continuous in t a.e.) Since $\Phi_1, \Phi_2, \Phi_3 \ll \Phi$, by theorem 3.8 in [4] there exist functions ℓ, m, n in $B^+(V)$ such that

(30) $$\Phi_1 = \ell\Phi, \quad \Phi_2 = m\Phi, \quad \text{and} \quad \Phi_3 = n\Phi.$$

Let $Q = nQ^*$. Recalling the definition of Φ_i's and (26), we have

(31) $$\begin{aligned} &\ell\cdot\Phi \approx \chi_V \cdot T, \\ &E_x\left(\int_0^\infty e^{-\gamma t} m\, d\Phi\right) = E_x\left(\sum_{T_c} \int_{\tau(s-)}^{\tau(s)} e^{-\gamma t}\, dt\right) && \text{for any } x, \\ &Qh\cdot\Phi \approx \chi_V P_D h g_\gamma \cdot L && \text{for any} \quad h \in B(D). \end{aligned}$$

Let

(32) $$N(\Phi) = \{f : f\cdot\Phi \approx 0,\ f \in B^+(s)\},$$

and let ν be a measure such that

(33) $$N(\Phi) = \{f : f = 0, \text{ a.e. } \nu\}.$$

The existence of such a ν is proved in [4].

Let $\tilde{\mathbf{M}}$ be a Markov process on V. Let $\ell, m \in B^+(V)$ and let Q be a kernel on $V \times D$.

DEFINITION 5.1. *The system* $(\tilde{\mathbf{M}}, \ell, m, Q)$ *is called the* boundary system *of* $\mathbf{M}$ *if and only if* $\tilde{\mathbf{M}}$ *is the U-process of* $\mathbf{M}$, *and* ℓ, m, *and* Q *satisfy condition* (31).

For any boundary system $(\tilde{\mathbf{M}}, \ell, m, Q)$, $\Phi_1 \approx \ell\Phi$, $\Phi_2 \approx m\Phi$, and $\Phi_3 \approx Q(\cdot, D)\Phi$ hold.

PROPOSITION 1. *For* $\mathbf{M}$ *satisfying* $(\mathbf{M}.1)$, $(\mathbf{M}.2)$, *and* [2.2], *there exists a boundary system. For fixed* $\gamma > 0$, $\tilde{\mathbf{M}}$ *is uniquely determined by* $\mathbf{M}$ *and* ℓ; m *and* Qh (*for any* $h \in B(D)$) *are uniquely determined by* $\mathbf{M}$ *except for a set of* ν *measure* 0, *where* ν *is a measure which satisfies* (33).

Let $(\tilde{\mathbf{M}}, \ell, m, n, Q)$ be a boundary system of $\mathbf{M}$, then by definitions of Φ_i's, we have;

[5.7] $(\tilde{\mathbf{M}}.3)$ $\ell + m + Q(\cdot, D) = 1$, a.e. ν.

PROPOSITION 2. (i) *The paths of* $\mathbf{M}$ *have no sojourn on* V a.e. *if and only if* $\ell = 0$, a.e. ν.

(ii) *The paths of* $\mathbf{M}$ *have no excursion which starts from* V (*that is,* T_c *is empty*) a.e. *if and only if* $m = 0$, a.e. ν).

(iii) *The paths of* $\mathbf{M}$ *have no jump from* V *to* D a.e. *if and only if* $Q = 0$, a.e. ν.

Finally, note that [5.6] can be written in the form

[5.6*] For any $f, h \in C(S)$

$$(34)\quad \lim_{k\to\infty} E_x\left(\sum e^{-\alpha\rho_n(k)} f(x_{\rho_n(k)-}) h(x_{\rho_n(k)}) g_\gamma(x_{\rho_n(k)})\right) = E_x\left(\int_0^\infty e^{-\alpha t} f(mh + Qh)\, d\Phi\right).$$

5.3. *Correspondence of* $\mathbf{M}$ *and its boundary system.*

DEFINITION 5.2. *Let*

$$(35)\quad \begin{gathered} K_\alpha^\lambda f(\xi) = E_\xi\left(\int_0^\infty e^{-\lambda\Phi(t)-\alpha t} f(x_t)\, d\Phi\right), \\ K^\lambda = K_0^\lambda \quad\text{and}\quad K_\alpha = K_\alpha^0, \qquad (\xi \in V). \end{gathered}$$

[5.8] The family $\{K_\alpha^\lambda\}$, $(\alpha, \lambda \geq 0,\ \alpha + \lambda > 0)$ is a system of bounded nonnegative kernels on $V \times V$ and

$$(36)\quad \begin{gathered} K^\lambda \equiv \tilde{G}_\lambda \text{ (Green kernel of } \tilde{\mathbf{M}}), \\ H_\alpha K_\alpha^\lambda f(x) = E_x\left(\int_0^\infty e^{-\lambda\Phi(t)-\alpha t} f(x_t)\, d\Phi\right). \end{gathered}$$

Hereafter we shall assume $(\mathbf{M}_0.1) \sim (\mathbf{M}_0.6)$ on $\mathbf{M}_0$ and $(\mathbf{M}.1) \sim (\mathbf{M}.3)$ on $\mathbf{M}$. Now, we shall prove the decomposition theorem for the Green kernel of $\mathbf{M}$.

THEOREM 3. *If* K_α, ℓ, m, *and* Q *are defined by* (31) *and definition* 5.2, *then*

$$(37)\quad G_\alpha f(x) = G_\alpha^0 f(x) + H_\alpha K_\alpha\{\ell + (m + Q)\hat{H}_\alpha\} f(x)$$

for any $f \in C^+(s)$.

PROOF. We can assume that $f \in C^+(S)$. Then

$$(38)\quad \begin{aligned} G_\alpha f(x) - G_\alpha^0 f(x) &= E_x\left(\int_\sigma^\infty e^{-\alpha t} f(x_t)\, dt\right) \\ &= E_x\left(\int_0^\infty e^{-\alpha t} \chi_V f\, dt\right) + E_x\left(\int_0^\infty e^{-\alpha t}\, d(\widetilde{\chi_D f T})_\alpha\right). \end{aligned}$$

By theorem 4.4 in [5], we see that

$$(39)\qquad E_x\Big(\int_0^\infty e^{-\alpha t}\, d(\widetilde{\chi_D f\cdot T})_\alpha\Big) = \lim_{k\to\infty} E_x\big(\sum e^{-\alpha\rho_n(k)} G_\alpha^0 f(x_{\rho_n(k)})\big)$$
$$= \lim_{k\to\infty} E_x\big(\sum e^{-\alpha\rho_n(k)} \hat{H}_\alpha f(x_{\rho_n(k)}) g_\gamma(x_{\rho_n(k)})\big)$$

for suitable $\{\rho_n(k)\}$. Now, applying [5.6*] to the right side, we obtain theorem 3.

Let

$$(40)\qquad C_\alpha = (\ell + (m + Q)\hat{H}_\alpha)$$

and

$$(41)\qquad U_\alpha = \alpha C_\alpha H = (\alpha\ell + (m + Q)\hat{H}_\alpha)H.$$

Then, C_α is a kernel on $V \times D$ and U_α is a kernel on $V \times V$. By (2), (3), and (5) in section 2, we have

$$(42)\qquad C_\alpha - C_\beta + (\alpha - \beta)C_\alpha G_\beta^0 = 0,$$

$$(43)\qquad U_\alpha - U_\beta = (\alpha - \beta)C_\alpha H_\beta$$

[5.9] The following relations hold:

$$(44)\qquad K_\alpha^\lambda - K_\alpha^\mu + (\lambda - \mu)K_\alpha^\lambda K_\alpha^\mu = 0,$$

$$(45)\qquad K_\alpha^\lambda - K_\beta^\mu + K_\alpha^\lambda(U_\alpha - U_\beta)K_\beta^\lambda = 0,$$

$$(46)\qquad G_\alpha = G_\alpha^0 + H_\alpha K_\alpha C_\alpha.$$

PROOF. Equality (46) is the same as (37). By [5.1], we can show that (44) holds and that

$$(47)\qquad K_\alpha^\lambda f(\xi) - K_\beta^\lambda f(\xi) + E_\xi\Big(\int_0^\infty e^{-\lambda\Phi(t)-\alpha t} H_\beta K_\beta f(x_t)(\alpha - \beta)\, dt\Big) = 0, \quad (\xi \in V).$$

Since $(\widetilde{h\cdot T})_\alpha \approx C_\alpha h\cdot\Phi$, by theorem 3, and $\Phi(t) \equiv 0$, $t \le \sigma$ a.e., by [5.2],

$$(48)\qquad E_\xi\Big(\int_0^\infty e^{-\lambda\Phi(t)-\alpha t} H_\beta K_\beta^\lambda f(x_t)(\alpha - \beta)\, dt\Big)$$
$$= (\alpha - \beta)E_\xi\Big(\int_0^\infty e^{-\lambda\Phi(t)-\alpha t} C_\alpha H_\beta K_\beta^\lambda f(x_t)\, d\Phi\Big)$$
$$= (\alpha - \beta)K_\alpha^\lambda C_\alpha H_\beta K_\beta^\lambda f.$$

Noting (43), we have (45).

Let $\ell, m \in B^+(V)$, and let Q be a bounded kernel on $V \times D$.

DEFINITION 5.3. *The system* $\{K_\alpha^\lambda\}$, $(\alpha, \lambda \ge 0, \alpha + \lambda > 0)$ *of bounded kernels on* $V \times V$ *is called an* (ℓ, m, Q)*-system if and only if the* K_α^λ*'s satisfy the relations* (44) *and* (45), *for* $U_\alpha = \alpha(\ell + (m + Q)\hat{H}_\alpha H)$.

Noting that $\|U_\alpha - U_\beta\| \le |\alpha - \beta|(\|\ell\| + \|m\| + \|Q\|)\|\hat{H}_\alpha\|$, the kernels K_α^λ and K_β^λ are uniquely expanded in a series which is uniformly convergent (in the norm of kernels); that is,

$$(49)\qquad K_\alpha^\mu = \sum_{n=1}^\infty (-1)^{n-1}(\mu - \lambda)^{n-1}(K_\alpha^\lambda)^{(n)},$$

(50) $$K_\beta^\lambda = \sum_{n=1}^{\infty} (-1)^{n-1}(K_\alpha^\lambda(U_\alpha - U_\beta))^{(n-1)}K_\alpha^\lambda,$$

if $|\mu - \lambda|$ and $|\beta - \alpha|$ are sufficiently small. Therefore we have

[5.10] $N(K) = \{f\colon K_\alpha^\lambda f = 0 \quad f \in B(V)\}$ is independent of α and λ.

[5.11] For some α_0 and λ_0, if $K_{\alpha_0}^{\lambda_0}$ is given, then the whole system $\{K_\alpha^\lambda\}$ is uniquely determined.

By (42) and ($\mathbf{M}_0$.4),

(51) $$R(C) = \{C_\alpha f\colon f \in C(S)\}$$

is independent of α. Noting (44) and (45),

[5.12] if the assertion,

(52) $$K_\alpha^\lambda f \in C(V) \qquad \text{for any } f \in C(V) \cup R(C),$$

holds for some α_0 and λ_0, then it also holds for every α and λ.

[5.13] Let ℓ, m, $\ell' m' \in B^+(S)$, let Q and Q' be kernels on $V \times D$ (bounded), and let $\{K_\alpha^\lambda\}$ be an (ℓ, m, Q)-system. If $\ell' - \ell$, $m' - m$, $Q'h - Qh \in N(K)$ (for any $h \in B(D)$), then $\{K_\alpha^\lambda\}$ is also an (ℓ', m', n', Q')-system.

Let $(\mathfrak{M}, \ell, m, Q)$ be the boundary system of the process $\mathbf{M}$, and let K_α^λ be defined by (35). Then $\{K_\alpha^\lambda\}$ is an (ℓ, m, Q)-system and $N(K) = N(\Phi)$. Now, we have the following uniqueness theorem.

