ACCESSIBLE TERMINAL TIMES

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1. Introduction

Let $X = (\Omega, \mathfrak{M}, P^x, X_t, \theta_t)$ be a Hunt process having a locally compact space E with a countable base as state space. We refer the reader to the expository paper ([4] or [1], pp. 133–134), for all concepts and notations which are not explicitly mentioned in the present paper.

A stopping time T for the process X is called *accessible* if for each initial measure μ on E there is a nondecreasing sequence $\{T_n\}$ of stopping times such that P^μ almost surely, $T_n \to T$ and $T_n < T$ for all n on $\{T > 0\}$. Meyer [7] has proved the remarkable result that a stopping time T is accessible if and only if the path $t \to X_t(\omega)$ is continuous at $T(\omega)$ almost surely on $\{T < \infty\}$. We will say that a stopping time T is *thin* if $P^x(T > 0) = 1$ for all x in E. As usual, an analytic subset A of E is thin if $P^x(T_A > 0) = 1$ for all x in E, where $T_A = \inf \{t > 0: X_t \in A\}$ is the hitting time of A. These definitions are consistent since clearly A is thin if and only if T_A is thin. Finally a stopping time T is called a *terminal time* if for each t

$$T = t + T \circ \theta_t, \qquad \text{almost surely on} \quad \{T > t\}. \tag{1.1}$$

If A is an analytic subset of E, then T_A is a terminal time and the phrase "almost surely" may even be dropped from statement (1.1).

Let us now assume that X satisfies Hunt's hypothesis (F). (See [5], [6], or [1], pp. 133–134.) It then follows from proposition 18.5 of [5] that T_A is an accessible terminal time whenever A is a thin analytic subset of E. Moreover, it is clear that $T_A = \infty$ on $\{T_A \geq \zeta\}$ if $A \subset E$. The main result of this paper is the following converse of the above statement.

THEOREM 1. *Assume X satisfies hypothesis (F). If T is a thin accessible terminal time with the property that $P^x[\zeta \leq T < \infty] = 0$ for all x, then there exists a thin Borel set $B \subset E$ such that $T = T_B$ almost surely.*

The proof of theorem 1 is given in section 2; then in section 3 we give some applications of theorem 1 to the structure of natural additive functionals.

Consider the following process: the state space $E = L \cup L_1 \cup L_2$ is the following subset of the Euclidean plane, $L = \{(x, y): x \leq 0, y = 0\}$ is the nonpositive x-axis, L_1 is the segment joining the points $(0, 1)$ and $(1, 0)$, whereas L_2 is the segment joining $(0, -1)$ and $(1, 0)$. The process consists of translation to the right at unit speed until $(0, 0)$ is reached. The point $(0, 0)$ is a holding point

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with parameter 1 from which the process jumps to $(0, 1)$ or $(0, -1)$ with probability $\frac{1}{2}$, respectively. The process then moves with unit speed along the appropriate segment L_1 or L_2 until it reaches $(1, 0)$ where it remains forever. Define T by $T(\omega) = \infty$ if the trajectory $t \to X_t(\omega)$ reaches $(1, 0)$ via the lower segment L_2 or if $X_0(\omega) = (1, 0)$; whereas, if the trajectory arrives at $(1, 0)$ via the upper segment L_1, let $T(\omega)$ be the time at which the process reaches $(1, 0)$. It is immediate that T is a thin accessible terminal time, and it is equally clear that if the initial measure μ attaches positive mass to L, then there is *no* thin set B such that $T = T_B$ almost surely P^μ, even if we allow the set B to depend on μ. This example is, of course, artificial, but it does show that theorem 1 is *not* valid for Hunt processes in general. One can construct examples which are less artificial.

