ALMOST SURE BEHAVIOR OF SUMS OF INDEPENDENT RANDOM VARIABLES AND MARTINGALES

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1. Introduction and summary

Toss a fair coin independently n times and let S_n be the number of heads minus the number of tails. Khintchine [26] proved the law of the iterated logarithm

$$\limsup_n (2n \log\log n)^{-1/2} S_n = 1, \qquad \text{a.s.} \tag{1}$$

Thus if the supremum of an empty set of natural numbers is understood to be 0, then

$$\hat{T}: = \sup \{n: (2n \log\log n)^{-1/2} S_n \geq c\} \tag{2}$$

is a random variable for $c > 1$ representing the speed of the upper half of the law of the iterated logarithm.

In the first part of this paper we will be mainly concerned with the distribution of $\hat{T}$ and of similar random variables. Our research has been initiated by a remark of Professor Alfred Rényi to the effect that $E\hat{T} = \infty$. Here is a simple proof:

$$\begin{aligned} E\hat{T} &\geq E \text{ (number of times } n \text{ such that } (2n \log\log n)^{-1/2} S_n \geq c) \\ &= \sum_n \Pr \{(2n \log\log n)^{-1/2} S_n \geq c\} = \infty \end{aligned} \tag{3}$$

by normal approximation.

Unfortunately, this argument seems to have a very limited scope. Replace, for instance, $(2n \log\log n)^{-(1/2)}$ in the definition of $\hat{T}$ by $(2n(\log n + \log\log n))^{-(1/2)}$ and put $c = 1$. Then $E\hat{T} = \infty$ (as will follow from corollary 4.7 of this paper), but

$$\sum_n \Pr \{(2n(\log n + \log\log n))^{-1/2} S_n \geq 1\} < \infty. \tag{4}$$

Moreover, the argument (3) fails if applied to moments other than the mean.

In order to state our first result in a more suggestive way, let us introduce a

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new motion of lim sup for random variables. Given a sequence $\theta_1, \theta_2, \cdots$ of random variables (such as $\theta_n = a_n S_n$ with positive constants a_n), put for any real c

$$\hat{T}_c = \sup \{n \colon \theta_n \geq c\}. \tag{5}$$

The statement

$$\text{“}\limsup \theta_n = x, \text{ a.s.,”} \tag{6}$$

where $x \in \langle -\infty, +\infty \rangle$ (we identify an almost constant measurable function with its almost sure value) may be expressed by means of the $\hat{T}_c$ in the following way:

$$\begin{aligned} \text{“}\hat{T}_c &< \infty \text{ a.s.} \quad \text{if} \quad c > x, \\ \hat{T}_c &= \infty \text{ a.s.} \quad \text{if} \quad c < x.\text{”} \end{aligned} \tag{7}$$

We define by analogy

$$\text{“}\limsup \theta_n = y \ (r\text{-quickly}),\text{”} \tag{8}$$

where $y \in \langle -\infty, +\infty \rangle$ and $r > 0$, to mean

$$\begin{aligned} \text{“}E\hat{T}_c^r &< \infty \quad \text{if} \quad c > y, \\ E\hat{T}_c^r &= \infty \quad \text{if} \quad c < y.\text{”} \end{aligned} \tag{9}$$

Thus y is the largest element of $\langle -\infty, +\infty \rangle$ such that any neighborhood of y will still be visited by θ_n with n so late that the time of the last visit has infinite r-th moment. Note that y always exists and is uniquely determined and that $x \leq y$ if x exists.

THEOREM 1.1. *Let $X_1, X_2, \cdots$ be a sequence of independent identically distributed random variables such that $EX_1 = 0$ and $EX_1^2 = 1$. Let $r > 0$, $z > 2(r + 1)$, and $E|X_1|^z < \infty$. Then if $S_n = \sum_{i \leq n} X_i$,*

$$\limsup (2n \log n)^{-1/2} S_n = \sqrt{r} \ (r\text{-quickly}). \tag{10}$$

This result follows from corollary 4.7 of the present paper in the same way as Khintchine's law of the iterated logarithm follows from the Kolmogorov-Petrovski-Erdös criterion (that is, the general law of the iterated logarithm (see Erdös [20] and Itô-McKean [25], pp. 161–164)).

Having answered the question about the existence of moments of random variables like $\hat{T}$, we may ask for a more detailed description of the distribution of such random variables. To simplify matters, let us first replace the coin-tossing process (or the more general random walks considered in the above theorem) by a standard Brownian motion ξ. For any positive function φ on R^+ (= the set of positive real numbers) such that $t^{-(1/2)}\varphi(t)$ increases with t, put

$$T_\varphi = \sup \{t \colon \xi(t) \geq \varphi(t)\}. \tag{11}$$

A well-known zero-one law assures that T_φ is either a random variable or $T_\varphi = \infty$, a.s. Because in the second case we know the distribution of T_φ perfectly, only the first case is of interest. The Kolmogorov-Petrovski-Erdös test tells us when this case happens,

$$\Pr \{T_\varphi < \infty\} = 1 \tag{12}$$

if and only if

$$\int_1^\infty t^{-3/2}\varphi(t)e^{-\varphi(t)^2/2t}\,dt < \infty. \tag{13}$$

It is natural to expect some information about the distribution of T_φ from the known proofs of this criterion. P. Lévy ([33], pp. 271–276) discusses this question and obtains upper bounds for the probabilities of the tails of T_φ, bounds which come out of the proof of the 'if' part. A special case of Lévy's results has been rediscovered recently by Baum and Katz (theorem 6 in [2], see [33], p. 275, (59)); we remark that P. Lévy gets the 'if' part and his upper bounds for a large class of martingales with individually bounded differences, and Baum and Katz prove their result for random walks satisfying Feller's condition). In their elegant proof of the 'if' part for Brownian motion, Itô and McKean [25] derive the following inequality:

$$\Pr\{T_\varphi > t\} \le 2\int_t^\infty \frac{\varphi(s)}{s}\,\frac{\exp\left\{\dfrac{-\varphi(s)^2}{2s}\right\}}{\sqrt{2\pi s}}\,ds, \tag{14}$$

which is valid under the slight additional assumption that $t^{-1}\varphi(t)$ decreases as t increases. Formula (14) contains P. Lévy's results (for Brownian motion).

Getting lower bounds seems to be more delicate, and the proofs (including Motoo's [36]) of the difficult 'only if' part of Kolmogorov-Petrovski-Erdös' criterion apparently do not help. The main result of the first part of this paper is the following.

THEOREM 1.2. (Compare with theorem 3.6.) *Let φ be a positive function on R^+ with a continuous derivative and such that $t^{-\delta}\varphi(t)$ increases in t for some $\delta > 0$. Assume that*

$$\frac{\varphi'(s)}{\varphi'(t)} \to 1 \qquad \text{as} \quad t \to \infty,\ \frac{s}{t} \to 1 \tag{15}$$

and that $T_\varphi < \infty$, a.s. Then T_φ has a continuous density D_φ (except possibly for some mass at 0) *and*

$$D_\varphi(t) \sim \varphi'(t)(2\pi t)^{-1/2}e^{-\varphi(t)^2/2t} \qquad \text{as} \quad t \to \infty. \tag{16}$$

Thus if, for example,

$$\varphi(t) = [2t(\log_2 t + \tfrac{3}{2}\log_3 t + \log_4 t + \cdots + (1+\epsilon)\log_k t)]^{1/2} \tag{17}$$

where $k > 3$, $\epsilon > 0$, and $\log_{i+1} t = \log(\log_i t)$, then

$$D_\varphi(t) \sim [\sqrt{4\pi}\, t \log t \log_2 t \cdots (\log_{k-1} t)^{1+\epsilon}]^{-1}, \tag{18}$$

or if

$$\varphi(t) = t^{(1/2)+a} \tag{19}$$

where $a > 0$, then

$$D_\varphi(t) \sim \frac{a+\frac{1}{2}}{\sqrt{2\pi}}\, t^{a-1}\exp\left\{-\frac{t^{2a}}{2}\right\}. \tag{20}$$

The slightly more general theorem 3.6 of this paper applies also to

$$\varphi(t) = e^{bt} \tag{21}$$

with $b > 0$, giving

$$D_\varphi(t) \sim b(2\pi t)^{-1/2} \exp\left\{-\frac{b^2 e^{2bt}}{2t} + bt\right\}. \tag{22}$$

We remark that if φ is concave in an interval $\langle r, \infty)$ (as the φ in (17), and (19) for $a \leq \frac{1}{2}$), then $\varphi'(t)(2\pi t)^{-(1/2)} \exp\{-\varphi(t)^2/2t\}$ overestimates $D_\varphi(t)$ in $\langle r, \infty)$; if φ is convex in $\langle r, \infty)$ (as in (19) for $a \geq \frac{1}{2}$, and (21)), then it underestimates $D_\varphi(t)$ in $\langle r, \infty)$. The above theorem not only gives us information about the distribution of T_φ, it also leads to a better understanding of the Kolmogorov-Petrovski-Erdös criterion. It is not hard to see that the assumptions of theorem 1.2 about φ imply that for $t \to \infty$, $\varphi'(t)$ has the same order of magnitude as $t^{-1}\varphi(t)$, so that for such functions the integrand of (13) has by (16) the same order of magnitude as the density of T_φ. Thus (13) simply states that this density is integrable. The restriction (15) on φ being easily removed (see theorem 3.6 and corollary 3.7) this actually leads to a proof of the 'only if' part of the Kolmogorov-Petrovski-Erdös criterion.