THEOREM 4. *Let* $\mathbf{M}$ *and* $\mathbf{M}'$ *be processes on* S *satisfying* ($\mathbf{M}$.1) $\sim$ ($\mathbf{M}$.3), *and* $(\mathfrak{M}, \ell, m, Q)$ *and* $(\mathfrak{M}', \ell', m', Q')$ *be their boundary systems respectively. Then,* $\mathbf{M} = \mathbf{M}'$ *if and only if* $\mathfrak{M} = \mathfrak{M}'$ *and* $\ell' = \ell$, $m' = m$, *and* $Q'h = Qh$ *(for any* $h \in B(D)$*)* a.e. ν, *where* ν *is a measure which satisfies* (33).

PROOF. The "only if" part is contained in proposition 1. If the conditions of the theorem are satisfied, let K_α^λ and $K_\alpha'^\lambda$ be kernels defined by (35) for $\mathbf{M}$ and $\mathbf{M}'$ respectively. Since $\mathfrak{M} = \mathfrak{M}'$, then $\mathbf{K}^\lambda \equiv \mathbf{K}'^\lambda$ and $N(K) = N(K')$, and $\{K_\alpha'^\lambda\}$ is also an (ℓ, m, n, Q) system by [5.13]. Therefore, by [5.10] $K_\alpha \equiv K'_\alpha$ and by theorem 3, $G_\alpha = G'_\alpha$, that is, $\mathbf{M} = \mathbf{M}'$.

For later use, we note the following:

[4.14] let K be a bounded kernel on $V \times V$, let ℓ, m, ℓ', $m' \in B^+(V)$, and let Q, Q' be bounded kernels on $V \times D$. For some $\alpha > 0$, if

(53) $$Km\hat{H}_\alpha\chi_V = 0 \quad \text{and} \quad Km'\hat{H}_\alpha\chi_V = 0$$

and

(54) $$K(\ell + (m + Q)\hat{H}_\alpha)f = K(\ell' + (m' + Q)\hat{H}_\alpha)f \qquad \text{for any } f \in C(S),$$

then $K\ell g = K\ell' g$, $Kmg = Km'g$, and $KQh = KQ'h$ for any $g \in C(V)$ and $h \in C(D)$.

PROOF. Since both sides of (54) are kernels, the relation holds for any $f \in B(S)$. Let $f = \chi_V g$ for $g \in B(V)$. Noting (53), we have $K\ell g = K\ell' g$. Therefore, for any $f \in C(S)$, $K(m + Q)\hat{H}_\alpha f = K(m' + Q')\hat{H}_\alpha f$. By ($\mathbf{M}_0$.6), the set $\{\hat{H}_\alpha f\colon f \in C(S)\}$ is dense in $C(S)$ and the carriers of Km and KQ, (Km' and KQ') are disjoint. So we have [5.14].

6. Properties of the boundary systems (Lévy's system of U-processes)

In this section, $\mathbf{M}_0$ and $\mathbf{M}$ on V are fixed. The process $\mathbf{M}_0$ is assumed to satisfy conditions $(\mathbf{M}_0.1) \sim (\mathbf{M}_0.6)$, and $\mathbf{M}$ is assumed to satisfy conditions $(\mathbf{M}.1) \sim (\mathbf{M}.3)$. Let $(\tilde{\mathbf{M}}, \ell, m, Q)$ be the boundary system of $\mathbf{M}$. Let $\tilde{G}_\lambda$ be the Green kernel of $\tilde{\mathbf{M}}$ and let

$$N(\tilde{G}) = \{f\colon f \in B(V), \tilde{G}_\lambda f = 0\}. \tag{1}$$

Then $N(\tilde{G}) = N(\Phi)$ in section 5. Let ν be a measure on V such that

$$N(\tilde{G}) = \{f\colon f = 0, \text{ a.e. } \nu\}, \tag{2}$$

where ν is a measure given in section 5 (33).

6.1. *Miscellaneous properties.* We have already seen that

$(\tilde{\mathbf{M}}.1)$ $\tilde{\mathbf{M}}$ is a Hunt process on V,

$(\tilde{\mathbf{M}}.2)$ $\tilde{\mathbf{M}}$ has a reference measure,

$(\tilde{\mathbf{M}}.3)$ $\ell, m \in B^+(S)$ and Q is a kernel on $V \times D$ such that $\ell(\xi) + m(\xi) + Q(\xi, D) = 1$, a.e. ν.

PROPOSITION 5. *The following relation holds:*

$(\tilde{\mathbf{M}}.4)$ $m\hat{H}_\alpha \chi_V = 0$, a.e. ν.

PROOF. Noting that $\hat{H}_\alpha \chi_V(x) = 0$ if $x \in D$, by theorem 3,

$$G_\alpha \chi_V(\xi) = K_\alpha(\ell + m\hat{H}_\alpha \chi_V)(\xi) \qquad \text{for any } \xi \in V. \tag{3}$$

On the other hand, by the definition of ℓ,

$$G_\alpha \chi_V(\xi) = E_\xi\Big(\int_0^\infty e^{-\alpha t}\chi_V\, dt\Big) = E_\xi\Big(\int_0^\infty e^{-\alpha t}\ell d\Phi\Big) = K_\alpha \ell(\xi) \quad \text{for any } \xi \in V. \tag{4}$$

Noting [5.10], we have $(\tilde{\mathbf{M}}.4)$.

PROPOSITION 6. *Let* $E = \{\xi \in V\colon \ell(\xi) + m(\xi) > 0\}$; *then*

$$(\tilde{\mathbf{M}}.5) \quad \int_0^t \left(\frac{1}{1 - \chi_E(\xi_s)} + Q\left(\frac{1}{g_\gamma}\right)(\xi_s)\right) ds = \infty \qquad \text{for any } t > 0, \text{ a.e. } (\tilde{\mathbf{M}}).$$

PROOF. Let $u = (1/1 - \chi_E) + Q(1/g_\gamma)$, $\rho = \inf\{t\colon \int_0^t u d\Phi = \infty\}$ and $\rho_1 = \inf\{t\colon \int_0^t u d\Phi > 1\}$ in the process $\mathbf{M}$. Then, by the continuity of Φ,

$$\int_0^{\rho_1} u d\Phi \le 1, \tag{5}$$

and since $\int_0^t u d\Phi$ is continuous for $t \in [0, \rho)$, $\rho_1 > 0$ if and only if $\rho > 0$. Moreover, by the definition of E and (5), we see that $\int_0^{\rho_1} (\ell + m)\, d\Phi = 0$. If $P_\xi(\rho_1 > 0) = 1$ for some ξ, noting that

$$\alpha\hat{H}_\alpha 1(x) = \frac{\alpha G_\alpha^0 1(x)}{g_\gamma(x)} \le \frac{1}{g_\gamma(x)}, \qquad (x \in D). \tag{6}$$

Further, by theorem 1,

$$\begin{aligned} \alpha G_\alpha 1(\xi) &= E_\xi\Big(\int_0^\infty \alpha e^{-\alpha t}(\ell + (m + Q)\hat{H}_\alpha 1)\, d\Phi\Big) \\ &\le E_\xi\Big(\int_0^{\rho_1} e^{-\alpha t}\alpha Q\hat{H}_\alpha 1\, d\Phi\Big) + E_\xi(e^{-\alpha\rho_1}\alpha H_\alpha G_\alpha 1(x_{\rho_1})) \\ &\le E_\xi\Big(\int_0^{\rho_1} e^{-\alpha t} Q\left(\frac{1}{g_\gamma}\right) d\Phi\Big) + E_\xi(e^{-\alpha\rho_1}). \end{aligned} \tag{7}$$

Noting (5), $P_\xi(\rho_1 > 0) = 1$, and the right continuity of $\mathbf{M}$, one obtains $1 = \lim \alpha G_\alpha 1(\xi) = 0$, which is a contradiction. Therefore,

$$P_\xi(\rho_1 > 0) = 0 \quad \text{and} \quad P_\xi(\rho > 0) = 0 \qquad \text{for any } \xi \in V. \tag{8}$$

Now,

$$\begin{aligned} \tilde{P}_\xi\Big(\int_0^t u(\xi_s)\,ds < \infty \text{ for some } t > 0\Big) &= P_\xi\Big(\int_0^t u(x_{\tau(s)})\,ds < \infty \text{ for some } t > 0\Big) \\ &= P_\xi\Big(\int_0^{\tau(t)} u(x_s)\,d\Phi < \infty \text{ for some } t > 0\Big) \\ &\leq P_\xi(\rho > 0) = 0. \end{aligned} \tag{9}$$

Therefore, ($\tilde{\mathbf{M}}$.5) is proved.

6.2. *The Lévy system of* $\tilde{\mathbf{M}}$. First, we shall note that

[6.1] $\beta\hat{H}_{\alpha+\beta}H_\alpha f$ is increasing in β and decreasing in α for any $f \in B^+(V)$.

PROOF. Since $\beta G^0_{\alpha+\beta}H_\alpha f = H_\alpha f - H_{\alpha+\beta}f = E.\,(e^{-\alpha\sigma}(1 - e^{-\beta\sigma})f(x_\sigma))$ is increasing in β and decreasing in α, the same holds for $\beta\hat{H}_{\alpha+\beta}H_\alpha f$ for $f \in C^+(V)$ by ($\mathbf{M}_0$.5) and it also holds for $f \in B^+(V)$.

DEFINITION 6.1. Θ_α, $(\alpha \geq 0)$ *is a (not necessarily bounded) kernel on* $S \times V$ *such that* $\Theta_\alpha(\xi, \{\xi\}) = 0$ *if* $\xi \in V$ *and*

$$\Theta_\alpha(\xi, E) = \lim_{\beta\to\infty} \beta\hat{H}_{\alpha+\beta}H_\alpha(x, E) \qquad \text{for } E < V \text{ and } x \notin E. \tag{10}$$

By [6.1], Θ_α is well defined and

$$\Theta = \Theta_0 = \lim_{\alpha\to 0} \Theta_\alpha, \tag{11}$$

$$\Theta_\alpha(x, E) = \frac{H_\alpha(x, E)}{g_\gamma(x)} \qquad \text{if} \quad x \in D. \tag{12}$$

Let (P, L) and $(\tilde{P}, \tilde{L})$ be the Lévy system of $\mathbf{M}$ and $\tilde{\mathbf{M}}$ respectively. Then, by the definitions of U-process and Lévy system, for any $f \in B_0^+(V \times V)$,

$$E_\xi\big(\sum_{s\leq t} f(x_{\tau(s-)-}, x_{\tau(s)})\big) = \tilde{E}_\xi\big(\sum_{s\leq t} f(\xi_{s-}, \xi_s)\big) = \tilde{E}_\xi\Big(\int_0^t \tilde{P}f(\xi_s)\,d\tilde{L}\Big). \tag{13}$$

[6.2] For $f \in B_0^+(V \times V)$,

$$\begin{aligned} E_x\big(\sum_{s>0} e^{-\alpha\tau(s)}f(x_{\tau(s-)-}, x_{\tau(s)})\big) \\ = E_x\Big(\int_0^\infty e^{-\alpha t}\chi_V P_V f(x_t)\,dL\Big) + E_x\Big(\sum_{s\in T} e^{-\alpha\tau(s)}f(x_{\tau(s-)-}, x_{\tau(s)})\Big), \end{aligned} \tag{14}$$

where $P_V f(\xi) = \int_V P(\xi, d\eta)f(\xi, \eta)$.