2. Proof of theorem 1

In the rest of this paper we will assume that X satisfies Hunt's hypothesis (F). We will break up the proof of theorem 1 into several lemmas. In this section (for typographical convenience) we will write U and P_B, rather than U^1 and P_B^1, for the potential kernel and hitting distributions obtained by taking the auxiliary parameter λ to be 1; that is, for any bounded Borel measurable f,

$$\begin{aligned} \int U(x, y)f(y)\,dy &= E^x \int_0^\infty e^{-t}f(X_t)\,dt, \\ P_Bf(x) &= E^x\{e^{-T_B}f(X_{T_B})\}. \end{aligned} \tag{2.1}$$

Here, as in Hunt, $\xi(dy) = dy$ denotes the basic measure on E. We will also need the fact that if T satisfies the hypotheses of theorem 1 and R is any stopping time, then $R + T \circ \theta_R = T$ almost surely on $\{R < T\}$. This follows easily from the strong Markov property for multiplicative functionals [6].

From now on T will always satisfy the hypotheses of theorem 1. Let $\phi(x) = E^x(e^{-T})$. Since T is a thin terminal time it is easy to see that ϕ is 1-excessive and that ϕ is strictly less than 1. According to theorem 18.7 of [5], we may write $\phi = U\mu + \psi$ where μ is a measure on E and ψ is a 1-excessive function with the property that $P_F\psi = \psi$ whenever F is the complement of a compact subset of E.

The following notation will be used in the remainder of this section. Let $K_n = \{\phi \geq 1 - 1/n\}$. Each K_n is a finely closed Borel set, and the K_n are decreasing with empty intersection. Let $T_n = T_{K_n}$ be the hitting time of K_n.

LEMMA 1. *For each n, $P_{K_n}\phi = \phi$, and almost surely $T_n \uparrow T$ with $T_n < T$ on $\{T < \infty\}$.*

PROOF. Fix an x and let $\{R_n\}$ be an increasing sequence of stopping times such that $P^x(R_n \to T, R_n < T \text{ for all } n) = 1$. Such a sequence exists, since T is a thin accessible time. Now

$$E^x\{e^{-R_n}\phi(X_{R_n})\} = E^x\{\exp(-R_n - T \circ \theta_{R_n}\} = E^x\{e^{-T}\}. \tag{2.2}$$

But $R_n \uparrow T$, and hence $\lim_n \phi(X_{R_n})$, which exists on $\{T < \infty\}$ since ϕ is 1-excessive, must equal one on $\{T < \infty\}$, all of these statements holding P^x almost surely.

Recalling the definition of K_n, it is clear that $T_n \leq T$ almost surely P^x. Moreover, on $\{T_n = T < \infty\} = \{T_n = T < \zeta\}$, one has for each $m > n$ that $\phi(X_T) = \phi(X_{T_n}) \geq 1 - (1/m)$, and this contradicts the fact that ϕ is strictly less than one. Consequently, $T_n < T$ almost surely P^x on $\{T < \infty\}$. In particular this yields

$$(2.3) \qquad P_{K_n}\phi(x) = E^x\{\exp(-T_n - T \circ \theta_{T_n})\} = E^x(e^{-T}) = \phi(x).$$

Finally, the relationship

$$(2.4) \qquad E^x\{e^{-(T-T_n)}; T_n < \infty\} = E^x\{\phi(X_{T_n}); T_n < \infty\} \geq \left(1 - \frac{1}{n}\right) P^x(T_n < \infty)$$

implies that $T_n \uparrow T$ almost surely P^x. Since x is arbitrary, this completes the proof of lemma 1.

Define $L_n = \{x: x \text{ is left regular for } K_n\}$ and let B be the intersection of the L_n. Each L_n is a countable intersection of open sets because $L_n = \{x: \hat{E}^x(e^{-\hat{T}_n}) = 1\}$, and excessive functions are lower semicontinuous. It will turn out that B is the set we are looking for; that is, $T = T_B$ almost surely.

LEMMA 2. *The measure* μ *is carried by* B.

PROOF. Since $P_{K_n}u \leq u$ for any 1-excessive function u and $P_{K_n}\phi = \phi$, it follows that $P_{K_n}U\mu = U\mu$ and $P_{K_n}\psi = \psi$. The equality $U\mu = P_{K_n}U\mu = U\hat{P}_{K_n}\mu$ and the uniqueness theorem for potentials of measures imply that $\mu = \hat{P}_{K_n}\mu$. Therefore μ is carried by $K_n \cup L_n$ for each n. But the intersection of the K_n is empty, and hence lemma 2 is established.