In the second part of this paper we extend the previous results to more general stochastic processes by means of almost sure invariance principles, which have been introduced in [42] in analogy with Donsker's distribution type invariance principle (see Donsker [13], Prohorov [39], Billingsley [4], Krickeberg [30]). As in [42], our method here is based on Skorohod's representation of sums of independent random variables by means of stopping times for the Brownian motion (Skorohod [40], p. 180; see also Knight [28]). This representation may be generalized to discrete parameter martingales in a straightforward manner (for a generalization to continuous parameter martingales with continuous sample paths, see Dubins-Schwarz [18], [17] and Dambis [9]).

We prove three almost sure invariance principles, the first of which applies to martingales. Lester Dubins and David Freedman also obtained an almost sure invariance principle for martingales $\sum_{i \leq n} X_i$ with uniformly bounded $|X_i|$, and they informed me in a letter about their result and its proof while the present paper was being written.

THEOREM 1.3. (Compare with theorem 4.4.) *Let $S_n = \sum_{i \leq n} X_i$ be a martingale with finite second moments. Assume that*

$$V_n = \sum_{i \leq n} E(X_i^2 | X_1, \cdots, X_{i-1}) \to \infty, \qquad \text{a.s.} \tag{23}$$

as $n \to \infty$ and that

$$\sum_{n \geq 1} f(V_n)^{-1} \int_{x^2 > f(V_n)} x^2 \, d \Pr\{X_n \leq x | X_1, \cdots, X_{n-1}\} < \infty, \qquad \text{a.s.,} \tag{24}$$

where f is a positive nondecreasing function on R^+, which increases slower than t. Then, if the underlying probability space is rich enough, there is a Brownian motion ξ such that

$$S(t) = \xi(t) + o((tf(t))^{1/4} \log t), \qquad \text{a.s. as} \quad t \to \infty, \tag{25}$$

where S is obtained by linearly interpolating S_n at V_n.

One consequence (corollary 4.5) of theorem 1.3 is the extension of the Kolmogorov-Petrovski-Erdös criterion to martingales satisfying a condition which for sums of independent random variables is almost as weak as Feller's condition ([22], p. 399). In the same way Chung's criterion [7] may also be generalized to martingales.

Our second invariance principle (theorem 4.6) applies to random walks under conditions on the moments and yields theorem 1.1 and the more general corollary 4.7. The assumptions for this invariance principle have been reduced to a minimum with the help of recent results of Baum and Katz [1], [2].

To extend as much of theorem 1.2 as possible to other random walks we have to assume the existence of a moment generating function (compare Baum-Katz-Read [3], theorem 1). Doing this we get by a third invariance principle (theorem 4.8) the following theorem.

THEOREM 1.4. (Compare with theorem 4.9.) *Let $X_1, X_2, \cdots$ be independent, identically distributed with mean 0, variance 1, and a finite moment generating function in a neighborhood of 0, and let φ be as in theorem 1.2 with $\delta = \frac{1}{2}$ and the additional restriction that $\varphi(t) \leq t^h$ for all t and some $h < \frac{3}{5}$. Then*

$$\Pr\{S_m \geq \varphi(m) \quad \text{for some} \quad m > n\} \sim \int_n^\infty \varphi'(t) \exp\left\{-\frac{\varphi(t)^2}{2t}\right\} \frac{dt}{\sqrt{2\pi t}} \qquad \text{as} \quad n \to \infty. \tag{26}$$

Thus, for instance, if

$$\varphi(t) = (2t(\log\log t + \tfrac{3}{2}\log_3 t + \log_4 t + \cdots + (1+\epsilon)\log_k t)^{1/2}, \tag{27}$$

then

$$\Pr\{S_m \geq \varphi(m) \quad \text{for some} \quad m > n\} \sim \frac{1}{\sqrt{4\pi\epsilon}(\log_{k-1} n)^\epsilon}. \tag{28}$$

We conclude with some remarks and problems. It is natural to ask for an r-quick analogue of theorem 2 of [42]. Under the assumptions of theorem 1.1 of the present paper, the following seems likely. If η_n is obtained by linearly interpolating $(2n \log n)^{-(1/2)} S_i$ at i/n, $(1 \leq i \leq n)$, then the sequence $(\eta_n)_{n \geq 2}$ is r-quickly relatively norm compact (in $C\langle 0, 1\rangle$), and the set of its r-quick norm limit points is $\{x\colon x \in C\langle 0, 1\rangle, x(0) = 0, x \text{ is absolutely continuous, and } \frac{1}{2}\int_0^1 \dot{x}^2\, dt \leq r\}$. Here we have used the following definitions: $(\eta_n)_{n\geq 2}$ is r-quickly relatively norm compact iff for any $\epsilon > 0$ there is a finite union U of ϵ-spheres in $C\langle 0, 1\rangle$ such that

$$E(\sup\{n\colon \eta_n \notin U\})^r < \infty; \tag{29}$$

and x is an r-quick norm limit point of $(\eta_n)_{n\geq 2}$ iff for any open $U \ni x$,

$$E(\sup\{n\colon \eta_n \in U\})^r = \infty. \tag{30}$$

The results of Dvoretzky-Erdös [19], Spitzer [41], Takeuchi [44], Takeuchi-

Watanabe [45] and Wiener-Itô-McKean (see Lamperti [32], Itô-McKean [25], pp. 255–257) give rise to a series of straightforward conjectures of the type (16) up to an order of magnitude. Most of these, however, seem to be hard to verify (see also Ciesielski-Taylor [8]).

Theorem 1.3 (4.4) may be sharpened by repeating its proof using corollary 4.5 in place of lemma 4.1. Then, however, 4-th conditional moments will appear in the assumptions as well as in the conclusion. For random walks it is easy to get the following.

THEOREM 1.5. *Let $X_1, X_2, \cdots$ be independent identically distributed such that*

$$EX_1 = 0, \qquad EX_1^2 = 1, \qquad EX_1^4 < \infty. \tag{31}$$

Then if the underlying probability space is rich enough, there is a Brownian motion ξ such that

$$S_n = \xi(n) + O((n \log\log n)^{1/4}(\log n)^{1/2}) \qquad \text{a.s. as} \quad n \to \infty. \tag{32}$$

It would be interesting to know whether (32) with o instead of O implies that X_1 is Gaussian, and also whether (32) and the assumptions of theorem 1.5, except (31), imply (31) (by [43], this is true for the first two statements of (31)).

Blackwell's paper [5] suggests that theorems 4.6 and 4.8 also hold for certain martingales.

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2. Notation and conventions

The following definitions are used throughout the paper:

(a) $R^+ = \{t\colon t \text{ real and positive}\}$,
(b) $\Pi = \{\psi\colon \psi \text{ is a positive real function on } R^+\}$,
(c) $\uparrow\ = \{\psi\colon \psi \text{ is a nondecreasing real function on } R^+\}$,
(d) $\downarrow\ = \{\psi\colon \psi \text{ is a nonincreasing real function on } R^+\}$,
(e) τ = identity function on R^+,
(f) $(u, v\rangle = \{r\colon u < r \le v\}$ for any real $u \le v$ ((u, v), $\langle u, v)$, and $\langle u, v\rangle$ are defined similarly),
(g) $G_\psi = (\exp\{-\psi^2/2\tau\}/\sqrt{2\pi\tau}) \in \Pi$ for any real function ψ on R^+.

If f and g are real-valued functions on any set, then $f \vee g$ and $f \wedge g$ denote the pointwise maximum and minimum respectively of f and g.