PROOF. Noting that $f(\xi, \xi) = 0$, by [4.14] we have

$$\begin{aligned} E_x\big(\sum_{s>0} e^{-\alpha\tau(s)}f(x_{\tau(s-)-}, x_{\tau(s)})\big) &= E_x\big(\sum e^{-\alpha t}\chi_V(x_{t-})\chi_V(x_t)f(x_{t-}, x_t)\big) \\ &\quad + E_x\big(\sum_{s\in T} e^{-\alpha\tau(s)}f(x_{\tau(s-)-}, x_{\tau(s)})\big). \end{aligned} \tag{15}$$

The relation [6.2] is a consequence of the definition of the Lévy system.

[6.3] Let $f \in B^+(V \times V)$; then

$$(16)\quad E_x\Big(\sum_{s \in T} e^{-\alpha\tau(s)}(1 - e^{-\beta(\tau(s)-\tau(s-))})f(x_{\tau(s-)-}, x_{\tau(s)})\Big) = E_x\Big(\int_0^\infty e^{-\alpha t}\{(m + Q)\beta \hat{H}_{\alpha+\beta}H_\alpha\}f \cdot d\Phi\Big).$$

PROOF. It is sufficient to prove the result for $f(\xi, \eta) = h_1(\xi)h_2(\eta)$, for $\xi, \eta \in V$ and $h_i \in C^+(S)$. Notations used will be the same as in section 4. Put

$$(17)\quad I(k) = E_x\Big(\sum_{n=1}^\infty e^{-\alpha\sigma_{n+1}(k)}\beta \int_{\rho_n(k)}^{\sigma_{n+1}} e^{-\beta\rho_n(k)}\, dt h_1(x_{\rho_n(k)-}) \cdot h_2(x_{\sigma_{n+1}(k)})\chi(x_{\rho_n(k)} \in D)\Big).$$

Since,

$$(18)\quad e^{-\alpha\tau(s)}\beta \int_{\rho(k,s)}^{\tau(s)} e^{-\beta(t-\rho(k,s))}\, dt\, h_1(x_{\rho(k,s)-})h_2(x_{\tau(s)})\chi(s \in T_k) \leq \beta\|h_1h_2\|e^{-\alpha\tau(s)} \int_{\tau(s-)}^{\tau(s)} dt$$

for any $s \in T$, and

$$(19)\quad \sum \beta\|h_1h_2\|e^{-\alpha\tau(s)} \int_{\tau(s-)}^{\tau(s)} dt \leq \frac{\beta}{\alpha}\|h_1h_2\|,$$

we have by [4.8] and [4.9], that the limit

$$(20)\quad \lim_{k\to\infty} I(k) = E_x\Big(\sum_T e^{-\alpha\tau(s)}\beta \int_{\tau(s-)}^{\tau(s)} e^{-\beta(t-\tau(s-))}\, dt\, h_1(x_{\tau(s-)})h_2(x_{\tau(s)})\Big)$$

is equal to the left side of (16). On the other hand, by [5.6*]

$$\begin{aligned}(21)\quad \lim_{k\to\infty} I(k) &= \lim_{k\to\infty} E_x\Big(\sum e^{-\alpha\rho_n}\beta \int_{\rho_n}^{\sigma_n} e^{-(\alpha+\beta)(t-\rho_n)} \cdot h_1(x_{\rho_n-})H_\alpha h_2(x_t)\, dt\, \chi(x_{\rho_n} \in D)\Big)\\
&= \lim_{k\to\infty} E_x(\sum e^{-\alpha\rho_n}h_1(x_{\rho_n-})\beta G^0_{\alpha+\beta}H_\alpha h_2(x_{\rho_n}))\\
&= \lim_{k\to\infty} E_x(\sum e^{-\alpha\rho_n}h_1(x_{\rho_n-})\beta \hat{H}_{\alpha+\beta}H_\alpha h_2(x_{\rho_n})g_\gamma(x_{\rho_n}))\\
&= E_x\Big(\int_0^\infty e^{-\alpha t}h_1(m + Q)\beta\hat{H}_{\alpha+\beta}H_\alpha h_2\, d\Phi\Big),\end{aligned}$$

which proves [6.3].

Noting that $1 - e^{-\beta(\tau(s)-\tau(s-))} \uparrow 1$ for $s \in T$ and $\beta\hat{H}_{\alpha+\beta}H_\alpha f \uparrow \Theta_\alpha f$ for $f \in B_0^+(V \times V)$ when $\beta \to \infty$, we obtain from [6.3] that

$$(22)\quad E_x\Big(\sum_{s \in T} e^{-\alpha\tau(s)}f(x_{\tau(s-)-}, x_{\tau(s)})\Big) = E_x\Big(\int_0^\infty e^{-\alpha t}\{(m + Q)\Theta_\alpha f \cdot d\Phi\}\Big)$$

for $f \in B_0^+(V \times V)$. Combining [6.2] and (22), we obtain

$$(23)\quad E_x(\sum e^{-\alpha\tau(s)}f(x_{\tau(s-)-}, x_{\tau(s)})) = E_x\Big(\int_0^\infty e^{-\alpha t}\chi_V P_V f(x_t)\, dt\Big) + E_x\Big(\int_0^\infty e^{-\alpha t}\{(m + Q)\Theta_\alpha\}f\, d\Phi\Big)$$

for any $f \in B_0^+(V \times V)$. Noting that $\tau(s)$ is a Markov time and $\tau(s+u) = \tau(s) + \tau(u, w_{\tau(s)}^+)$, we have from (23),

$$(24) \quad E_x\Big(\sum_{s \le t} e^{-\alpha\tau(s)} f(x_{\tau(s-)-}, x_{\tau(s)})\Big) = E_x\Big(\int_0^{\tau(t)} e^{-\alpha s}(\chi_V P_V f \cdot dL + \{(m+Q)\Theta\} f \cdot d\Phi)\Big).$$

Letting $\alpha \downarrow 0$ and noting (13), we have

[6.4] for $f \in B_0^+(V \times V)$

$$(25) \quad \tilde{E}_\xi\Big(\int_0^t \tilde{P}f(\xi_s)\, d\tilde{L}\Big) = E_\xi\Big(\int_0^{\tau(t)} \chi_V P_V f\, dL\Big) + \tilde{E}_\xi\Big(\int_0^t \{(m+Q)\Theta\} f\, ds\Big).$$

Hence, we have

$$(26) \qquad \tilde{E}_\xi\Big(\int_0^t \tilde{P}f\, dL\Big) \ge \tilde{E}_\xi\Big(\int_0^t \{(m+Q)\Theta\} f\, ds\Big)$$

for any t, and consequently (by theorem 3.8 in [4]),

$$(27) \qquad \tilde{P}f \cdot \tilde{L} \gg \{(m+nQ)\Theta\} f \cdot T \qquad \text{for any } f \in B_0^+(V \times V).$$

PROPOSITION 7. *Let $(\tilde{P}, \tilde{L})$ be the Lévy system of $\tilde{\mathbf{M}}$; then*

($\tilde{\mathbf{M}}$.6) $(\tilde{P}, \tilde{L}) \gg ((m+Q)\Theta, T)$.

By [6.4], $(\tilde{P}, \tilde{L}) \approx ((m+Q)\Theta, T)$ if and only if $\chi_V P_V f \cdot L \approx 0$ for any $f \in B_0^+(V \times V)$. So, we have the following proposition.

PROPOSITION 8. *The paths of $\mathbf{M}$ have no jump from V to V a.e. if and only if $(\tilde{P}, \tilde{L}) \approx ((m+nQ)\Theta, T)$.*

Combining this proposition and proposition 2 (iii) with [4.20], we have the following result.

PROPOSITION 9. *The paths of $\mathbf{M}$ are continuous for $t \in [0, \zeta)$ if and only if $Q = 0$, a.e. ν, $(\tilde{P}, \tilde{L}) \approx (m\Theta, T)$, and the sample paths of $\mathbf{M}_0$ are continuous for $t \in [0, \zeta)$.*

6.3. *Killing functionals of $\tilde{\mathbf{M}}$.* Let $\tilde{A}_\infty$ be a killing functional of the process $\tilde{\mathbf{M}}$. First we note that

[6.5] $\alpha\hat{H}_\alpha(I-H)1(x)$ is increasing in α.

PROOF. Since

$$(28) \quad \begin{aligned} \alpha G_\alpha^0(I-H)1(x) &= E_x^0\Big(\alpha \int_0^{\sigma\wedge\zeta} e^{-\alpha t}\chi(\sigma(w_t^+) = \infty)\, dt\Big) \\ &= E_x^0\Big(\alpha \int_0^{\sigma\wedge\zeta} e^{-\alpha t}\, dt:\ \sigma = \infty\Big) \\ &= E_x\Big(\alpha \int_0^{\zeta} e^{-\alpha t}\, dt:\ \sigma = \infty\Big) \\ &= E_x(1 - e^{-\alpha\zeta}:\ \sigma = \infty), \end{aligned}$$

$\alpha G_\alpha^0(I-H)1(x)$ is increasing in α. By ($\mathbf{M}_0$.5), [6.5] is proved.

DEFINITION 6.2. *Let*

$$(29) \qquad \theta(x) = \lim_{\alpha\to\infty} \alpha\hat{H}_\alpha(I-H)1(x), \qquad x \in S.$$

Then,

$$\theta(x) = \frac{(I - H)1(x)}{g_\gamma(x)} \qquad \text{if} \quad x \in D. \tag{30}$$

Let

$$\xi = \inf \{s \colon x_{\tau(s)} = \partial\} \tag{31}$$

as in section 4; then from [4.17]

[6.6] $P_x(\xi < \infty) = P_x(x_{\zeta-} \in V, \zeta < \infty) + P_x(\xi \in T)$,

[6.7] $P_x(\xi \in T) = E_x\Big(\int_0^\infty (m + Q)\theta \, d\Phi\Big)$.

PROOF. Let

$$I(k) = E_x\Big(\sum_{s \in T} \chi(s \in T_k) e^{-\alpha\tau(s-)} \chi(\tau(s) = \infty)(1 - e^{-\beta(\zeta - \rho(k,s))})\Big). \tag{32}$$

Then, by the monotone convergence theorem,

$$\lim_{k \to \infty} I(k) = E_x\Big(\sum_{s \in T} e^{-\alpha\tau(s-)} \chi(\tau(s) = \infty)(1 - e^{-\beta(\zeta - \tau(s-))})\Big). \tag{33}$$

On the other hand, by (28),

$$\begin{aligned} I(k) &= E_x\big(\sum \chi(x_{\rho_n} \in D) e^{-\alpha\rho_n} \chi(\sigma_{n+1} = \infty)(1 - e^{-\beta(\zeta - \rho_n)})\big) \\ &= E_x\big(\sum e^{-\alpha\rho_n} \beta G_\beta^0 (I - H)|(x_{\rho_n})\big) \\ &= E_x\big(\sum e^{-\alpha\rho_n} \beta \hat{H}_\beta (I - H)|(x_{\beta_n}) g_\gamma(x_{\rho_n})\big). \end{aligned} \tag{34}$$

And by [5.6*], we have

$$\lim_{k \to \infty} I(k) = E_x\Big(\int_0^\infty e^{-\alpha t}(m + Q)\beta \hat{H}_\beta (I - H)1 \, d\Phi\Big). \tag{35}$$

Hence,

$$\begin{aligned} &E_x\Big(\sum_{s \in T} e^{-\alpha\tau(s-)} \chi(\tau(s) = \infty)(1 - e^{-\beta(\zeta - \tau(s-))})\Big) \\ &\qquad = E_x\Big(\int_0^\infty e^{-\alpha t}(m + Q)\beta \hat{H}_\beta (I - H)1 \cdot d\Phi\Big). \end{aligned} \tag{36}$$

Letting $\alpha \downarrow 0$ and then $\beta \to \infty$, we have

$$P_x(\xi \in T) = E_x\Big(\sum_{s \in T} \chi(\tau(s) = \infty)\Big) = E_x\Big(\int_0^\infty (m + Q)\theta \cdot d\Phi\Big). \tag{37}$$

From [6.6] and [6.7]

$$P_x(\xi < \infty) = P_x(x_{\zeta-} \in V, \zeta < \infty) + E_x\Big(\int_0^\infty (m + Q)\theta \, d\Phi\Big). \tag{38}$$

Noting that $\{t < \xi\} \in \{\xi = t + \xi(w_{\tau(t)}^+)\}$,

$$P_x(\xi \leq t) = P_x(x_{\zeta-} \in V, \zeta \leq \tau(t), \zeta < \infty) + E_x\Big(\int_0^{\tau(t)} (m + Q)\theta \, d\Phi\Big). \tag{39}$$

Since $\tilde{E}_\xi(\tilde{A}_\infty(t)) = \tilde{P}_\xi(\zeta \leq t) = P_\xi(\xi \leq t)$, we have

[6.8] $\tilde{E}_\xi(\tilde{A}_\infty(t)) = P_\xi(x_{\zeta-} \in V, \zeta \leq \tau(t), \zeta < \infty) + \tilde{E}_\xi\Big(\int_0^t (m + Q)\theta \, dt\Big)$.