According to theorem 18.8 of [5], if $\{G_n\}$ is an increasing sequence of open subsets of E whose union is E, then letting F_n denote the complement of G_n we have $P_{F_n}U\mu \to 0$ as $n \to \infty$. It is not difficult to conclude from this that $U\mu$ is a potential of class (D) (see [4] or [6] for the definition), and consequently, according to Meyer's result [6], there exists a unique natural additive functional A of X such that $U\mu(x) = E^x \int_0^\infty e^{-t}\, dA(t)$ for all x.

LEMMA 3. *Let* $R = \inf\{t: A(t) > 0\}$; *then* $R = T$ *almost surely.*

PROOF. We have

$$(2.5) \qquad U\mu(x) = P_{K_n}U\mu(x) = E^x \int_{(T_n,\infty)} e^{-t}\, dA(t),$$

and hence $A(T_n) = 0$ almost surely. However, $T_n \uparrow T$, and this implies that $T \leq R$ almost surely. We turn now to the opposite inequality.

As a first step, we will show that $\psi(X_{T_n}) \to \psi(X_T)$ almost surely on $\{T < \infty\}$. By assumption, $T = \infty$ almost surely on $\{\zeta \leq T\}$, hence it suffices to prove the convergence on $\{T < \zeta\}$. Let $\{D_n\}$ be an increasing sequence of compact subsets of E whose union is E. If S_n is the hitting time of the complement of D_n, then $S_n \uparrow \zeta$ as $n \to \infty$. For a fixed k define $Q_n = \min(T_n, S_k)$ and $Q = \min(T, S_k)$. Clearly $Q_n \uparrow Q$, and it suffices to show that $\psi(X_{Q_n}) \to \psi(X_Q)$ on $\{Q < \infty\}$. If x is fixed, $\{e^{-Q_n}\psi(X_{Q_n}), P^x\}$ is a bounded nonnegative supermartingale, therefore $H = \lim_n e^{-Q_n}\psi(X_{Q_n})$ exists almost surely P^x.

Recall the basic fact that $P_{S_k}\psi = \psi$. But $Q_n \leq S_k$, and consequently

$$\psi \geq P_{Q_n}\psi \geq P_{S_k}\psi = \psi, \tag{2.6}$$

with a similar relationship for $P_Q\psi$. Hence, $\psi = P_{Q_n}\psi = P_Q\psi$ for all n, and this yields

$$E^x(H) = \lim_n E^x[e^{-Q_n}\psi(X_{Q_n})] = E^x[e^{-Q}\psi(X_Q)]. \tag{2.7}$$

On the other hand, $Q \geq Q_n$; hence, if Λ is in $\mathfrak{F}_{Q_m}$, then for all $n \geq m$

$$E^x\{e^{-Q_n}\psi(X_{Q_n}); \Lambda\} \geq E^x\{e^{-Q}\psi(X_Q); \Lambda\}, \tag{2.8}$$

and letting $n \rightarrow \infty$

$$E^x\{H; \Lambda\} \geq E^x\{e^{-Q}\psi(X_Q); \Lambda\}. \tag{2.9}$$

Using the characterization of $\mathfrak{F}_Q$ given in [2], it is immediate that (2.9) holds for all Λ in $\mathfrak{F}_Q$, and H and $e^{-Q}\psi(X_Q)$ being $\mathfrak{F}_Q$ measurable, it follows that $H \geq e^{-Q}\psi(X_Q)$ almost surely P^x. In view of (2.7) and the definition of H, this implies that $\psi(X_{Q_n}) \rightarrow \psi(X_Q)$ almost surely on $\{Q < \infty\}$. Thus we have shown that $\psi(X_{T_n}) \rightarrow \psi(X_T)$ almost surely on $\{T < \infty\}$.