The process ξ is always a standard Brownian motion (Doob [14], Krickeberg [31], Itô-McKean [25]), but need not denote the same individual copy throughout the paper. We treat ξ as a continuous function on $\{0\} \cup R^+$ which depends in a Borel measurable way on the points ω of the underlying probability space. This dependence on ω, however, will not be expressed explicitly. We will use the strong Markov property of ξ without mentioning it (see Hunt [24], Itô-McKean

[25], p. 22). Measurability arguments are skipped because only standard ones are needed (Itô-McKean [25], pp. 12–17, Nelson [37]). The underlying probability space is always assumed to be complete. The abbreviation 'a.s.' means 'almost surely'. If ξ is given and if ψ is a piecewise continuous real function on R^+, then we define

(h) $S_\psi = \inf \{s\colon s > 0 \text{ and } \xi(s) \geq \psi(s)\}$,

which is assumed to be ∞ if the set on the right is empty;

(j) $T_\psi = \sup \{s\colon s > 0 \text{ and } \xi(s) \geq \psi(s)\}$,

which is assumed to be 0 if the set on the right is empty. Both S_ψ and T_ψ are measurable functions on the underlying probability space which take values in $R^+ \cup \{0, +\infty\}$. A standard zero-one law (Itô-McKean [25], p. 25, problem 2, together with p. 18, problem 3; Hewitt-Savage [23]) assures that either T_ψ is a random variable (namely, $T_\psi < \infty$, a.s.) or $T_\psi = \infty$, a.s. Also, either $S_\psi = 0$, a.s. or $S_\psi > 0$, a.s. By F_ψ we will denote the (sub)distribution function of S_ψ. Thus,

(k) $F_\psi(t) = \Pr \{\xi(u) \geq \psi(u) \text{ for some } u \in (0, t\rangle\}$

if ψ is right-continuous at $t > 0$. Generalizing, if $0 \leq s < t$, x real, and ψ again right-continuous at t, we define

(m) $$\begin{aligned} F_{\psi|\xi(s)=x}(t) &= \Pr \{\xi(u) \geq \psi(u) \text{ for some } \quad u \in (s, t\rangle | \xi(s) = x\} \\ &= F_{\psi_1}(t - s), \end{aligned}$$

where $\psi_1 = \psi(s + \tau) - x$.

Finally, we remark that in the last part of the paper random variables T_n and $S_n = \sum_{i \leq n} X_i$ will occur, which have nothing to do with T_ψ, S_ψ. Possible confusion will be prevented by the context.

3. Brownian motion

LEMMA 3.1. *Let a and b be piecewise continuous real functions on R^+ and $t > 0$ such that $a \leq b$ in $(0, t\rangle$ and $a \geq b$ in $\langle t, t + \delta\rangle$, where $\delta > 0$, a and b are differentiable at t. Then*

$$\limsup_{h,k \to 0+0} \left[\frac{F_a(t + k) - F_a(t - h)}{h + k} - \frac{F_b(t + k) - F_b(t - h)}{h + k}\right] \leq 0. \tag{33}$$

PROOF. (A correct proof of this lemma was first given by Professor A. Dvoretzky.) Let both $h > 0$ and $0 < k < \delta$ be so small that a and b are continuous in $\langle t - h, t + k\rangle$, and for $0 < \epsilon \leq h$, put

$$c(\epsilon) = \sup \{b(u)\colon t - \epsilon \leq u \leq t\} - a(t - \epsilon). \tag{34}$$

Then $c(\epsilon) = O(\epsilon)$ as $\epsilon \to 0$. We have

$$
\begin{aligned}
(35)\qquad & F_a(t+k) - F_a(t-h) - (F_b(t+k) - F_b(t-h)) \\
&= \Pr\{S_a \in (t-h, t+k\rangle\} - \Pr\{S_b \in (t-h, t+k\rangle\} \\
&\leq \Pr\{S_a \in (t-h, t\rangle, \xi < b \text{ in } (0, t\rangle\} \\
&\leq \int_{(t-h,t\rangle} \Pr\{\xi < b \text{ in } \langle s, t\rangle | \xi(s) = a(s)\}\, dF_a(s) \\
&\leq \int_{(t-h,t\rangle} \Pr\{\xi < c(t-s) \text{ in } (0, t-s\rangle\}\, dF_a(s) \\
&= \int_{(t-h,t\rangle} \Pr\{|\xi(t-s)| < c(t-s)\}\, dF_a(s) \text{ (by lemma 3.2, (37) below)} \\
&\leq \sup \Pr\left\{|\xi(1)| < \frac{c(\epsilon)}{\sqrt{\epsilon}} \cdot \Pr\{S_a \in (t-h, t\rangle\right\} \\
&= 0(h^{1/2}) \Pr\{S_a \in (t-h, t\rangle\}.
\end{aligned}
$$

We have to show that $\Pr\{S_a \in (t-h, t\rangle\} = o(h^{1/2})$. Let $\delta > 0$, L the straight line through the point $(t, a(t))$ with slope $a'(t) - \delta$ (so that $L > a$ in a left neighborhood of t, say in $\langle t-H, t)$). Then if $x < a(t-H)$ and $h < H$,

$$
\begin{aligned}
(36)\qquad & \Pr\{S_a \in (t-h, t\rangle | \xi < a \text{ in } (0, t-H\rangle, \xi(t-H) = x\} \\
&= \Pr\{S_{a(\tau+t-H)-x} \in (H-h, H\rangle\} \\
&\leq \Pr\{S_{L(\tau+t-H)-x} \in (H-h, H\rangle\} \\
&+ \Pr\{S_{a(\tau+t-H)-x} \in (H-h, H\rangle, \xi < L(\tau+t-H) - x \\
&\qquad \text{in } (H-h, H\rangle\}.
\end{aligned}
$$

The first summand here is $O(h)$ uniformly in x, because by lemma 3.2, (38) below with $l = L(\tau + t - H) - x$, $S_{L(\tau+t-H)-x}$ has a density in $(0, H\rangle$, bounded uniformly for $x \leq a(t-H) < L(t-H)$. The second summand in (36) is $\leq \sup_{u\in(H-h,H\rangle} \Pr\{\xi < L(\tau + u + t - H) - a(u + t - H) \text{ in } (0, H-u\rangle\} \leq 4\delta h^{1/2}$ for small h, using lemma 3.2, (37) below and $L(v) - a(v) \leq 2\delta(t-v)$ for small $t - v$. Thus the left side of (36) is $\leq 5\delta h^{1/2}$ for small h uniformly in x. Integration yields $\Pr\{S_a \in (t-h, t\rangle\} \leq 5\delta h^{1/2}$ for small h, so that the lemma follows because δ is arbitrary.

The following lemma is well known (Lévy [34], p. 211, (6) together with Cameron-Martin [6]; a simple direct proof may be found in Dinges [12]).

LEMMA 3.2. *Let $t > 0$, $m > 0$, q real, $l = m + q\tau$ and $x \leq l(t)$. Then*

$$
\begin{aligned}
(37)\qquad & \Pr\{\xi < l \text{ in } (0, t\rangle, \xi(t) \leq x\} \\
&= \Pr\{\xi(t) \leq x\} - e^{-2qm} \Pr\{\xi(t) \leq x - 2m\}.
\end{aligned}
$$

Thus (by $\Pr\{S_l > s\} = \Pr\{\xi < l \text{ in } (0, s\rangle\}$),

$$
(38)\qquad F_l'(s) = \frac{m}{s} \frac{\exp\left\{-\frac{1}{2s}(m + qs)^2\right\}}{\sqrt{2\pi s}} = \frac{m}{s} G_l(s) \qquad \text{for any} \quad s > 0\cdot
$$

Also

(39) $$\Pr\{\max_{s \le t} |\xi(s)| > m\} \le 4 \Pr\{\xi(t) > m\}.$$

LEMMA 3.3. *Let ψ be a piecewise continuous real function on R^+ with continuous derivative in (u, v) where $0 < u < v$. Then F_ψ has a continuous derivative in (u, v).*

PROOF. Denote by F^l_ψ the lower, and by F^u_ψ the upper derivative of F_ψ (thus, for example,

(40) $$F^l_\psi(t) = \liminf_{h,k \to 0+0} \left(\frac{F_\psi(t+k) - F_\psi(t-h)}{h+k} \right).$$

Then if the lemma is wrong, there are sequences t_n, r_n in (u, v) converging to the same $t \in (u, v)$ such that

(41) $$\liminf_{n \to \infty} (F^l_\psi(t_n) - F^l_\psi(r_n)) > \epsilon > 0$$

for some ϵ. Let $\delta > 0$ be so small that $u < t - \delta < t + \delta < v$ and $\alpha - \beta < \epsilon/4$, where

(42) $$\alpha = \sup\{\psi'(s) : |s - t| < \delta\}, \qquad \beta = \inf\{\psi'(s) : |s - t| < \delta\}.$$

Let y_1 (resp. y_2) be such that the linear function with slope α (resp. β) and with value y_1 (resp. y_2) at $t - \delta$ takes the value $\psi(t + \delta)$ at $t + \delta$. It is clear that $y_1 \le \psi(t - \delta) \le y_2$ and that