PROPOSITION 10. *Let $\tilde{A}_\infty$ be a killing functional of* $\tilde{\mathbf{M}}$. *Then*

($\tilde{\mathbf{M}}$.7) $\tilde{A}_\infty \gg (m + Q)\theta \cdot T$.

From (39), the following result is easily obtained.

PROPOSITION 11. *The relation $x_{\zeta-} \notin V$ when $\zeta < \infty$, a.e.* (**M**) *holds if and only if*

$$\tilde{A}_\infty \approx (m + Q)\theta \cdot T(\tilde{\mathbf{M}}). \tag{40}$$

Since $\theta = 0$ if $\mathbf{M}_0$ is conservative, combining [4.21] with proposition 11, we have the following result.

PROPOSITION 12. *The process* **M** *is conservative if and only if* $\mathbf{M}_0$ *is conservative and* $\tilde{A}_\infty \approx (m + Q) \cdot T$ *for* $\tilde{\mathbf{M}}$.

In conclusion, we have the next theorem.

THEOREM 13. *Suppose* $\mathbf{M}_0$ *satisfies* ($\mathbf{M}_0$.1) $\sim$ ($\mathbf{M}_0$.6) *and* **M** *satisfies* (**M**.1) $\sim$ (**M**.3). *Then the boundary system* ($\tilde{\mathbf{M}}$, ℓ, m, Q) *of* **M** *satisfies* ($\tilde{\mathbf{M}}$.1) $\sim$ ($\tilde{\mathbf{M}}$.7).

6.4. *Remarks on condition* ($\tilde{\mathbf{M}}$.C). Suppose $\mathbf{M}_0$ satisfies the conditions ($\mathbf{M}_0$.1) $\sim$ ($\mathbf{M}_0$.6) and **M** satisfies the condition (**M**.1) $\sim$ (**M**.3) and

(**M**.C) $G_\alpha f \in C(S)$ if $f \in C(S)$.

By theorem 3,

$$G_\alpha f(\xi) = K_\alpha \hat{H}_\alpha f(\xi) \quad \text{for} \quad \xi \in V \quad \text{if} \quad \ell = Q = 0, \quad \text{a.e. } \nu, \tag{41}$$

$$G_\alpha f(\xi) = K_\alpha(\ell + m\hat{H}_\alpha) f(\xi) \quad \text{for} \quad \xi \in V \quad \text{if} \quad Q = 0, \quad \text{a.e. } \nu. \tag{42}$$

If $l = Q = 0$, a.e., from (41) and ($\mathbf{M}_0$.6), $K_\alpha f \in C(V)$ for $f \in C(V)$. And if $Q = 0$, a.e. and $\mathbf{M}_0$ satisfies the condition

($\mathbf{M}_0$.6*) $\{\hat{H}_\alpha f\colon f \in C(S) \text{ and } f(\xi) = 0 \text{ for } \xi \in V\}$ is dense in $C(S)$;

[6.9] $K_\alpha f$, $K_\alpha \ell f$ and $K_\alpha m f \in C(V)$ for $f \in C(V)$.

Noting [5.12] and proposition 2, we have the following result.

PROPOSITION 14. *Suppose that* **M** *satisfies* (**M**.C) *in addition to* (**M**.1) $\sim$ (**M**.3).

(i) *If the paths of* **M** *have no sojourn on V and no jump from V to D, then*

($\tilde{\mathbf{M}}$.C) $\tilde{G}_\lambda f \in C(V)$ for $f \in C(V)$.

(ii) *Suppose that* $\mathbf{M}_0$ *satisfies* ($\mathbf{M}_0$.6*) *for* (**M**.6). *If the paths of* **M** *have no jump from V to D, then*

($\tilde{\mathbf{M}}$.C) $\tilde{G}_\lambda f$ and $\tilde{G}_\lambda(\ell + m\hat{H}_\alpha) f \in C(V)$ for $f \in C(V)$.

REMARK. The converse of proposition 14 is in general true. Namely, if **M** satisfies

($\tilde{\mathbf{M}}$.C) $\tilde{G}_\lambda f$ and $\tilde{G}_\lambda(\ell + m\hat{H}_\alpha + Q\hat{H}_\alpha) f \in C(V)$ for $f \in C(V)$,

then, by [5.12] and theorem 3, we can easily see that **M** satisfies condition (**M**.C).

7. Construction of M for a given boundary system

7.1. *Lemmas on right continuous functionals.* Let **M** be any process on S satisfying (**M**.1) and (**M**.2) and let (P, L) be the Lévy system of **M**. Let A, B, and C be continuous additive functionals of **M**, $\alpha, \beta \in B_0^+(S \times S)$, and $0 \leq \alpha$, $\beta \leq 1$. Let

$$a(t) = \prod_{s \leq t} (1 - \alpha(x_{s-}, x_s)),$$

(1)

$$b(t) = \prod_{s \leq t} (1 - \beta(x_{s-}, x_s)).$$

The following lemmas are proved in [6].

[7.1] Suppose that $E_x(\int_0^\infty e^{-A(t)}\, dC)$ and $E_x(\int_0^\infty e^{-B(t)}\, dC)$ are bounded in x. Let

$$\begin{aligned} K_1f(x) &= E_x\Big(\int_0^\infty e^{-A(t)}a(t)f(x_t)\, dC\Big), \\ K_2f(x) &= E_x\Big(\int_0^\infty e^{-B(t)}b(t)f(x_t)\, dC\Big), \\ U_1f(x) &= \int P(x, dy)\alpha(x, y)f(y), \\ U_2f(x) &= \int P(x, dy)\beta(x, y)f(y), \end{aligned} \tag{2}$$

and let $E_x(\int_0^\infty U_iI(x_t)\, dL)$ be bounded in $x (i = 1, 2)$. Then

$$\begin{aligned} &K_1f(x) - K_2f(x) \\ &\quad + E_x\Big(\int_0^\infty e^{-A(t)}a(t)\{K_2f(x_t)(dA - dB) + (U_1 - U_2)K_2f(x_t)\, dL\}\Big) = 0. \end{aligned} \tag{3}$$

[7.2] For any Markov time σ such that $\{t < \sigma\} \subset \{\sigma(w_t^+) = \sigma - t\}$ and any $f \in B_0^+(S \times S)$,

$$E_x\Big(\int_0^\sigma e^{-A(s)}Pf(x_s)\, dL\Big) = E_x\Big(\sum_{s \leq \sigma} e^{-A(s)}f(x_{s-}, x_s)\Big). \tag{4}$$

Let A_∞ be a killing functional of $\mathbf{M}$; then we have the following analogue of [7.2].

[7.3] For any Markov time σ such that $\{t < \sigma\} \subset \{\sigma(w_t^+) = \sigma - t\}$ and for any $f \in B^+(S)$,

$$E_x\Big(\int_0^\sigma e^{-A(t)}f(x_t)\, dA_\infty\Big) = E_x(f(x_{\zeta-})e^{-A(\zeta)}; \zeta \leq \sigma, \zeta < \infty). \tag{5}$$

7.2. *Preliminaries.* In the following, $\gamma > 0$ and $\mathbf{M}_0$ satisfying $(\mathbf{M}_0.1) \sim (\mathbf{M}_0.6)$ are fixed. Let $\tilde{\mathbf{M}}$ be a process on V which satisfies $(\tilde{\mathbf{M}}.1)$ and $(\tilde{\mathbf{M}}.2)$. Let

$$N(\tilde{G}) = \{f \in B^+(V), \tilde{G}_\lambda f = 0\} \tag{6}$$

where $\tilde{G}_\lambda$ is a Green kernel of $\mathbf{M}$. Let ν be a measure on V such that

$$N(\tilde{G}) = \{f \in B^+(V) \quad f = 0, \quad \text{a.e. } \nu\} \tag{7}$$

as before. Let $(\tilde{P}, \tilde{L})$ be the Lévy system of $\tilde{\mathbf{M}}$. In section 7.2, we shall assume that a system $(\tilde{\mathbf{M}}, \ell, m, Q)$ satisfies the conditions $(\tilde{\mathbf{M}}.4) \sim (\tilde{\mathbf{M}}.7)$ and the following (which is weaker than $(\tilde{\mathbf{M}}.3)$):

$(\tilde{\mathbf{M}}.3^*)$ $\ell, m \in B^+(V)$, Q is a kernel on $V \times D$ such that $\ell + m + Q(\cdot, D) \geq 1$, a.e. ν.

Let

$$C_\alpha = \ell + (m + Q)\hat{H}_\alpha, \tag{8}$$

$$U_\alpha = \alpha C_\alpha H,$$

$$V_\alpha(\xi, E) = (m + Q)\alpha\hat{H}_\alpha H(\xi, E - \{\xi\}), \qquad (E \subset V),$$

$$v_\alpha(\xi) = (m + Q)\alpha\hat{H}_\alpha H(\xi, \{\xi\}), \tag{9}$$

$$w_\alpha(\xi) = \alpha\ell + v_\alpha.$$

Then

$$\alpha C_\alpha = U_\alpha + \alpha C_\alpha(I - H), \qquad U_\alpha = w_\alpha + V_\alpha, \tag{10}$$

and by ($\tilde{\mathbf{M}}$.6), the definition of Θ, and [6.1],

$$V_\alpha f \cdot T \ll \tilde{P}f \cdot \tilde{L} \qquad \text{for any } f \in B_0^+(V \times V). \tag{11}$$

Therefore, by appendix I, we have that

[7.4] there exists $k_\alpha \in B_0^+(V \times V)$ such that $0 \leq k_\alpha \leq 1$ and

$$V_\alpha f \cdot T = \tilde{P}(k_\alpha f) \cdot \tilde{L} \qquad \text{for any } f \in B_0^+(V \times V). \tag{12}$$

Let

$$K_\alpha^\lambda f(\xi) = \tilde{E}_\xi\Big(\int_0^\infty e^{-\lambda t - w_\alpha \cdot t} \prod_{s \leq t} (1 - k_\alpha(\xi_{s-}, \xi_s))f(\xi_t)\, dt\Big); \tag{13}$$

then

$$K^\lambda = K_0^\lambda = \tilde{G}_\lambda. \tag{14}$$

[7.5] The operator K_α^λ is a positive kernel and $\alpha K_\alpha^\lambda C_\alpha 1 \leq 1$, $(\alpha, \lambda \geq 0,$ $\alpha + \lambda > 0)$.