Let Λ be in $\mathfrak{F}_{T_m}$ and let $n > m$; then we have

$$E^x\{e^{-T_n}\phi(X_{T_n}); \Lambda\} = E^x\{e^{-T_n}U\mu(X_{T_n}); \Lambda\} + E^x\{e^{-T_n}\psi(X_{T_n}); \Lambda\}, \tag{2.10}$$

with a similar expression in which T replaces T_n. Since $\psi(X_{T_n}) \rightarrow \psi(X_T)$ and $\phi(X_{T_n}) \rightarrow 1$ almost surely on $\{T < \infty\}$, by letting $n \rightarrow \infty$ and then by subtracting the corresponding expression involving T, one obtains

$$E^x\{e^{-T}[1 - \phi(X_T)]; \Lambda\} = E^x\{e^{-T}A(T); \Lambda\}. \tag{2.11}$$

It now follows that this must hold for all Λ in $\mathfrak{F}_T$, and consequently,

$$A(T) = 1 - \phi(X_T) > 0 \tag{2.12}$$

almost surely on $\{T < \infty\}$. But this implies that $R \leq T$ almost surely, and so lemma 3 is established.

LEMMA 4. *The inequality $T \geq T_B$ holds almost surely.*

PROOF. By construction, $U\mu(x) = E^x \int_0^\infty e^{-t}\, dA(t)$, and hence a result of Meyer [6] (see also [1]) implies that

$$\int U(x, y)f(y)\mu(dy) = E^x \int_0^\infty e^{-t}f(X_t)\, dA_t \tag{2.13}$$

for all bounded Borel measurable f. Taking f to be the characteristic function of $E \setminus B$ and using lemmas 2 and 3, one finds

$$0 = E^x \int_{[T,\infty)} e^{-t}f(X_t)\, dA(t) \geq E^x\{e^{-T}f(X_T)A(T)\}. \tag{2.14}$$

But $A(T) > 0$ on $\{T < \infty\}$, according to (2.12), and so X_T is in B almost surely on $\{T < \zeta\} = \{T < \infty\}$. Since T is thin we may conclude $T_B \leq T$ almost surely, thus completing the proof of lemma 4.

Let $B_n = \{\phi < 1 - 1/n\}$. Each B_n is a finely open Borel set, and the B_n increase to E as $n \rightarrow \infty$. Of course, $B_n = E \setminus K_n$ for each n.

LEMMA 5. *Let ν be a positive measure on E such that $g = U\nu$ is bounded, then $P_T g \geq P_B g$.*

PROOF. Recalling the definition of L_n and B, one has $\hat{P}^x[\hat{T}_{L_n} < \hat{T}_{K_n}] = 0$ for all x and $\hat{T}_{L_n} \leq \hat{T}_B$. Consequently,

$$P_{K_n}P_{B_m}g = U\hat{P}_{K_n}\hat{P}_{B_m}\nu \geq U\hat{P}_B\hat{P}_{B_m}\nu = P_BP_{B_m}g. \tag{2.15}$$

Let $R_{n,m} = T_n + T_{B_m} \circ \theta_{T_n}$, then

$$P_{K_n}P_{B_m}g(x) = E^x\{\exp(-R_{n,m})g[X(R_{n,m})]\}. \tag{2.16}$$

We now claim that for m, fixed $R_{n,m}$ coincides with $T + T_{B_m} \circ \theta_T$ for sufficiently large n almost surely on $\{T < \infty\}$. To prove this we note that since B_m is finely open, we have

$$\begin{aligned} P^x[T_n + T_{B_m} \circ \theta_{T_n} < T + T_{B_m} \circ \theta_T; T < \infty] \\ \leq P^x[X_t \in B_m \quad \text{for some} \quad t \in [T_n, T); T < \infty]. \end{aligned} \tag{2.17}$$

However $T_n \to T$ and $\phi(X_t) \to 1$ as $t \uparrow T$ on $\{T < \infty\}$, and hence this last expression approaches zero as $n \to \infty$. Therefore, $R_{n,m} = T + T_{B_m} \circ \theta_T$ for sufficiently large n almost surely on $\{T < \infty\}$. It now follows from (2.16) that $P_{K_n}P_{B_m}g \to P_TP_{B_m}g$ as $n \to \infty$ for each fixed m, and hence $P_TP_{B_m}g \geq P_BP_{B_m}g$. But $P_{B_m}g$ increases to g as $m \to \infty$, and this establishes lemma 5.