(43) $$\frac{y_2 - y_1}{2\delta} = \frac{\psi(t+\delta) - y_1}{2\delta} - \frac{\psi(t+\delta) - y_2}{2\delta} = \alpha - \beta < \frac{\epsilon}{4}.$$

It is also clear that any linear function on $\langle t - \delta, t + \delta\rangle$ which is y_2 at $t - \delta$ and $\psi(s)$ at some $s \in \langle t - \delta, t + \delta\rangle$ lies above ψ in $\langle t - \delta, s\rangle$ and below ψ in $\langle s, t + \delta\rangle$. Thus, if we call b_n the function which coincides with ψ on $(0, t - \delta)$ and which is linear on $\langle t - \delta, \infty)$, and such that $b_n(t - \delta) = y_2$ and $b_n(t_n) = \psi(t_n)$ where n is large enough to ensure $|t_n - t| < \delta$, then we may apply lemma 3.1 to ψ in place of a, b_n in place of b, and t_n in place of t to get

(44) $$F^u_\psi(t_n) \le \left[\frac{d}{ds} F_{b_n}(s)\right]_{s = t_n} = \int_{-\infty}^{y_2} \frac{y_2 - x}{t_n - (t - \delta)} \frac{\exp\left\{-\dfrac{(\psi(t_n) - x)^2}{2(t_n - (t - \delta))}\right\}}{\sqrt{2\pi(t_n - (t - \delta))}} \, d_x \Pr\{\xi(t - \delta) \le x, \xi < \psi \text{ in } (0, t - \delta)\}$$

by (38). Thus

(45) $$\limsup F^u_\psi(t_n) \le \int_{-\infty}^{y_2} \frac{y_2 - x}{\delta} \frac{\exp\left\{-\dfrac{(\psi(t) - x)^2}{2\delta}\right\}}{\sqrt{2\pi\delta}} \, d_x \Pr\{\xi(t - \delta) \le x, \xi < \psi \text{ in } (0, t - \delta)\} = \int_{-\infty}^{y_2} \frac{y_2 - x}{\delta} \, dH(x),$$

where H is a subdistribution function ($\lim_{x\to\infty} H(x) \leq 1$). Similarly,

$$(46) \qquad \liminf F^l_\psi(r_n) \geq \int_{-\infty}^{y_1} \frac{y_1 - x}{\delta}\, dH(x) \geq \int_{-\infty}^{y_2} \frac{y_1 - x}{\delta}\, dH(x)$$

$$= \int_{-\infty}^{y_2} \frac{y_2 - x}{\delta}\, dH(x) - \int_{-\infty}^{y_2} \frac{y_2 - y_1}{\delta}\, dH(x) \geq \limsup F^u_\psi(t_n) - \frac{\epsilon}{2}$$

by (45) and (43). But this contradicts (41).

Now lemma 3.1 may be formulated in a neater way.

LEMMA 3.4. *Let a and b be piecewise continuous functions on R^+, both of which have a continuous derivative in a neighborhood of some $t > 0$. Assume $a \leq b$ in $(0, t\rangle$ and $a \geq b$ in $\langle t, t + \delta)$ for some $\delta > 0$. Then F_a and F_b have a continuous derivative in a neighborhood of t and $F'_a(t) \leq F'_b(t)$.*

PROOF. The proof is immediate from lemmas 3.1 and 3.3.

THEOREM 3.5. *Let $0 < \alpha < 1$, $\psi \in \Pi$ such that ψ has a continuous derivative and*

$$(47) \qquad \tau^{-\alpha}\psi \in \downarrow.$$

Assume

$$(48) \qquad S_\psi > 0, \text{ a.s.}$$

Then

$$(49) \qquad t^{-1/2}\psi(t) \to \infty \qquad \text{as} \quad t \to 0.$$

Also F_ψ has a continuous derivative f_ψ such that if $0 < \epsilon < 1 - \alpha$ and

$$(50) \qquad \begin{matrix}\lambda(t)\\ \Lambda(t)\end{matrix} = \begin{matrix}\inf\\ \sup\end{matrix}\left\{\psi(u) - u\psi'(u) \colon t\left(1 - \frac{t^\epsilon}{\psi(t)^{2\epsilon}}\right) \leq u \leq t\right\}$$

for $t > 0$ small enough to ensure $t^\epsilon\psi(t)^{-2\epsilon} < 1$ by (49), then

$$(51) \qquad \limsup_{t\to 0} f_\psi(t) \left(\frac{\Lambda(t)}{t}\, G_\psi(t)\right)^{-1} \leq 1$$

and

$$(52) \qquad \liminf_{t\to 0} f_\psi(t) \left(\frac{\lambda(t)}{t}\, G_\psi(t)\right)^{-1} \geq 1.$$

Notice that geometrically, $\lambda(t)$ and $\Lambda(t)$ are the lowest and highest point of intersection of the tangents of ψ in the interval $\langle t(1 - t^\epsilon\psi(t)^{-2\epsilon}), t\rangle$ with the vertical coordinate axis. Note also that $\lambda(t) > 0$, because $\psi' \leq \alpha\psi/\tau$ by (47), and therefore,

$$(53) \qquad \psi - \tau\psi' \geq (1 - \alpha)\psi.$$

PROOF. Inequality (48) alone implies $t^{-1/2}\psi(t) \to \infty$ as $t \to 0$, for otherwise, $t^{-(1/2)}\psi(t) < A < \infty$ for suitable arbitrarily small $t > 0$, so that for such t,

$$(54) \qquad \Pr\{S_\psi < t\} \geq \Pr\{\xi(t) > At^{1/2}\} = \Pr\{\xi(1) > A\} > 0,$$

contradicting (48). Lemma 3.3 asserts that F_ψ has a continuous derivative f_ψ.

To prove (51), we apply lemma 3.4 to ψ in place of a and ψ_1 in place of b, where ψ_1 is defined as follows. Let $t > 0$, $\Lambda_1 > \Lambda(t)$ be a real number and l be the linear function with $l(0) = \Lambda_1$, $l(t) = \psi(t)$, and let $s = t(1 - t^{\epsilon}\psi(t)^{-2\epsilon})$. Then it is easy to see that $l \geq \psi$ in $\langle s, t\rangle$ and $l \leq \psi$ in a right neighborhood of t, say in $\langle t, t + \delta)$, where $\delta > 0$; ψ_1 is now defined to coincide with ψ on $(0, s) \cup \langle t + \delta, \infty)$ and with l on $\langle s, t + \delta)$. Lemma 3.4 yields

$$\begin{aligned}(55)\qquad f_{\psi}(t) &= F'_{\psi}(t) \leq F'_{\psi_1}(t)\\ &= \int_{-\infty}^{\psi_1(s)} F'_{\psi_1|\xi(s)=x}(t)\, d_x \Pr\{\xi < \psi \text{ in } (0, s),\ \xi(s) \leq x\}\\ &\leq \int_{-\infty}^{l(s)} \frac{l(s) - x}{t - s}(2\pi(t - s)2\pi s)^{-1/2} \exp\left\{-\frac{(\psi(t) - x)^2}{2(t - s)} - \frac{x^2}{2s}\right\} dx\end{aligned}$$

by lemma 3.2, (38). Substituting $y = x - (s/t)\psi(t)$ and using

$$(56)\qquad \frac{l(s) - \dfrac{s}{t}\psi(t)}{t - s} = \frac{\Lambda_1}{t}$$

(definition of l), we get

$$\begin{aligned}(57)\qquad &f_{\psi}(t)\\ &\leq \frac{\Lambda_1}{t} G_{\psi}(t) \int_{-\infty}^{(t-s)\Lambda_1 t^{-1}} \left(1 - \frac{ty}{(t - s)\Lambda_1}\right)\left(2\pi(t - s)\frac{s}{t}\right)^{-1/2} \exp\left\{-\frac{y^2 t}{2(t - s)s}\right\} dy\\ &\leq \frac{\Lambda_1}{t} G_{\psi}(t)\left(1 + \frac{t}{\Lambda_1}\int_0^{\infty} \frac{y}{t - s}\left(2\pi(t - s)\frac{s}{t}\right)^{-1/2} \exp\left\{-\frac{y^2 t}{2(t - s)s}\right\} dy\right)\\ &\leq \frac{\Lambda_1}{t} G_{\psi}(t)\left(1 + \frac{t}{\Lambda_1\sqrt{t - s}}\right).\end{aligned}$$

Using the definition of s and the fact that

$$(58)\qquad \Lambda_1 > \Lambda(t) \geq \psi(t) - t\psi'(t) \geq (1 - \alpha)\psi(t)$$

by (53), we get

$$(59)\qquad f_{\psi}(t) \leq (\Lambda_1/t)G_{\psi}(t)(1 + (1/1 - \alpha)(\sqrt{t}/\psi(t))^{1-\epsilon}),$$

which implies (51) by (49) and the fact that Λ_1 was arbitrary greater than $\Lambda(t)$.