Proof. Since $\lim_{\lambda \to 0} K_\alpha^\lambda 1 = K_\alpha 1$, it is sufficient to prove [7.5] for $\lambda > 0$. Let

$$V_\alpha(\xi) = V_\alpha(\xi, V) = V_\alpha(\xi, V - \{\xi\}), \qquad V_\alpha^\epsilon(\xi) = \int_{r(\xi,\eta) \geq \epsilon} V_\alpha(\xi, d\eta), \tag{15}$$

$$\alpha C_\alpha^\epsilon(\xi) = w_\alpha(\xi) + V_\alpha^\epsilon(\xi) + \alpha C_\alpha(I - H)1(\xi), \tag{16}$$

$$\rho = \rho(\epsilon) = \inf s\colon r(\xi_{s-}, \xi_s) \geq \epsilon, \qquad \rho_0 = 0 \quad \text{and} \quad \rho_{n+1} = \rho_n + \rho(w_{\rho_n}^+), \tag{17}$$

then $\lim_{n \to \infty} \rho_n = \infty$. Let

$$J_\epsilon(\xi) = \alpha\tilde{E}_\xi\Big(\int_0^\infty e^{-(\lambda + w_\alpha) \cdot t} \prod_{\rho_n \leq t} \gamma(n) C_\alpha^\epsilon(\xi_t)\, dt\Big) \tag{18}$$

where $\gamma(n) = (1 - k_\alpha(\xi_{\rho_n -}, \xi_{\rho_n})\chi(\rho_n < \zeta))$. Since

$$|\alpha \prod_{\rho_n \leq t} \gamma(n) C_\alpha^\epsilon(\xi)| \leq \alpha(\|\ell\| + \|m\| + \|Q\|)\|\hat{H}_\alpha 1\|, \qquad \lim_{\epsilon \downarrow 0} \alpha \prod_{\rho_n \leq t} \gamma(n) C_\alpha^\epsilon(\xi) = \alpha \prod_{s \leq t} (1 - k(\xi_{s-}, \xi_s)) C_\alpha 1(\xi), \tag{19}$$

noting that $\lambda > 0$, we have

(20)
$$\lim_{\epsilon \downarrow 0} J_\epsilon(\xi) = \alpha K_\alpha^\lambda C_\alpha 1(\xi),$$

(21)
$$J_\epsilon(\xi) \leq \alpha \tilde{E}_\xi \Big(\sum_n \int_{\rho_n}^{\rho_{n+1}} e^{-w_\alpha \cdot t} \prod_{m \leq n} \gamma(m) C_\alpha^\epsilon(\xi_t)\, dt \Big)$$
$$= \tilde{E}_\xi \Big(\sum_n e^{-w_\alpha \cdot \rho_n} \prod_{m \leq n} \gamma(m) E_{\xi_{\rho_n}} \Big(\int_0^\rho e^{-w_\alpha \cdot t} \alpha C_\alpha^\epsilon(\xi_t)\, dt \Big)\Big).$$

Let

(22)
$$I_\epsilon(\xi) = \tilde{E}_\xi \Big(\int_0^\rho e^{-w_\alpha \cdot t} \alpha C_\alpha^\epsilon(\xi_t)\, dt \Big).$$

Since $\alpha C_\alpha(I - H)1 = (m + Q)\alpha \hat{H}_\alpha(I - H) \leq (m + Q)\theta$, by [7.4] and ($\mathfrak{M}$.7),

(23)
$$I_\epsilon(\xi) = \tilde{E}_\xi \Big(\int_0^\rho e^{-w_\alpha \cdot t}(w_\alpha + V_\alpha^\epsilon + \alpha C_\alpha(I - H)1)\, dt \Big)$$
$$\leq \tilde{E}_\xi \Big(\int_0^\rho e^{-w_\alpha \cdot t} \Big(w_\alpha\, dt + \int_{r(\xi_t, \eta) \geq \epsilon} k_\alpha(\xi_t, \eta) P(\xi_t, d\eta)\, dL + d\tilde{A}_\infty \Big)\Big).$$

Noting that $r(\xi_{s-}, \xi_s) < \epsilon$ and $\xi_s \in V$, if $\xi_0 \in V$ and $s < \rho$, by [7.2] and [7.3],

(24)
$$I_\epsilon(\xi) \leq \tilde{E}_\xi(1 - e^{-w_\alpha \cdot \rho} + e^{-w_\alpha \cdot \rho} \chi(r(\xi_{\rho-}, \xi_\rho) \geq \epsilon) k_\alpha(\xi_{\rho-}, \xi_\rho) + e^{-w_\alpha \cdot \rho}) \qquad (\zeta = \rho < \infty))$$
$$\leq \tilde{E}_\xi(1 - e^{-w_\alpha \cdot \rho}(1 - k_\alpha(\xi_{\rho-}, \xi_\rho))\chi(\rho < \zeta)).$$

Therefore,

(25)
$$J_\epsilon(\xi) \leq \tilde{E}_\xi(\sum e^{-\alpha \rho_n} \prod_{m \leq n} \gamma(m)(1 - e^{-\alpha(\rho_{n+1} - \rho_n)} \gamma(n + 1)) \leq 1.$$

From (20), [7.5] is proved.

Since $1 = \hat{H}_\gamma 1 = \hat{H}_\alpha(1 + (\alpha - \gamma)G_\gamma^0 1)$,

(26)
$$K_\alpha 1 \leq K_\alpha(\ell + m + Q(\cdot, D)) = K_\alpha C_\alpha(1 + (\alpha - \gamma)G_\gamma^0 1)$$
$$\leq \frac{1}{\alpha} \|1 + |\alpha - \gamma| G_\gamma^0 1\|$$

for $\alpha > 0$. So, we have that

[7.6] K_α^λ is a bounded kernel ($\alpha \geq 0$, $\lambda \geq 0$, $\alpha + \lambda > 0$).

[7.7] The kernel K_α satisfies the relations

(27)
$$K_\alpha^\lambda - K_\alpha^\mu + (\lambda - \mu) K_\alpha^\lambda K_\alpha^\mu = 0,$$

(28)
$$K_\alpha^\lambda - K_\beta^\lambda + K_\alpha^\lambda(U_\alpha - U_\beta) K_\beta^\mu = 0.$$

PROOF. For $\lambda > 0$, we can apply [7.1]. Note that $\tilde{P}(k_\alpha f) \cdot \tilde{L} \approx V_\alpha f \cdot T$. For $\lambda = 0$, we can obtain (27) and (28) by letting $\lambda \to 0$.

Let

(29)
$$G_\alpha = G_\alpha^0 + H_\alpha K_\alpha C_\alpha, \qquad (\alpha > 0);$$

then G_α is a kernel on $S \times S$.

[7.8] The kernel G_α satisfies the inequality $\alpha G_\alpha 1 \leq 1$.

PROOF. By [7.5], $\alpha G_\alpha 1 = \alpha G_\alpha^0 1 + \alpha H_\alpha K_\alpha C_\alpha 1 \leq \alpha G_\alpha^0 1 + H_\alpha 1 \leq 1$.

The following is an immediate consequence of [7.7] and (29):

[7.9] $G_\alpha - G_\beta + (\alpha - \beta)G_\alpha G_\beta = 0$.

7.3. *The case $\ell \geq p > 0$.* In section 7.3, for the system $(\tilde{\mathbf{M}}, \ell, m, Q)$, we shall assume $(\tilde{\mathbf{M}}.C)$ and

(M.3**) $\ell \geq p$ where p is a positive constant, in addition to $(\tilde{\mathbf{M}}.1)$, $(\tilde{\mathbf{M}}.2)$, $(\tilde{\mathbf{M}}.3^*)$, and $(\tilde{\mathbf{M}}.4) \sim (\tilde{\mathbf{M}}.7)$. Notations used are the same as in section 7.2. From (M.3**), we can see by the definition of K_α^λ that

$$K_\alpha^\lambda 1 \leq \min\left(\frac{1}{\alpha p}, \frac{1}{\lambda}\right). \tag{30}$$

From $(\tilde{\mathbf{M}}.C)$ and [5.12] we have that

[7.10] $G_\alpha f \in C(S)$ if $f \in C(S)$.

From $(\tilde{\mathbf{M}}.C)$ and $(\tilde{\mathbf{M}}.1)$

$$R(K) = \{K^\lambda f \colon f \in C(V)\} \text{ is dense in } C(V). \tag{31}$$

Since $G_\alpha^0 + (1/\alpha)\, H_\alpha$ is a Green kernel of $\mathbf{M}_0$,

$$\lim_{\alpha \to \infty} \|\alpha G_\alpha^0 f + H_\alpha f - f\| = 0 \qquad \text{for} \quad f \in C(S).. \tag{32}$$

In particular,

$$\lim_{\alpha \to \infty} \|\alpha G_\alpha^0 f - f\| = 0 \tag{33}$$

for $f \in C(S)$ and $f(\xi) = 0$, $\xi \in V$.

[7.11] One has $\lim_{\alpha\to\infty} \|K_\alpha U_\alpha g - g\|_V = 0$.

PROOF. Since $K_\alpha U_\alpha = \alpha K_\alpha C_\alpha H$ is a uniformly bounded kernel in α, it is sufficient to prove [7.11] for $g = K^\lambda f$ with $f \in C(V)$. Since

$$\begin{aligned} K_\alpha U_\alpha g - g &= (K_\alpha - K_\alpha^\lambda)U_\alpha g - K_\alpha^\lambda U_\alpha K^\lambda f - K^\lambda f \\ &= \lambda K_\alpha K_\alpha^\lambda U_\alpha g - K_\alpha^\lambda f, \end{aligned} \tag{34}$$

then

$$\|K_\alpha U_\alpha g - g\| \leq \frac{\lambda}{\alpha p}\|g\| - \frac{1}{\alpha p}\|f\| \to 0, \qquad (\alpha \to \infty). \tag{35}$$

[7.12] If $f \in C(S)$ and $f = 0$ on V, then $\lim_{\alpha\to\infty} \|\alpha K_\alpha C_\alpha f\|_V = 0$.

PROOF. By (33), it is sufficient to prove the result for $f = G_\beta^0 g$, for some $\beta > 0$ and $g \in C(S)$. Since

$$C_\alpha G_\beta^0 g = (m + Q)\hat{H}_\alpha G_\beta^0 g = (m + Q)\hat{H}_\beta G_\alpha^0 g, \tag{36}$$

$$\begin{aligned} \|\alpha K_\alpha C_\alpha f\| &= \|\alpha K_\alpha C_\alpha G_\beta^0 g\| \\ &\leq (\|m\| + \|Q\|)\alpha\|K_\alpha \hat{H}_\beta G_\alpha^0 g\| \frac{\|m\| + \|Q\|}{\alpha p}\|\hat{H}_\beta 1\|\,\|g\| \to 0, \quad (\alpha \to \infty) \end{aligned} \tag{37}$$

[7.13] For every $f \in C(S)$, $\lim_{\alpha\to\infty} \|\alpha K_\alpha C_\alpha f - f\|_V = 0$.

PROOF. For any $f \in C(S)$, by [7.11] and [7.12], we have

$$\|\alpha K_\alpha C_\alpha f - f\|_V \leq \|K_\alpha U_\alpha f - f\|_V + \|\alpha K_\alpha C_\alpha (I - H)f\|_V \to 0, \quad (\alpha \to \infty). \tag{38}$$

[7.14] For every $f \in C(S)$, $\lim_{\alpha\to\infty} \|\alpha G_\alpha f - f\| = 0$.

PROOF. Since $\alpha G_\alpha f - f = \alpha G_\alpha^0 f + H_\alpha f - f + H_\alpha(\alpha K_\alpha C_\alpha f - f)$, the relation [7.14] is a consequence of (32) and [7.13].

From [7.8], [7.9], [7.10], and [7.14], G_α is a Green kernel of a strongly continuous contraction semigroup on $C(S)$. Therefore, there exists a (unique) Hunt process $\mathbf{M}$ on S whose Green kernel is G_α given by (29). (See [3].)

[7.15] The following equalities hold:

$$H_\alpha f = E_\cdot(e^{-\alpha\sigma}f(x_\sigma): \sigma < \infty), \tag{39}$$

$$G_\alpha^0 f = E_\cdot\left(\int_0^\sigma e^{-\alpha t}f(x_t)\, dt\right); \tag{40}$$

that is, $\mathbf{M}_0$ is a stopped process of $\mathbf{M}$.