We may now easily complete the proof of theorem 1. We have just shown that $E^x\{e^{-T}g(X_T)\}$ dominates $E^x\{e^{-T_B}g(X_{T_B})\}$ whenever g is a bounded potential $U\nu$. But the function identically equal to one on E is the limit of an increasing sequence of such potentials, hence $E^x(e^{-T}) \geq E^x(e^{-T_B})$. Combining this with lemma 4 completes the proof of theorem 1.

REMARK 1. Naturally the set B is finely closed since it is thin. In addition, the particular set B constructed above is cofinely closed in as much as each L_n is cofinely closed, that is, closed in the fine topology for the dual process $\hat{X}$.

3. Natural additive functionals

In this section U and P_B will have their usual meanings; that is, they are the potential kernel and the hitting distributions for $\lambda = 0$. Let $A = A(t)$ be a natural additive functional of X, and for simplicity, we assume that A has a finite potential, that is, $u(x) = E^x\{A(\infty)\} < \infty$. (The following results are valid with the obvious modifications if one only assumes that A has a finite λ-potential for some strictly positive λ.)

In [6], Meyer has shown that A can be decomposed into the sum of a continuous additive functional, C, and a purely discontinuous natural additive functional, D, in the following manner. For a given n let $T^{(n)}(\omega) = T(\omega)$ be the smallest value of t such that $|u[X_t(\omega)] - u[X_{t-}(\omega)]| > 1/n$ and the path $X(\cdot, \omega)$ is continuous at t. Here $u(X_{t-})$ denotes $\lim_{s \uparrow t} u(X_s)$ and *not* $u(\lim_{s \uparrow t} X_s)$. It is known that T satisfies the hypotheses of theorem 1 for each n (see [6] or [3]). Let $G_n(\omega)$ be the magnitude of the jump at T, that is

$$G_n(\omega) = u[X(T(\omega) -, \omega)] - u[X(T(\omega), \omega)] \geq 0. \tag{3.1}$$

It is clear that $G_n = 0$ on $\{T = \infty\}$ and $G_n > 1/n$ on $\{T < \infty\}$.

We now define the successive jumping times $T_k^{(n)} = T_k$ by $T_0 = 0$ and $T_{k+1} = T_k + T \circ \theta_{T_k}$ for $k \geq 0$. Next define the additive functional $D_n(t)$ by

$$D_n(t, \omega) = \sum G_n(\theta_{T_k}\omega), \tag{3.2}$$

the sum being taken over those k for which $T_k(\omega) \leq t$. It is easy to see that D_n is a natural additive functional for each n, and Meyer has shown that $\lim_n D_n(t)$ exists uniformly on $[0, \infty)$ almost surely and defines a natural additive functional $D(t)$. Finally the difference $u(x) - E^x\{D(\infty)\}$ is the potential of a continuous additive functional $C(t)$, and consequently $A(t) = C(t) + D(t)$. This decomposition is valid for general Hunt processes and so does not depend on hypothesis (F).

From now on we assume that (F) holds; then for each n there exists by theorem 1 a thin set B_n such that $T^{(n)} = T_{B_n}$ almost surely. Let B_d be the union of the B_n, so that B_d is semipolar, and let $B_c = E \setminus B_d$. Moreover, the $T^{(n)}$ are decreasing, and therefore we may assume the B_n are increasing.

THEOREM 2. *Let I_d and I_c be the indicator functions of B_d and B_c respectively; then*

$$D(t) = \int_0^t I_d(X_u)\, dA(u) \quad \text{and} \quad C(t) = \int_0^t I_c(X_u)\, dA(u), \tag{3.3}$$

where, as usual, the equality of additive functionals means equivalence.

PROOF. If R is an accessible terminal time, a standard argument, like the one used in the proof of lemma 3, shows that

$$u(X_{R-}) - u(X_R) = A(R) - A(R-) \tag{3.4}$$

almost surely on $\{R < \infty\}$. Therefore, it follows that $T^{(n)}$ is the first t such that $A(t) - A(t-) > 1/n$ and that G_n is the jump in A at that point. If $T^{(n)}$ is finite, $X(T^{(n)})$ is in B_n almost surely since B_n is thin. Consequently, we may write

$$D_n(t) = \int_0^t I_{B_n}(X_u)\, dA(u), \tag{3.5}$$

and letting $n \to \infty$ we obtain the assertion about $D(t)$. The one about $C(t)$ is then obvious since $I_c + I_d = 1$.