To prove (52), we again apply lemma 3.4, but we need a little preparation. We have $\alpha + \epsilon < 1$, so there is a β such that

$$(60)\qquad \alpha + \epsilon < 2\beta - 1 < \beta < 1.$$

Let $s = t(1 - t^{\epsilon}\psi(t)^{-2\epsilon})$ be as above and r be any function of s (and thus of t) for which (see (49)).

$$(61)\qquad \frac{\psi(s)^2}{s}\left(\frac{r}{s}\right)^{2\beta-1} \to \infty \qquad \text{as} \quad s \to 0$$

and

$$(62)\qquad \frac{\psi(s)^2}{s}\left(\frac{r}{s}\right)^{\beta} \to 0 \qquad \text{as} \quad s \to 0.$$

Using (49), we see that for small t (to which we will restrict ourselves) we have $0 < r < s < t$. In the rest of the proof all asymptotic statements are understood to hold as $t \to 0$ (and therefore as $s \to 0$, $r \to 0$).

Notice that

$$s \sim t \tag{63}$$

and

$$r = o(s). \tag{64}$$

Now let g be the linear function with

$$g(r) = \psi(s)\left(\frac{r}{s}\right)^{\alpha} \quad \text{and} \quad g(s) = \psi(s) \tag{65}$$

(more rigorously we should write g_t, because g depends via r and s, on the parameter t). We have $g \leq \psi$ on $\langle r, s\rangle$ (this is easily seen using an auxiliary function $\eta = \psi(s)(\tau/s)^{\alpha}$; in fact, $\eta \leq \psi$ on $\langle r, s\rangle$ by (47), and $g \leq \eta$ on $\langle r, s\rangle$ because η is concave). Next let h be the linear function with

$$h(0) = \nu\cdot\lambda(t) \quad \text{and} \quad h(t) = \psi(t) \tag{66}$$

where $\nu = \nu(t)$ will be chosen below such that $\nu < 1$ and $\nu \to 1$ as $t \to 0$. From the definition of $\lambda(t)$, it follows easily that $h \leq \psi$ in $\langle s, t\rangle$ and $h \geq \psi$ in $\langle t, t + \delta)$ for some $\delta > 0$. Now we define ψ_2 to be ψ on $(0, r)$, g on $\langle r, s)$, h on $\langle s, t + \delta)$, and ψ on $\langle t + \delta, \infty)$. We apply lemma 3.4 to ψ in place of b and ψ_2 in place of a and get

$$f_{\psi}(t) \geq F'_{\psi_2}(t) \geq \int_k^{h(s)} F'_{h|\xi(s)=x}(t)I(x)\,dx \tag{67}$$

for any k, where

$$I(x) = \frac{d}{dx}\Pr\{\xi < \psi \text{ in } (0, r),\ \xi < g \text{ in } \langle r, s),\ \xi(s) \leq x\}. \tag{68}$$

We choose $k = \psi(s)(r/s)^{\beta}$ and estimate $I(x)$ for $x \in \langle k, h(s)\rangle$, using lemma 3.2, (37):

$$\begin{aligned}
I(x) &= \int \frac{d}{dx}\Pr\{\xi < g \text{ in } \langle r, s),\ \xi(s) \leq x | \xi(r) = y\} \\
&\qquad\qquad d_y \Pr\{\xi < \psi \text{ in } (0, r),\ \xi(r) \leq y\} \\
&= \int_{(-\infty, g(r))} (2\pi(s - r))^{-1/2} \exp\left\{-\frac{(x - y)^2}{2(s - r)}\right\} \\
&\qquad \left(1 - \exp\left\{-\frac{2}{s - r}(\psi(s) - x)\left(\psi(s)\left(\frac{r}{s}\right)^{\alpha} - y\right)\right\}\right) \\
&\qquad\qquad d_y \Pr\{\xi < \psi \text{ in } (0, r),\ \xi(r) \leq y\} \\
&\geq \int_{\langle -k, k\rangle} (2\pi s)^{-1/2} \exp\left\{-\frac{(x + k)^2}{2(s - r)}\right\} \\
&\qquad \left(1 - \exp\left\{-\frac{2}{s}(\psi(s) - h(s))\left(\psi(s)\left(\frac{r}{s}\right)^{\alpha} - k\right)\right\}\right) \\
&\qquad\qquad d_y \Pr\{\xi < \psi \text{ in } (0, r),\ \xi(r) \leq y\},
\end{aligned} \tag{69}$$

by first replacing $(-\infty, g(r)\rangle$ by $\langle -k, k\rangle$ and then estimating the integrand using $k \leq \psi(s)\ (r/s)^\alpha = g(r)$ and $k \leq x \leq h(s) \leq \psi(s)$. Now

$$\exp\left\{-\frac{(x+k)^2}{2(s-r)}\right\} \sim \exp\left\{-\frac{x^2}{2s}\right\} \tag{70}$$

uniformly in $x \in \langle k, h(s)\rangle$, by (64) and (62) and $h(s) \leq \psi(s)$.

Furthermore,

$$0 \leq \psi(s) - h(s) = \psi(s) - \frac{t-s}{t}\nu\lambda(t) - \frac{s}{t}\psi(t), \tag{71}$$

so that in particular, because $\nu < 1$ is still arbitrary $(t - s/t)\lambda(t) \leq \psi(s) - (s/t)\psi(t)$, and therefore,

$$\psi(s) - h(s) = \psi(s) - \frac{t-s}{t}\nu\lambda(t) - \frac{s}{t}\psi(t) \geq (1-\nu)\left(\psi(s) - \frac{s}{t}\psi(t)\right). \tag{72}$$

Now (47) implies that

$$\psi(t) \leq \left(\frac{t}{u}\right)^\alpha \psi(u) \tag{73}$$

for any $u < t$, so that

$$\psi(s) - h(s) \geq (1-\nu)\psi(s)\left(1 - \left(\frac{s}{t}\right)^{1-\alpha}\right) \geq (1-\nu)\psi(s)\,\epsilon\,\frac{t-s}{t} \tag{74}$$

for small t by (63) and $\epsilon < 1 - \alpha$, and therefore, using the definition of s,

$$\psi(s) - h(s) \geq (1-\nu)\psi(s)\epsilon t^\epsilon \psi(t)^{-2\epsilon}. \tag{75}$$

On the other hand,

$$2\left(\psi(s)\left(\frac{r}{s}\right)^\alpha - k\right) \geq \psi(s)\left(\frac{r}{s}\right)^\alpha \geq \psi(s)\left(\frac{r}{s}\right)^{2\beta-1}\left(\frac{s}{r}\right)^\epsilon \tag{76}$$

for small t by (64) and the definition of k and β. Now by (62), (73), and (63),

$$\left(\frac{s}{r}\right)^\epsilon \geq 2\left(\frac{\psi(s)^2}{s}\right)^\epsilon \geq \left(\frac{\psi(t)^2}{t}\right)^\epsilon \tag{77}$$

for small t, so that

$$2\left(\psi(s)\left(\frac{r}{s}\right)^\alpha - k\right) \geq \psi(s)\left(\frac{r}{s}\right)^{2\beta-1}\left(\frac{\psi(t)^2}{t}\right)^\epsilon. \tag{78}$$

This together with (75) yields

$$\begin{aligned}\exp&\left\{-\frac{2}{s}(\psi(s) - h(s))\left(\psi(s)\left(\frac{r}{s}\right)^\alpha - k\right)\right\} \\ &\leq \exp\left\{-(1-\nu)\frac{\psi(s)^2}{s}\left(\frac{r}{s}\right)^{2\beta-1}\epsilon\right\} \to 0\end{aligned} \tag{79}$$

by (61) and a proper choice of $\nu < 1$, $\nu \to 1$. Finally, according to (48), one has $\Pr\{\xi < \psi \text{ in } (0, r), |\xi(r)| \leq k\} \to 1$ and

$$\frac{k^2}{r} = \frac{\psi(s)^2}{s}\left(\frac{r}{s}\right)^{2\beta-1} \to \infty \tag{80}$$