PROOF. For any $f \in R(C) = \{C_\alpha g: g \in C(S)\}$, $f/\ell \in B(V)$, and noting ($\tilde{\mathbf{M}}$.4), $G_\alpha(f/\ell) = H_\alpha K_\alpha f$. On the other hand,

$$G_\alpha\left(\frac{f}{\ell}\right) = E_\cdot\left(\int_0^\infty e^{-\alpha t}\frac{f}{\ell}\, dt\right) = E_\cdot\left(e^{-\alpha\sigma}G_\alpha \frac{f}{\ell}(x_\sigma)\right) = E_\cdot(e^{-\alpha\sigma}K_\alpha f(x_\sigma)). \tag{41}$$

However, by [7.13], $\{K_\alpha C_\alpha g: g \in C(S)\}$ is dense in $C(S)$ and we have (39). Since

$$\begin{aligned} G_\alpha^0 f &= G_\alpha f - H_\alpha K_\alpha C_\alpha f = G_\alpha f - H_\alpha G_\alpha f = G_\alpha f - E_\cdot(e^{-\alpha\sigma}G_\alpha f(x_\sigma)) \\ &= E_\cdot\left(\int_0^\sigma e^{-\alpha t}f(x_t)\, dt\right), \end{aligned} \tag{42}$$

equation (40) is proved.

[7.16] The process $\mathbf{M}$ satisfies ($\mathbf{M}$.2).

PROOF. Let ν on V and m_0 on D be the measures appearing in ($\tilde{\mathbf{M}}$.2) and ($\mathbf{M}_0$.3) respectively. Let $m(E) = m_0(E) + \nu C_\alpha(E)$, $(E \subset S)$. Then, $m_0(E) = 0$ if and only if $G_\alpha^0(x, E) = 0$ for any x in S, and $\nu C_\alpha(E) = 0$ if and only if $K_\alpha C_\alpha(\xi, E) = 0$ for any ξ in V (by [5.10]). Therefore, $m(E) = 0$ if and only if $G_\alpha(x, E) = 0$ for any x in S.

PROPOSITION 15. *Let the system* ($\tilde{\mathbf{M}}, \ell, m, Q$) *satisfy* ($\tilde{\mathbf{M}}$.1), ($\tilde{\mathbf{M}}$.2), ($\tilde{\mathbf{M}}$.3*), ($\tilde{\mathbf{M}}$.3**), ($\tilde{\mathbf{M}}$.4) $\sim$ ($\tilde{\mathbf{M}}$.7), *and* ($\tilde{\mathbf{M}}$.C). *Then* K_α^λ, *given by* (16), *is an* (ℓ, m, Q)-*system, and* G_α, *given by* (29), *is the Green kernel of a process* $\mathbf{M}$ *which satisfies* ($\mathbf{M}$.1) $\sim$ ($\mathbf{M}$.3). *Let* ($\tilde{\mathbf{M}}^*, \ell^*, m^*, Q^*$) *be the boundary system of* $\mathbf{M}$; *then* $N(K) = N(K^*)$, *and* $K_\alpha g = K_\alpha^*(1/\ell + m + n)g$, $\ell^* = (\ell/\ell + m + n)$, $m^* = (m/\ell + m + n)$, *and* $Q^*f = (1/\ell + m + n)Qf$, *for any* $g \in B(V)$ *and* $f \in B(D)$, *except for functions in* $N(K) = N(K^*)$. *Here* $\{K_\alpha^{*\lambda}\}$ *is a system of kernels defined by* $\mathbf{M}$ (*as in definition* 5.2) *and* $n = Q(\cdot, D)$.

PROOF. The first part has already been proved. Since

$$\begin{aligned} G_\alpha f &= G_\alpha^0 f + H_\alpha K_\alpha(\ell + (m + Q)\hat{H}_\alpha)f \\ &= G_\alpha^0 f + H_\alpha K_\alpha^*(\ell^* + (m^* + Q^*)\hat{H}_\alpha)f, \end{aligned} \tag{43}$$

for any $g \in B(V)$ and $K_\alpha g = K_\alpha^*(\ell^*/\ell)g$. Therefore, $N(K) = N(K^*)$ and

$$K_\alpha^*\left(\frac{\ell^*}{\ell}(\ell + (m + Q)\hat{H}_\alpha)\right) = K_\alpha^*(\ell^* + (m^* + Q^*)\hat{H}_\alpha). \tag{44}$$

Noting ($\tilde{\mathbf{M}}$.4), by [5.14] we have $m = m^*$ and $Qf = Q^*f$, except for functions in

$N(K^*)$. Since $\ell^* + m^* + Q^*(\cdot, D) = 1(N(K^*))$, $\ell^*/\ell = (1/\ell + m + n)(N(K^*))$, and $N(K) = N(K^*)$. Thus, we have the last part.

7.4. *The general case.* Let the system $(\tilde{\mathbf{M}}, \ell, m, Q)$ satisfy the conditions $(\tilde{\mathbf{M}}.1)\sim(\tilde{\mathbf{M}}.7)$ and $(\mathbf{M}.C)$. Let $\tilde{G}_\lambda$ be the Green kernel of $\tilde{\mathbf{M}}$. Then the system $(\tilde{\mathbf{M}}, 1 + \ell, m, Q)$ satisfies the conditions in proposition 15. By proposition 15, there exist a $(1 + \ell, m, Q)$-system $\bar{K}^\lambda_\alpha$ such that $\bar{K}^\lambda = \tilde{G}_\lambda$ and a Markov process $\bar{\mathbf{M}}$ on S whose boundary system $(\tilde{\mathbf{M}}, \bar{\ell}, \bar{m}, \bar{Q})$ satisfies

$$\bar{\ell} = \frac{1+\ell}{2}, \quad \bar{m} = \frac{m}{2}, \quad \bar{Q}f = \frac{Qf}{2} \qquad \text{(for any } f \in B(D)), \quad \text{a.e. } \nu, \tag{45}$$

$$\bar{E}_\xi\left(\int_0^\infty e^{-\alpha t} f(x_t)\, d\bar{\Phi}\right) = 2\bar{K}_\alpha f(\xi), \qquad (f \in B(V), \xi \in V), \tag{46}$$

where ν is a measure appearing in $(\tilde{\mathbf{M}}.2)$ and $\bar{\Phi} = \tilde{T}_\gamma$ (in $\bar{\mathbf{M}}$).

[7.17] Let $\Phi = \frac{1}{2}\bar{\Phi}$; then $\bar{K}^\lambda_\alpha f(\xi) = \bar{E}_\xi\left(\int_0^\infty e^{-\alpha t - \lambda\Phi(t)} f(x_t)\, d\Phi\right)$.

Proof. Let $K'^\lambda_\alpha f = \bar{E}_\cdot(\int_0^\infty e^{-\alpha t - \lambda\Phi(t)} f\, d\Phi)$. Then by [5.1], and the definition of Φ, $\{K'^\lambda_\alpha\}$ is a $(1 + \ell, m, Q)$-system and $K'_\alpha = \bar{K}_\alpha$. Thus, by [5.11], $K'^\lambda_\alpha = \bar{K}^\lambda_\alpha$ for any α and λ.

In particular, we have

[7.18] $\tilde{G}_\lambda f(\xi) = \bar{E}_\xi\left(\int_0^\infty e^{-\lambda\Phi t} f\, d\Phi\right) = \tilde{\tilde{E}}_\xi\left(\int_0^\infty e^{-\frac{\lambda}{2}t} f\, \frac{1}{2}dt\right)$, $(\xi \in V)$. Therefore, $\tilde{\mathbf{M}}$ is a process whose velocity is exactly twice times that of $\tilde{\mathbf{M}}$.

Let

$$u(\cdot) = \frac{1}{1 - \chi_E} + Q\left(\frac{1}{g_\gamma}\right), \tag{47}$$

where $E = \{\xi \colon \ell + m > 0\}$.

[7.19] $\bar{P}_\xi\left(\int_0^t u\, d\Phi = \infty, t > 0\right) = 1$, for any $\xi \in V$.

Proof. Let $\bar{\tau}(t) = \sup\{s \colon \bar{\Phi}(s) \leq t\}$. By $(\tilde{\mathbf{M}}.5)$ and [7.18],

$$\begin{aligned} \bar{P}_\xi\left(\int_0^{\bar{\tau}(t)} u(x_s)\, d\Phi = \infty, \text{ for any } t > 0\right) \\ = \tilde{\tilde{P}}_\xi\left(\tfrac{1}{2}\int_0^t u(\xi_s)\, ds = \infty, \text{ for any } t > 0\right) \\ = \tilde{P}_\xi\left(\int_0^t u(\xi_s)\, ds = \infty, \text{ for any } t > 0\right) = 1, \end{aligned} \tag{48}$$

for $\xi \in V$. Noting that $\bar{P}_\xi(\bar{\tau}(0) = \sigma = 0) = 1$, we have [7.19].

[7.20] Let $(\bar{P}, \bar{L})$ be a Lévy system of $\bar{\mathbf{M}}$ and $\bar{\sigma} = \sigma_D$. Then

$$\bar{E}_\xi\left(\int_0^{\bar{\sigma}} e^{-\gamma t} Q\left(\frac{f}{g_\gamma}\right) d\Phi\right) = \bar{E}_\xi(e^{-\gamma\bar{\sigma}} f(x_{\bar{\sigma}}) \colon x_{\bar{\sigma}} \in D), \tag{49}$$

$$\bar{E}_\xi\left(\int_0^{\bar{\sigma}} e^{-\gamma t} m\, d\Phi\right) = 0, \tag{50}$$

where $\xi \in V$ and $f \in B(S)$.

Proof. Since $x_t \in V$ for $0 \leq t < \bar{\sigma}$. By [7.2] and (31) in section 5,

(51) $$\overline{E}_\xi(e^{-\gamma\bar{\sigma}}f(x_{\bar{\sigma}}) \colon x_{\bar{\sigma}} \in D) = \overline{E}_\xi\Big(\sum_{s \leq \bar{\sigma}} e^{-\gamma s}\chi_V(x_{s-})\chi_D(x_s)f(x_s)\Big)$$
$$= \overline{E}_\xi\left(\int_0^{\bar{\sigma}} e^{-\gamma t}\chi_V\overline{P}_D f\, dL\right)$$
$$= \overline{E}_\xi\left(\int_0^{\bar{\sigma}} e^{-\gamma t}\overline{Q}\left(\frac{f}{g_\gamma}\right) d\overline{\Phi}\right)$$
$$= \overline{E}_\xi\left(\int_0^{\bar{\sigma}} e^{-\gamma t}Q\left(\frac{f}{g_\gamma}\right) d\Phi\right).$$

Thus (50) is proved. Letting $f = g_\gamma$ in (49),

(52) $$\overline{E}_\xi\left(\int_0^{\bar{\sigma}} e^{-\gamma t}Q(\cdot, D)d\Phi\right) = \overline{E}_\xi(e^{-\gamma\bar{\sigma}}g_\gamma(x_{\bar{\sigma}}) \colon x_{\bar{\sigma}} \in D)$$
$$= \overline{E}_\xi\left(\int_{\bar{\sigma}}^{\sigma} e^{-\gamma t}\chi_D\, dt\right)$$
$$= \overline{E}_\xi\left(\int_0^{\sigma} e^{-\gamma t}\chi_D\, dt\right)$$
$$= \overline{E}_\xi\left(\int_0^{\sigma} e^{-\gamma t}(m + Q)\hat{H}_\gamma 1\, d\Phi\right)$$

Since $\hat{H}_\gamma 1 = 1$, we see that

(53) $$E\left(\int_0^{\bar{\sigma}} e^{-\gamma t}Q(\xi_t, D)\, d\Phi\right) = E\left(\int_0^{\bar{\sigma}} e^{-\gamma t}(m + Q)\, d\Phi\right).$$

Therefore, we have proved (50).