REMARK. If the only semipolar sets are polar, then it is an immediate consequence of theorem 2 that the only natural additive functionals (with finite potential) are continuous. One can find simple examples to show that this is *not* the case if we assume only Hunt's hypothesis (A).

We will close this section with one more comment on the structure of $D(t)$. Fix n and consider the approximating functional $D_n(t)$ and also the natural additive functional $J_n(t)$ defined to be the largest k such that $T_k^{(n)} \leq t$. Clearly

$$E^x \int_0^\infty e^{-t}\, dJ_n(t) \leq n E^x \int_0^\infty e^{-t}\, dD_n(t), \tag{3.6}$$

because the jumps of D_n all exceed $1/n$. It follows from theorem 18.7 of [5] that

$$U_A^1(x) = E^x \int_0^\infty e^{-t}\, dA(t) = U^1\mu(x), \tag{3.7}$$

where μ is an appropriate measure, and that $U_{J_n}^1 = U_{\nu_n}^1$, $U_{D_n}^1 = U_{\mu_n}^1$ where μ_n is the restriction of μ to B_n in view of (3.5). Now

$$\int U^1(x, y) f(y) \nu_n(dy) = E^x \int_0^\infty e^{-t} f(X_t)\, dJ_n(t), \tag{3.8}$$

with a similar statement for μ_n and D_n, whenever f is bounded and measurable. Thus if for some $f \geq 0$ the left side of (3.8) vanishes, then using the relationship between J_n and D_n, one sees that $\int U^1(x, y) f(y) \mu_n(dy)$ also vanishes. But this, together with the uniqueness of potentials, implies that μ_n is absolutely continuous with respect to ν_n.

Letting $d\mu_n = g_n\, d\nu_n$, we have

$$D_n(t) = \int_0^t g_n(X_u)\, dJ_n(u). \tag{3.9}$$

In particular, $G_n = g_n[X(T^{(n)})]$ almost surely, and g_n may be assumed to vanish off B_n. Moreover, if $m > n$, we may write

$$J_n(t) = \int_0^t I_{B_n}(X_u)\, dJ_m(t), \tag{3.10}$$

because $A(t)$ jumps by more than $1/n$ at time $T_k^{(m)}$ if and only if $X(T_k^{(m)})$ is in B_n. Hence ν_n is the restriction of ν_m to the set B_n. Thus we may assume that g_n and g_m agree on B_n; that is, we may define a nonnegative Borel measurable function g on E vanishing off B_d such that g_n is the restriction of g to B_n for each n.

Now we may write the approximating functionals D_n as

$$D_n(t) = \sum_k g[X(T_k^{(n)})] \tag{3.11}$$

where the sum is over those k satisfying $T_k^{(n)} \leq t$. Thus the purely discontinuous additive functional D is completely determined by the function g and the increasing sequence of thin sets $\{B_n\}$. Finally if one defines $A_1 = B_1$, $A_{n+1} = B_{n+1} - \bigcup_{j=1}^n A_j$ and the times $R_k^{(n)}$ to be the successive hitting times of A_n, that is, $R_0^{(n)} = 0$ and

$$R_{k+1}^{(n)} = R_k^{(n)} + T_{A_n} \circ \theta_{R_k^{(n)}}, \qquad \text{for} \quad k \geq 0, \tag{3.12}$$

then one has the following representation of D:

$$D(t) = \sum_n \sum_k g[X(R_k^{(n)})], \tag{3.13}$$

where again the inner summation is over those k satisfying $R_k^{(n)} \leq t$.

One can show by simple examples that the jump G_n will not always be expressible as a function of the position $X(T^{(n)})$ alone, if one merely assumes Hunt's hypothesis (A). In particular, the representations (3.11) and (3.13) are *not* valid for arbitrary Hunt processes.

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