(see (61)). Putting this together with (70) and (69) we get

$$I(x) \geq \frac{\exp\left\{-\frac{x^2}{2s}\right\}}{\sqrt{2\pi s}}\,\eta, \tag{81}$$

where $\eta \leq 1$, but $\eta \to 1$ uniformly for $x \in \langle k, h(s)\rangle$. From (67) we get, using (38) and (66)

$$\begin{aligned} f_\psi(t) \geq & \int_{-\infty}^{h(s)} F'_{h|\xi(s)=x}(t) \frac{\exp\left\{-\frac{x^2}{2s}\right\}}{\sqrt{2\pi s}}\,\eta\,dx \\ & - \int_{-\infty}^{k} F'_{h|\xi(s)=x}(t)(2\pi s)^{-1/2} \exp\left\{-\frac{x^2}{2s}\right\} dx \\ \geq & F'_h(t)\eta - \int_{-\infty}^{k} \frac{h(s)-x}{t-s}(2\pi(t-s)2\pi s)^{-1/2} \exp\left\{-\frac{(\psi(t)-x)^2}{2(t-s)} - \frac{x^2}{2s}\right\} dx \\ = & F'_h(t)\eta - J \text{ (say).} \end{aligned} \tag{82}$$

Now $h(s) = (t - s/t)\nu\lambda(t) + (s/t)\psi(t) \geq (s/t)\psi(t)$, so that $(h(s) - x)(\psi(t) - x) > t - s$ for $x \leq 0$ and small t because $(\psi(t)^2/t) \to \infty$. But this implies that $(h(s) - x) \exp\{-(\psi(t) - x)^2/2(t - s)\}$ increases with x for $x \leq 0$. Thus

$$J \leq \frac{h(s)}{t-s}(2\pi(t-s))^{-1/2} \exp\left\{-\frac{(\psi(t)-k)^2}{2(t-s)}\right\}. \tag{83}$$

Because

$$\frac{k^2}{s} = \frac{\psi(s)^2}{s}\left(\frac{r}{s}\right)^{2\beta} \to 0, \tag{84}$$

we have by (49) and (63)

$$k = o(\sqrt{s}) = o(\psi(t)), \tag{85}$$

so that

$$\exp\left\{-\frac{(\psi(t)-k)^2}{2(t-s)}\right\} < \exp\left\{-\frac{\psi(t)^2}{4(t-s)}\right\} \tag{86}$$

for small t. Also by (53) and (73),

$$\psi(t) \leq \frac{2}{1-\alpha}(\psi(u) - u\psi'(u)) \tag{87}$$

for small t and $u \in \langle s, t\rangle$, so that

$$\psi(t) \leq \frac{2}{1-\alpha}\lambda(t). \tag{88}$$

Thus,

$$h(s) = \frac{t-s}{t}\nu\lambda(t) + \frac{s}{t}\psi(t) \leq \frac{2}{1-\alpha}\lambda(t). \tag{89}$$

Therefore, we may continue (83) using (86), (89), and (63) and (49),

$$J \leq \frac{\lambda(t)}{t} 2t(1-\alpha)^{-1}(t-s)^{-3/2} \exp\left\{-\frac{\psi(t)^2}{4(t-s)}\right\} = o\left(\frac{\lambda(t)}{t} G_\psi(t)\right). \tag{90}$$

From (82) we get by lemma 3.2 using $\nu \to 1$, $\eta \to 1$ and the definition of h, $f_\psi(t) \geq (\lambda(t)/t)G_\psi(t)(1 + o(1))$, which proves (52).

THEOREM 3.6. *Let* $0 < \delta < 1$, $\varphi \in \Pi$ *such that* φ *has a continuous derivative and* $\tau^{-\delta}\varphi \in \uparrow$. *Assume that*

$$T_\varphi < \infty, \quad \text{a.s.} \tag{91}$$

Then

$$t^{-1/2}\varphi(t) \to \infty \qquad \text{as} \quad t \to \infty. \tag{92}$$

Except perhaps for some mass at 0, T_φ *has a continuous density* D_φ (*so that* $\int_0^\infty D_\varphi(t)\,dt = \Pr\{T_\varphi > 0\} \leq 1$), *and if* $0 < \theta < \delta$ *and*

$$\begin{matrix}\gamma(t)\\ \Gamma(t)\end{matrix} = \begin{matrix}\inf\\ \sup\end{matrix}\left\{\varphi'(s)\colon t \leq s \leq t\left(1 + \left(\frac{t}{\varphi(t)^2}\right)^\theta\right)\right\}, \tag{93}$$

then

$$\limsup_{t\to\infty} D_\varphi(t)(\Gamma(t)G_\varphi(t))^{-1} \leq 1, \tag{94}$$

$$\liminf_{t\to\infty} D_\varphi(t)(\gamma(t)G_\varphi(t))^{-1} \geq 1. \tag{95}$$

Thus, if

$$1 \leq \frac{s}{t} \leq 1 + \left(\frac{t}{\varphi(t)^2}\right)^\theta \quad \text{implies} \quad \varphi'(s) \sim \varphi'(t) \qquad \text{as} \quad s \to \infty,\ t \to \infty, \tag{96}$$

then $D_\varphi(t) \sim \varphi'(t)G_\varphi(t)$ as $t \to \infty$.

PROOF. Let

$$\alpha = 1 - \delta, \qquad \epsilon \in (\theta, 1-\alpha), \qquad \psi = \tau\cdot\varphi\left(\frac{1}{\tau}\right), \qquad \hat{\xi} = \tau\cdot\xi\left(\frac{1}{\tau}\right). \tag{97}$$

The process $\hat{\xi}$ is again a standard Brownian motion (Lévy [34], p. 246 or Itô-McKean, p. 18, problem 3). We apply theorem 3.5 to $\hat{\xi}$ instead of ξ. With the help of $S_\psi = 1/T_\varphi$, all the assumptions are easily verified. Using $\psi - \tau\cdot\psi' = \varphi'(1/\tau)$ and $\epsilon > \theta$, one gets $\gamma(t) \leq \lambda(1/t) \leq \Lambda(1/t) \leq \Gamma(t)$ for large t. This together with $\varphi = \tau\psi(1/\tau)$ and $D_\varphi = (1/\tau^2)f_\psi(1/\tau)$ allows one to conclude (94) and (95) from (51) and (52). The rest is trivial.

It should be noticed that the assumptions $\varphi > 0$ and $\tau^{-\delta}\varphi \in \uparrow$ of the theorem may be weakened to $\varphi(t) > 0$ eventually as $t \to \infty$ and $t^{-\delta}\varphi(t)$ increases eventually as $t \to \infty$, because one may then replace φ on a finite interval appropriately. By (92) (the proof of (92) depends only on (91)!) this can always be done, and it does not affect the distribution of T_φ outside that finite interval.

COROLLARY 3.7. *Let* $\varphi \in \Pi$ *and* $\delta > 0$ *such that*

$$\tau^{-\delta}\varphi \in \uparrow. \tag{98}$$

Then

$$\Pr\{\xi(t) < \varphi(t) \text{ eventually as } t \to \infty\} = 1 \tag{99}$$

implies

$$\int_1^\infty t^{-1}\varphi(t)G_\varphi(t)\,dt < \infty. \tag{100}$$

PROOF. We may assume that

$$\varphi \leq \tau, \tag{101}$$

because if φ satisfies (98) and (99), then so does $\varphi \wedge \tau$ (by the law of large numbers), and if $\varphi \wedge \tau$ satisfies (100), then so does φ. We also may assume that $\delta < 1$. The proof proceeds by regularization.

Let σ be a nonnegative continuously differentiable function on R^1 such that $\sigma(t) = 0$, except if $t \in (\frac{1}{2}, 1)$ and such that $\int_{-\infty}^{\infty} \sigma(t)\,dt = 1$. Put $\varphi_0(t) = \int_0^\infty \varphi(s)\sigma(s - t)\,ds$. Then φ_0 has a continuous derivative satisfying, for $t > 0$,

$$\begin{aligned} \varphi_0'(t) &= -\int_0^\infty \varphi(s)\sigma'(s - t)\,ds \\ &= \int_0^\infty \sigma(s - t)\,d\varphi(s) \\ &= \int_0^\infty \sigma(s - t)s^\delta d\left(\frac{\varphi(s)}{s^\delta}\right) + \delta \int_0^\infty \sigma(s - t)\,\frac{\varphi(s)}{s}\,ds. \end{aligned} \tag{102}$$

Here the first integral on the right side is ≥ 0 and the integrand of the second vanishes for $s > t + 1$. Thus if $t > 1$,

$$\varphi_0'(t) \geq \frac{\delta}{t + 1}\,\varphi_0(t) > \frac{\delta}{2}\,\frac{\varphi_0(t)}{t}, \tag{103}$$

or equivalently,

$$t^{-\delta/2}\varphi(t) \quad \text{increases in } t \text{ for } t > 1. \tag{104}$$