Let

(54) $$\Psi = \ell \cdot \Phi + \chi_D \cdot T,$$

then

(55) $$T = \frac{1 + \ell}{2}\overline{\Phi} + \chi_D \cdot T = \Phi + \Psi.$$

[7.21] Let $\rho = \sup\{t \colon \Psi(t) = 0\}$; then $\rho = 0$, a.e. ($\overline{\mathbf{M}}$).

PROOF. If $t > \bar{\sigma} = \sigma_D$, $\Psi(t) \geq \int_{\bar{\sigma}}^t \chi_D\, dt > 0$ and $\rho \leq \bar{\sigma}$, a.e. Therefore, $\overline{P}_x(\rho = 0) = 1$ for $x \in D$. For $\xi \in V$, $\overline{E}_\xi(\int_0^\rho \ell\, d\Phi) \to \overline{E}_\xi(\Psi(\rho)) = 0$, and by (50)

(56) $$\overline{E}_\xi\left(\int_0^\rho e^{-\gamma t}m\, d\Phi\right) \leq \overline{E}_\xi\left(\int_0^\rho e^{-\gamma t}m\, d\Phi\right) = 0.$$

Thus, $\overline{P}_\xi(\int_0^t (1/1 - \chi_E)\, d\Phi < \infty$ for any $t < \rho) = 1$. On the other hand, by (49),

(57) $$\overline{E}_\xi\left(\int_0^\rho e^{-\gamma t}Q\left(\frac{1}{g_\gamma}\right) d\Phi\right) \leq \overline{E}_\xi(e^{-\gamma\sigma_D} \colon x_{\sigma_D} \in D) \leq 1,$$

and $\overline{P}_\xi(\int_0^t Q(1/g_\gamma)\, d\Phi < \infty$ for any $t < \rho) = 1$. Therefore,

(58) $$\overline{P}_\xi\left(\int_0^t u\, d\Phi < \infty \text{ for any } t < \rho\right) = 1$$

and, comparing this equality with [7.19], we have $\overline{P}_\xi(\rho = 0) = 1$.

[7.22] The function $\Psi(t)$ is strictly increasing for $t \in [0, \zeta)$ a.e. (The proof of [7.22] was given by K. Sato.)

PROOF. By [7.21], we have

$$\overline{P}_x(t < \zeta, \rho(w_t^+) = 0) = \overline{E}_x(\chi(t < \zeta)\overline{P}_{x_t}(\rho = 0) = P_x, \qquad (t < \zeta) \tag{59}$$

for every fixed t. Hence

$$P_x(\rho(w_t^+) = 0 \text{ for all rational } t < \zeta) = 1. \tag{60}$$

Relation [7.22] is proved.

Therefore, by [9] there exists a right-continuous Markov process $\mathbf{M}$ on S such that, for each $\mathbf{A}$ belonging to the Borel field generated by cylinder sets,

$$P_x(\mathbf{A}) = \overline{P}_x(\tilde{w} \in \mathbf{A}), \tag{61}$$

where the mapping $w \to \tilde{w}$ is defined by $x_s(\tilde{w}) = x_{\mu(s)}(w)$ and $\mu(s) = \sup\{t\colon \Psi(t) \leq s\}$. Moreover, the Green kernel G_α of $\mathbf{M}$ is given by

$$G_\alpha f(x) = \overline{E}_x\left(\int_0^\infty e^{-\alpha\Psi(t)} f(x_t)\, d\Psi(t)\right). \tag{62}$$

Noting $\overline{E}_x(\int_\sigma^\infty e^{-\alpha t} f\chi_D\, dt) = \overline{E}_x(\int_0^\infty e^{-\alpha t}(m + Q)\hat{H}_\alpha f\, d\Phi)$ and $\Psi = T - \Phi$, by [5.2], we have

$$\begin{aligned} \overline{E}_x\left(\int_\sigma^\infty e^{-\alpha\Psi(t)} f\, d\Psi\right) &= \overline{E}_x\left(\int_\sigma^\infty e^{-\alpha\Psi(t)} f\ell\, d\Phi\right) + \overline{E}_x\left(\int_\sigma^\infty e^{-\alpha\Psi(t)} f\chi_D\, dt\right) \\ &= \overline{E}_x\left(\int_0^\infty e^{-\alpha\Psi}(\ell + (m + Q)\hat{H}_\alpha) f\, d\Phi\right). \end{aligned} \tag{63}$$

Since $\Psi(t) = t$ for $t \leq \sigma \wedge \zeta$, we have

$$G_\alpha f = G_\alpha^0 f + H_\alpha K_\alpha(\ell + (m + Q)\hat{H}_\alpha) f \tag{64}$$

where $K_\alpha f(\xi) = \overline{E}_\xi\left(\int_0^\infty e^{-\alpha\Psi(t)} f(x_t)\, d\Phi\right)$.

Letting

$$K_\alpha^\lambda f(\xi) = E_\xi\left(\int_0^\infty e^{-\alpha\Psi(t) - \lambda\Phi(t)} f(x_t)\, d\Phi\right), \tag{65}$$

by [5.1], we can easily see that $\{K_\alpha^\lambda\}$ is an (ℓ, m, Q)-system and $K^\lambda = \overline{K}^\lambda$ is the Green kernel of $\hat{\mathbf{M}}$. Therefore by [5.10] and ($\hat{\mathbf{M}}$.C),

[7.23] $G_\alpha f \in C(S)$ if $f \in C(S)$.

Combining [7.23] and the right continuity of $\mathbf{M}$, we see that $\mathbf{M}$ is a Hunt process. By (62), for a measure such that $f \cdot \Psi \approx 0$ is equivalent to $f = 0$ a.e., ($f \in B^+(S)$), we also see that $\mathbf{M}$ satisfies ($\mathbf{M}$.2). (Such a measure exists by [4].) Finally, by the definition of Ψ, the stopped process of $\mathbf{M}$ coincides with that of $\hat{\mathbf{M}}$. Hence, we have that

[7.24] $\mathbf{M}$ satisfies the conditions ($\mathbf{M}$.1) $\sim$ ($\mathbf{M}$.3).

Let $(\hat{\mathbf{M}}^*, \ell^*, m^*, Q^*)$ be the boundary system of $\mathbf{M}$, and let $\Phi^* = \tilde{T}_\gamma$ (in $\mathbf{M}$), that is, $E_x(\int_\sigma^\infty e^{-\gamma t}\, dt) = E_x(\int_0^\infty e^{-\gamma t}\, d\Phi)$. Let $K_\alpha^{*\lambda} f(\xi) = E_\xi(\int_0^\infty e^{-\lambda\Psi(t) - \alpha t} f(x_t)\, d\Phi)$; then $K^{*\lambda}$ is the Green kernel of $\hat{\mathbf{M}}^*$ and

$$G_\alpha f = G_\alpha^0 f + H_\alpha K_\alpha^*(\ell^* + (m^* + Q^*)\hat{H}_\alpha) f. \tag{66}$$

Comparing this with (64), we have

$$K_\alpha(\ell + (m + Q)\hat{H}_\alpha)f = K^*_\alpha(\ell^* + (m^* + Q^*)\hat{H}_\alpha)f. \tag{67}$$

Letting $f = (I + (\alpha - \gamma)G^0_\gamma)1$ (hence, $\hat{H}_\alpha f = \hat{H}_\gamma 1 = 1$) and noting ($\tilde{\mathbf{M}}$.3), we obtain $K_\alpha 1 = K^*_\alpha 1$; that is,

$$\tilde{E}_x\Big(\int_0^\infty e^{-\alpha\Psi(t)}\,d\Phi\Big) = E_x\Big(\int_0^\infty e^{-\alpha t}\,d\Phi^*\Big). \tag{68}$$

[7.25] For any $f \in B(V)$, $K_\alpha f = K^*_\alpha f$.

PROOF. It is sufficient to prove the result for $f \in C(S)$. Let

$$\rho = \inf t\colon |f(x_t) - f(x_0)| \geq \epsilon, \quad \rho_0 = 0, \quad \rho_{n+1} = \rho_n + \rho(w^+_{\rho_n}). \tag{69}$$

Since $\Psi(t)$ is strictly increasing, $\rho(\tilde{w}) = \Psi(\rho(w), w)$ and $\rho(w) = \mu(\rho(\tilde{w}), w)$; hence, $E_x(g(x_\rho)e^{-\alpha\rho}) = \tilde{E}_x(g(x_\rho)e^{-\alpha\Psi(t)})$. Since $\Psi(\rho_{n+1}) = \Psi(\rho_n) + \Psi(\rho(w^+_{\rho_n}), w^+_{\rho_n})$, we can prove by induction that

$$E_x(g(x_{\rho_n})e^{-\alpha\rho_n}) = \tilde{E}_x(g(x_{\rho_n})e^{-\alpha\Psi(\rho_n)}), \qquad g \in B(S). \tag{70}$$

Therefore, writing $\varphi(x)$ for both sides of (68), we have

$$\begin{aligned} (71)\quad E_\xi\Big(\sum_{n=0}^\infty e^{-\alpha\rho_n}f(x_{\rho_n})\int_{\rho_n}^{\rho_{n+1}} e^{-\alpha t}\,d\Phi^*\Big) \\ &= E_\xi\big(\textstyle\sum e^{-\alpha\rho_n}f(x_{\rho_n})(\varphi(x_{\rho_n}) - E_{x_{\rho_n}}(\varphi(x_\rho)e^{-\alpha\rho}))\big) \\ &= \tilde{E}_\xi\big(\textstyle\sum e^{-\alpha\Psi(\rho_n)}f(x_{\rho_n})(\varphi(x_{\rho_n}) - \tilde{E}_{x_{\rho_n}}(\varphi(x_\rho)e^{-\alpha\Psi(\rho)}))\big) \\ &= \tilde{E}_\xi\Big(\textstyle\sum e^{-\alpha\Psi(\rho_n)}f(x_{\rho_n})\displaystyle\int_{\rho_n}^{\rho_{n+1}} e^{-\alpha\Psi(t)}\,d\Phi\Big). \end{aligned}$$

Write I_1 for the first member of the above equality and I_2 for the last member. Noting

$$\Big|E_\xi\Big(\int_0^\infty e^{-\alpha t}f(x_t)\,d\Phi^*\Big) - I_1\Big| \leq \epsilon K^*_\alpha 1 \tag{72}$$

and

$$\Big|\tilde{E}_\xi\Big(\int_0^\infty e^{-\alpha\Psi(t)}f(x_t)\,d\Phi\Big) - I_2\Big| \leq \epsilon K_\alpha 1, \tag{73}$$

we have $K_\alpha f = K^*_\alpha f$.

Now, by [5.13], we have $\ell = \ell^*$, $m = m^*$, and $Qf = Q^*f$, ($f \in B(D)$) up to functions of $N(K) = N(K^*)$. And by [5.11] and [5.13], $K^\lambda = K^{*\lambda}$; that is, $\tilde{\mathbf{M}}^*$ coincides with the given process $\tilde{\mathbf{M}}$.

THEOREM 16. *Let* ($\tilde{\mathbf{M}}, \ell, m, Q$) *be a system satisfying* ($\tilde{\mathbf{M}}$.1) $\sim$ ($\tilde{\mathbf{M}}$.7) *and* ($\tilde{\mathbf{M}}$.C). *Then, there exists a (unique) process* $\mathbf{M}$ *on* S *which satisfies* ($\mathbf{M}$.1) $\sim$ ($\mathbf{M}$.3) *and* ($\mathbf{M}$.C) *and whose boundary system is* ($\tilde{\mathbf{M}}, \ell, m, Q$).

Noting proposition 2 and proposition 14, the following theorems are consequences of theorem 16.

THEOREM 17. *Let* $\tilde{\mathbf{M}}$ *be a process on* V. *Then* $\tilde{\mathbf{M}}$ *is a U-process of a certain process* $\mathbf{M}$ *on* S *which satisfies* ($\mathbf{M}$.1) $\sim$ ($\mathbf{M}$.5) *and* ($\mathbf{M}$.C) *if and only if* $\tilde{\mathbf{M}}$ *satisfies* ($\tilde{\mathbf{M}}$.1), ($\tilde{\mathbf{M}}$.2), *and*

($\mathfrak{M}$.4) $\tilde{G}_\lambda \hat{H}_\alpha \chi_V = 0$ for some $\lambda > 0$ and $\alpha > 0$,
($\mathfrak{M}$.6) $(\tilde{P}, \tilde{L}) \gg (\Theta, T)$,
($\mathfrak{M}$.7) $\tilde{A}_\infty \gg \theta \cdot T$,
($\mathfrak{M}$.C) $\tilde{G}_\lambda f \in C(V)$ if $f \in C(V)$.

The process **M** *is uniquely determined by* $\mathfrak{M}$.

THEOREM 18. *Suppose that* $\mathbf{M}_0$ *satisfies* ($\mathbf{M}_0$.1) $\sim$ ($\mathbf{M}_0$.5) *and*

($\mathbf{M}_0$.6*) the set $\{H_\alpha f : f \in C(S) \text{ and } f = 0 \text{ on } V\}$ *is dense in* $C(S)$.

Then, $(\mathfrak{M}, \ell, m, 0)$ *is a boundary system of a certain process* **M** *which satisfies* (**M**.1) $\sim$ (**M**.3), (**M**.5) *and* (**M**.C) *if and only if* $(\mathfrak{M}, \ell, m, 0)$ *satisfies* ($\mathfrak{M}$.1), ($\mathfrak{M}$.2), *and*

($\mathfrak{M}$.3) $\ell + m = 1$, a.e. ν,
($\mathfrak{M}$.4) $m\hat{H}_\alpha \chi_V = 0$, a.e. ν,
($\mathfrak{M}$.6) $(\tilde{P}, \tilde{L}) \gg (m\Theta, T)$,
($\mathfrak{M}$.7) $\tilde{A}_\infty \gg m\theta \cdot T$,
($\mathfrak{M}$.C) $\tilde{G}_\lambda \ell f, \tilde{G}_\lambda m f \in C(V)$ if $f \in C(V)$.

(In the above theorems, $\tilde{G}_\lambda$ is the Green kernel of $\mathfrak{M}$, ν is a measure appearing in ($\mathfrak{M}$.2), and $(\tilde{P}, \tilde{L})$ and $\tilde{A}_\infty$ are the Lévy system and the killing functionals of $\mathfrak{M}$ respectively.) Combining these theorems with propositions 2, 8, and 9, we can obtain many alternatives, of which we state the following one alone.

THEOREM 19. *Let* **M** *satisfy* (**M**.1) $\sim$ (**M**.5).

(i) *The path of* **M** *is continuous for* $t \in [0, \zeta)$ *if and only if the path of* $\mathbf{M}_0$ *is continuous and* $(\tilde{P}, \tilde{L}) \approx (\Theta, T)$;

(ii) **M** *is conservative if and only if* $\mathbf{M}_0$ *is conservative and* $\tilde{A}_\infty \approx \theta \cdot T$.

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APPENDIX I. *Proof of* [7.4]

Let **M** be a process satisfying (**M**.1) and (**M**.2). Let P and Q be (not necessarily bounded) kernels on $S \times S$, and let L and M be continuous additive functionals. Suppose that

(*) there exists an increasing sequence of sets $\{F_n\}$, $(F_n \subset S \times S)$ such that $\chi_{F_n} \uparrow 1$, $(n \to \infty)$ and $E_x(\int_0^\infty e^{-\alpha t} P\chi_{F_n}\, dL)$ are bounded in n for a fixed $\alpha > 0$.

(The condition (*) is satisfied if (P, L) is the Lévy system of **M** (see [11] and [2.3]).)

[I.1] If $Pf \cdot L \gg Qf \cdot M$ for all $f \in B^+(S \times S)$, then there exists $k \in B^+(S \times S)$ such that $P(kf) \cdot L \approx Qf \cdot M$ for all $f \in B^+(S \times S)$.

Before proving [I.1], we note the following.

[I.2] Let m be a measure appearing in (**M**.2), and let $\tilde{L}$ and $\tilde{M}$ be any continuous additive functionals. If

$$(1) \qquad E_m\Big(\int_0^\infty e^{-\alpha t} f\, d\tilde{L}\Big) = E_m\Big(\int_0^\infty e^{-\alpha t} f\, d\tilde{M}\Big) < \infty$$

for all $f \in B^+(S)$ and $x \in S$, then $\tilde{L} \approx \tilde{M}$. Here $\alpha \geq 0$ is fixed.

PROOF. Let $\tilde{N} = \tilde{L} + \tilde{M}$; then $\tilde{L} \approx k \cdot \tilde{N}$ and $\tilde{M} \approx \ell \cdot \tilde{C}$ for some k and $\ell \in B^+(S)$ (see [4]). Let

$$u(x) = E_x\Big(\int_0^\infty e^{-\alpha t}\chi(k \geq \ell)(d\tilde{L} - d\tilde{M})\Big) = E_x\Big(\int_0^\infty e^{-\alpha t}\chi(k \geq \ell)(k - \ell)\, d\tilde{N}\Big). \tag{2}$$

Then $u(x)$ is an α-excessive function and, by assumption, $u = 0$, a.e. m. Thus, $u(x) = 0$ for all $x \in S$. Similarly, we can see that

$$E_x\Big(\int_0^\infty e^{-\alpha t}\chi(k < \ell)(d\tilde{L} - d\tilde{M})\Big) = 0 \tag{3}$$

for all x, and that

$$E_x\Big(\int_0^\infty e^{-\alpha t}\, d\tilde{L}\Big) = E_x\Big(\int_0^\infty e^{-\alpha t}\, d\tilde{M}\Big) \tag{4}$$

for all x, that is, $\tilde{L} \approx \tilde{M}$.

PROOF of [I.1]. For any $F \subset S \times S$, let

$$\mu(F) = E_m\Big(\int_0^\infty e^{-\alpha t}P\chi_F\, dL\Big), \quad \nu(F) = E_m\Big(\int_0^\infty e^{-\alpha t}Q\chi_F\, dL\Big). \tag{5}$$

Then $\mu(F) \geq \nu(F)$, and by (*), μ and ν are σ-finite measures on $S \times S$. Therefore there exists a function $k \in B(S \times S)$, $0 \leq k \leq 1$ such that $\nu(F) = \int \chi_F k\, d\mu$. Hence,

$$E_m\Big(\int_0 e^{-\alpha t}Qf\, dM\Big) = \int f\, d\nu = \int kf\, d\mu = E_m\Big(\int_0^\infty e^{-\alpha t}Q(kf)\, dM\Big). \tag{6}$$

Therefore, for any $g \in B^+(S)$ and $h \in B^+(S \times S)$ such that $E_x(\int_0^\infty e^{-\alpha t}Ph\, dL)$ s bounded in x, letting $f(x, y) = g(x)h(x, y)$, one obtains

$$E_m\Big(\int_0^\infty e^{-\alpha t}gP(kh)\, dL\Big) = E_m\Big(\int_0^\infty e^{-\alpha t}gQh\, dM\Big). \tag{7}$$

By (*) and [I.2], we have [I.1].

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APPENDIX II. *Dependency of the boundary system on* γ

Let $\mathbf{M}$ be a process satisfying ($\mathbf{M}$.1) $\sim$ ($\mathbf{M}$.3), and let γ and γ^* be two positive constants. Let $(\tilde{\mathbf{M}}, \ell, m, Q)$ and $(\tilde{\mathbf{M}}^*, \ell^*, m^*, Q^*)$ be boundary systems of $\mathbf{M}$ corresponding to γ and γ^*, respectively, and $\Phi \approx T_\gamma$ and $\Phi^* \approx \tilde{T}_{\gamma^*}$. Let

$$c(x) = c(\gamma, \gamma^*; x) = \frac{g_\gamma(x)}{g_{\gamma^*}(x)}. \tag{1}$$

Then c can be considered as a function in $C(S)$ and

(2) $$\hat{H}_\alpha = c\hat{H}^*_\alpha.$$

By theorem 3,

(3) $$\begin{aligned} G_\alpha f &= G^0_\alpha f + H_\alpha K_\alpha(\ell + (m + Q)\hat{H}_\alpha) \\ &= G^0_\alpha f + H_\alpha K^*_\alpha(\ell^* + (m^* + Q^*)\hat{H}^*_\alpha). \end{aligned}$$

Letting $\alpha = \gamma^*$ and $f = 1$ in (3), we have

(4) $$\Phi^* \approx (\ell + (m + Q)c)\cdot\Phi$$

and

(5) $$K_\alpha g = K^*_\alpha\left(\frac{1}{\ell + (m + Q)c}\, g\right), \qquad (g \in B(V)).$$

Applying [5.14], we have

(6) $$\ell^* = \frac{\ell}{\ell + (m + Q)c}, \quad m^* = \frac{mc}{\ell + (m + Q)c},$$
$$Q^*h = \frac{Q(ch)}{\ell + (m + Q)c}, \quad (h \in B(D)),$$

except for functions in $N(K) = N(K^*)$.

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APPENDIX III. *Lateral conditions*

Let $\mathbf{M}$ be a process on S which satisfies ($\mathbf{M}$.1) $\sim$ ($\mathbf{M}$.3), and let $(\tilde{\mathbf{M}}, \ell, m, Q)$ be the boundary system of $\mathbf{M}$. In this appendix, we shall assume that $\mathbf{M}$ satisfies ($\mathbf{M}$.C) and $(\tilde{\mathbf{M}}, \ell, m, Q)$ satisfies

$$(\tilde{\mathbf{M}}.\mathrm{C}^*) \quad \begin{aligned} &\tilde{G}_\lambda g \in C(V) \quad \text{if} \quad g \in C(V), \\ &C_\alpha f \in C(V) \quad \text{if} \quad f \in C(S), \end{aligned}$$

where $C_\alpha = \ell + (m + nQ)\hat{H}_\alpha$ and $U_\alpha = \alpha C_\alpha H$ (see section 6.4).

Let A (or $\tilde{A}$) be an infinitesimal generator of the strongly continuous semi-group of $\mathbf{M}$ (or $\tilde{\mathbf{M}}$) on $C(S)$ (or $C(V)$), and $D(A)$ (or $D(\tilde{A})$) be its domain. Then by [5.9] and ($\mathbf{M}$.C*), we see that

[III.1] $D(\tilde{A}) = \{v = K^\lambda_\alpha g : g \in C(V)\}$
and $\tilde{A}v = (\lambda + U_\alpha)v - g$ if $v = K^\lambda_\alpha g$.

Let

(1) $$D_0 = \{u\colon u - H_\alpha u = G^0_\alpha f, f \in C(S)\},$$

and for $u \in D_0$, such that $u - H_\alpha u = G^0_\alpha f$, let

(2) $$\begin{aligned} A^*_0 u &= \alpha u - f, \\ Nu &= \hat{H}_\alpha(\alpha H u - f). \end{aligned}$$

By (2), (3) and (5) of section 2, we have that

[III.2] D_0, A^*_0, and N are independent of α.

Now, by theorem 3, we can easily prove

[3] (i) $A \leq A_0^*$; (ii) let $u \in D_0$. Then $u \in D(A)$ if and only if $u_V \in D(\tilde{A})$ and

(**) $\tilde{A}u_V = \ell A_0^* u + (m + Q)Nu$,

where u_V is the restriction of u on V.

By Appendix II, we can see that the equation (**) is independent of γ.

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