Obviously,

$$\varphi \leq \varphi_0 \leq \varphi(\tau + 1). \tag{105}$$

By the left inequality,

$$\Pr\{\xi(t) < \varphi_0(t) \text{ eventually as } t \to \infty\} = 1, \tag{106}$$

so that theorem 3.6, (95) applied to φ_0 and $\delta/2$ yields $\int_1^\infty \gamma(t)G_{\varphi_0}(t)\,dt < \infty$, because D_{φ_0} is integrable. Now

$$\gamma(t) = \inf\left\{\varphi_0'(s) : t \leq s \leq t\left(1 + \frac{t^\theta}{\varphi_0(t)^{2\theta}}\right)\right\} \geq \frac{\delta}{4}\,\frac{\varphi_0(t)}{t} \tag{107}$$

for large t by (103), (104), and (92). Therefore we get

$$\int_1^\infty t^{-1}\varphi_0(t)G_{\varphi_0}(t)\,dt < \infty. \tag{108}$$

Because $x \exp\{-(x^2/2t)\}$ is decreasing in x for $x > \sqrt{t}$, this together with (92) (applied to φ_0) and (105) implies

$$\int_1^\infty t^{-3/2}\varphi(t+1)\exp\left\{-\frac{\varphi(t+1)^2}{2t}\right\}dt < \infty, \tag{109}$$

and by substitution, using (101), we get (100).

COROLLARY 3.8. *Let $\varphi \in \mathrm{II}$ such that $\tau^{-\epsilon}\varphi \in \uparrow$ and $\tau^{-q}\varphi \in \downarrow$ for some $0 < \epsilon < q$ and let $r > 0$. Then*

$$ET_\varphi^r < \infty \tag{110}$$

implies

$$\int_1^\infty t^{r-1}\varphi(t)G_\varphi(t)\,dt < \infty. \tag{111}$$

Conversely, if $\epsilon = \frac{1}{2}$, then (111) *implies* (110).

PROOF. We may again assume (101), because the validity of the assumptions of the present corollary and the truth values of both (110) and (111) are not affected by the passage from φ to $\varphi \wedge \tau$ (for example, (110) implies $ET_{\varphi\wedge\tau}^r = E(T_\varphi^r \vee T_\tau^r) < \infty$, because $ET_\tau^r < \infty$ by (94)).

The first part of the present corollary follows now in the same way as corollary 3.7 and may be left to the reader.

The second part is proved similarly. Put $\varphi_1 = \varphi_0(\tau - 1)$ (remember that φ_0 was defined not only on R^+ but on R^1). Inequality (103) with $\delta = \frac{1}{2}$ yields $\varphi_1'(t) \geq (1/2t)\varphi_1(t)$, or equivalently, $t^{-(1/2)}\varphi_1(t)$ increases in t for $t > 2$. Formula (102) for $\delta = q$ and the remark that the first integral on its right side is now ≤ 0, and the integrand of the second vanishes for $s < t$, yields $\varphi_0'(t) \leq (q/t)\varphi_0(t)$ or

$$\varphi_1'(t) \leq \frac{2q}{t}\varphi_1(t) \tag{112}$$

for $t > 2$. Inequality (105) becomes $\varphi_1 \leq \varphi \leq \varphi_1(\tau + 1)$. Applying the right inequality here to (111) we see by a time substitution using (101) that

$$\int_1^\infty t^{r-1}\varphi_1(t)G_{\varphi_1}(t)\,dt < \infty. \tag{113}$$

Now the easy half of the law of the iterated logarithm (Erdös [20], theorem 1; Feller [22], pp. 197–198, problems 7 and 8; Itô-McKean [25], p. 34, under an additional restriction, or pp. 161–164) allows us to conclude that $T_{\varphi_1} < \infty$, a.s., so that we may apply theorem 3.6, (94) to φ_1. The inequality $\varphi_1 \leq \varphi$ implies $ET_\varphi^r \leq ET_{\varphi_1}^r$. In order to prove (110), it is therefore sufficient to show that

$$\int_1^\infty t^r\Gamma(t)G_{\varphi_1}(t)\,dt < \infty, \tag{114}$$

where for large t,

$$\Gamma(t) \leq \sup\{\varphi_1'(s) \colon t \leq s \leq 2t\} \leq \frac{2q}{t}\varphi_1(2t) \tag{115}$$

by (112). But (112) also implies $\varphi_1(2t) \leq 2^{2q}\varphi_1(t)$, so that (114) follows from (113).

4. Almost sure invariance principles

The first two lemmas below and their proofs are routine.

LEMMA 4.1. *Let $T_1, T_2, \cdots$ be nonnegative random variables, let $\mathfrak{A}_0 = \{\emptyset, \Omega\}$ where Ω is the underlying probability space, let $\mathfrak{A}_1, \mathfrak{A}_2, \cdots$ be a nondecreasing sequence of σ-algebras of events such that T_n is $\mathfrak{A}_n$-measurable and such that $E(T_n^2|\mathfrak{A}_{n-1})$ is defined (that is, the measure on $\mathfrak{A}_n$ which is obtained by integrating T_n^2 with the basic probability measure* Pr *remains σ-finite if restricted to $\mathfrak{A}_{n-1}$; this is the case, for example, if $T_n^2 \leq U$,* a.s., *where U is any $\mathfrak{A}_{n-1}$-measurable random variable). We do not assume, however, that $E(T_n^2) < \infty$. Put*

$$W_n = \sum_{i \leq n} E(T_i|\mathfrak{A}_{i-1}) \tag{116}$$

and let $0 < V_1 \leq V_2 \leq \cdots$ be random variables such that V_n is $\mathfrak{A}_{n-1}$-measurable. Finally, let $g \in \Pi \cap \uparrow$ be such that $g(t) \to \infty$ as $t \to \infty$. Assume that

$$\sum_{n \geq 1} g(V_n)^{-2} E(T_n^2|\mathfrak{A}_{n-1}) \in L^1 \tag{117}$$

or equivalently,

$$\sum_{n \geq 1} E(g(V_n)^{-2} E(T_n^2|\mathfrak{A}_{n-1})) = \sum_{n \geq 1} E(g(V_n)^2 T_n^2) \leq \infty, \tag{118}$$

and

$$W_n - V_n = o(g(V_n)) \quad \text{as} \quad n \to \infty, \qquad \text{a.s. on} \quad \{V_n \to \infty\}. \tag{119}$$

Then a.s. *on $\{V_n \to \infty\}$,*

$$\sum_{i \leq n} T_i - V_n = o(g(V_n)) \qquad \text{as} \quad n \to \infty. \tag{120}$$

PROOF. If $T'_n = g(V_n)^{-1}(T_n - E(T_n|\mathfrak{A}_{n-1}))$, then T'_n is $\mathfrak{A}_n$-measurable, $E(T'_n|\mathfrak{A}_{n-1}) = 0$, a.s. and

$$\sum_{n \geq 1} E(T_n'^2) \leq \sum_{n \geq 1} E(g(V_n)^{-2} E(T_n^2|\mathfrak{A}_{n-1})) < \infty; \tag{121}$$

consequently (see, for example, Loève [35], p. 387),

$$\sum_{n \geq 1} T'_n < \infty, \qquad \text{a.s.} \tag{122}$$

Thus by Kronecker's lemma (Loève [35], p. 238),

$$g(V_n)^{-1} \sum_{i \leq n} (T_i - E(T_i|\mathfrak{A}_{i-1})) \to 0, \qquad \text{a.s. on} \quad \{V_n \to \infty\}. \tag{123}$$

By (119) this implies (120).

LEMMA 4.2. *Let $f \in \Pi$ such that*

$$f \in \uparrow, \qquad \tau^{-1} f \in \downarrow, \tag{124}$$

and put

$$\begin{aligned} g &= (\tau f)^{1/2} \log \tau, \\ h &= (\tau f)^{1/4} \log \tau. \end{aligned} \tag{125}$$

Then

(126) $\Pr\{\xi(t_n) - \xi(s_n) = o(h(t_n))$ as $n \to \infty$ for any sequences t_n, s_n such that $t_n \to \infty,\ s_n \to \infty,\ t_n - s_n = o(g(t_n))$ as $n \to \infty\} = 1$.

PROOF. Let $\epsilon > 0$. Lemma 3.2, (38) yields

$$\sum_{n \geq 1} \Pr\{\max_{|t-n| \leq \epsilon g(n)} |\xi(t) - \xi(n - \epsilon g(n))| > \sqrt{8\epsilon}\, h(n)\} \tag{127}$$
$$\leq 4 \sum_{n\geq 1} \Pr\{\xi(n + \epsilon g(n)) - \xi(n - \epsilon g(n)) > \sqrt{8\epsilon}\, h(n)\}$$
$$= 4 \sum_{n \geq 1} \Pr\{\xi(1) > 2\sqrt{\log n}\} < \infty,$$

so that with probability one, eventually

$$\max\{|\xi(t) - \xi(s)|: |t - n| \leq \epsilon g(n),\ |s - n| \leq \epsilon g(n)\} \leq 4\sqrt{2\epsilon}\, h(n). \tag{128}$$

If $t_k \to \infty$, $s_k \to \infty$, and $t_k - s_k = o(g(t_k))$ as $k \to \infty$, and if n_k is such that $n_k - 1 < t_k \leq n_k$, then

$$|t_k - n_k| < \epsilon g(n_k), \qquad |s_k - n_k| < \epsilon g(n_k) \tag{129}$$

for sufficiently large k. Outside the set of measure zero where (128) fails we have, therefore,

$$|\xi(t_k) - \xi(s_k)| \leq 4\sqrt{2\epsilon}\, h(n_k) < 8\sqrt{\epsilon}\, h(t_k) \tag{130}$$

for large k by (124). Because $\epsilon > 0$ was arbitrary, the lemma follows.

The next theorem is a generalization to martingales of a result of A. B. Skorohod ([40], p. 180) on sums of independent random variables. The proof (which we omit) follows similar lines as Skorohod's and will be presented by F. Jonas, at Erlangen, in his Diplomarbeit (see also Dambis [9], theorem 7, and Dubins-Schwarz [18]). The phrase 'without loss of generality' in the next theorem is used in a specific sense, namely: there is a new probability space on which random variables

$$\hat{Y}_1, \hat{Y}_2, \cdots \tag{131}$$

are defined such that the sequence (132) and the sequence (131) have the same distribution, and such that theorem 4.3 (where now 'without loss of generality' is to be omitted) holds for the $\hat{Y}_n$ (ξ and the T_n are of course defined on the new space). This interpretation is valid for the rest of the paper.

THEOREM 4.3. *Let*

$$Y_1, Y_2, \cdots \tag{132}$$

be random variables such that for all n, $E(Y_n^2|Y_1, \cdots, Y_{n-1})$ *is defined and* $E(Y_n|Y_1, \cdots, Y_{n-1}) = 0$, a.s. *Then, without loss of generality, there is a Brownian motion* ξ *together with a sequence of nonnegative random variables* $T_1, T_2, \cdots$ *such that*

$$\sum_{i \leq n} Y_i = \xi\Big(\sum_{i \leq n} T_i\Big), \quad \text{a.s.} \tag{133}$$

for all n. *Moreover, if* $\mathfrak{A}_n$ *is generated by* $Y_1, \cdots, Y_n$ *and* $\xi(t)$ *for* $0 \leq t \leq \sum_{i \leq n} T_i$, *then* T_n *is* $\mathfrak{A}_n$*-measurable,* $\xi(\sum_{i\leq n} T_i + s) - \xi(\sum_{i\leq n} T_i)$ *is independent of* $\mathfrak{A}_n$ *for any* $s > 0$, $E(T_n|\mathfrak{A}_{n-1})$ *is defined and*

$$E(T_n|\mathfrak{A}_{n-1}) = E(Y_n^2|\mathfrak{A}_{n-1}) = E(Y_n^2|Y_1, \cdots, Y_{n-1}), \qquad \text{a.s.} \tag{134}$$

If k *is a real number* >1 *and* $E(Y_n^{2k}|Y_1, \cdots, Y_{n-1})$ *is defined, then* $E(T_n^k|\mathfrak{A}_{n-1})$ *is also defined and*

$$E(T_n^k|\mathfrak{A}_{n-1}) \leq L_k E(Y_n^{2k}|\mathfrak{A}_{n-1}) = L_k E(Y_n^{2k}|Y_1, \cdots, Y_{n-1}), \qquad \text{a.s.,} \tag{135}$$

where L_k *are constants which depend only on* k.

If the Y_n *are mutually independent, then the* T_n *are mutually independent. If in addition the* Y_n *are identically distributed or have a moment generating function in a neighborhood of* 0, *then the same holds for the* T_n *(with the same neighborhood).*

THEOREM 4.4. *Let* $X_1, X_2, \cdots$ *be random variables such that* $E(X_n^2|X_1, \cdots, X_{n-1})$ *is defined and* $E(X_n|X_1, \cdots, X_{n-1}) = 0$, a.s. *for all* n. *Put* $S_n = \sum_{i\leq n} X_i$ *and* $V_n = \sum_{i\leq n} E(X_i^2|X_1, \cdots, X_{i-1})$, *where, in order to avoid trivial complications, we assume* $V_1 = EX_1^2 > 0$. *Let* $f \in$ II *such that*

$$f \in \uparrow, \qquad \tau^{-1}f \in \downarrow. \tag{136}$$

Assume that

$$V_n \to \infty, \qquad \text{a.s.} \tag{137}$$

as $n \to \infty$ *and*

$$\sum_{n\geq 1} f(V_n)^{-1} \int_{x^2 > f(V_n)} x^2 d \Pr \{X_n \leq x|X_1, \cdots, X_{n-1}\} < \infty, \qquad \text{a.s.} \tag{138}$$

Let S *be the (random) function on* $R^+ \cup \{0\}$ *obtained by interpolating* S_n *at* V_n *in such a way that* $S(0) = 0$ *and* S *is constant in each* $\langle V_n, V_{n+1})$ *(or alternatively, is linear in each* $\langle V_n, V_{n+1}\rangle$*). Then without loss of generality there is a Brownian motion* ξ *such that*

$$S(t) = \xi(t) + o(\log t(tf(t))^{1/4}), \qquad \text{a.s. as} \quad t \to \infty. \tag{139}$$

PROOF. Let $r > 0$ and $\rho_r(x) = 2r^{1/2} - r/x$ for $x > r^{(1/2)}$. Then ρ_r maps $(r^{(1/2)}, \infty)$ topologically onto $(r^{(1/2)}, 2r^{(1/2)})$ such that always

$$\rho_r(x) \leq x, \tag{140}$$

and $\rho_r(x)$ is jointly Borel measurable in r, x for $0 < r^{(1/2)} < x$. Now let

$$\tilde{X}_n = \begin{cases} X_n & \text{if} \quad X_n^2 \leq f(V_n), \\ \operatorname{sign}(X_n)\rho_{f(V_n)}(|X_n|) & \text{if} \quad X_n^2 > f(V_n), \end{cases} \tag{141}$$

and $Y_n = \tilde{X}_n - E(\tilde{X}_n|\mathfrak{L}_{n-1})$, where $\mathfrak{L}_n$ is the σ-algebra generated by $X_1, \cdots, X_n$. Then the properties of ρ_r imply that $\mathfrak{L}_n$ is generated also by $\tilde{X}_1, \cdots, \tilde{X}_n$ and is therefore also the σ-algebra generated by $Y_1, \cdots, Y_n$. Moreover, $E(\tilde{X}_n^{2k}|\mathfrak{L}_{n-1})$ and $E(Y_n^{2k}|\mathfrak{L}_{n-1})$ are defined for any k, because

$$|\tilde{X}_n| \leq 2f(V_n)^{1/2} \quad \text{and} \quad |Y_n| \leq 4f(V_n)^{1/2} \tag{142}$$

and V_n is $\mathfrak{L}_{n-1}$-measurable. Also

$$E(Y_n|\mathfrak{L}_{n-1}) = 0, \qquad \text{a.s.} \tag{143}$$

and

$$(144) \qquad E(Y_n^4|\mathfrak{L}_{n-1}) = E(\tilde{X}_n^4|\mathfrak{L}_{n-1}) - 4E(\tilde{X}_n^3|\mathfrak{L}_{n-1})E(\tilde{X}_n|\mathfrak{L}_{n-1}) + 6E(\tilde{X}_n^2|\mathfrak{L}_{n-1})E(\tilde{X}_n|\mathfrak{L}_{n-1})^2 - 3E(\tilde{X}_n|\mathfrak{L}_{n-1})^4 \leq 44f(V_n)E(\tilde{X}_n^2|\mathfrak{L}_{n-1}),$$

again using $|\tilde{X}_n| \leq 2f(V_n)^{1/2}$, which is $\mathfrak{L}_{n-1}$-measurable. With the help of $\tilde{X}_n^2 \leq X_n^2$, one gets for $n \geq 1$,

$$(145) \qquad E(Y_n^4|\mathfrak{L}_{n-1}) \leq 44f(V_n)(V_n - V_{n-1}).$$

Here $V_0 = 0$. Because $E(X_n|\mathfrak{L}_{n-1}) = 0$, a.s., we have

$$(146) \qquad E(\tilde{X}_n|\mathfrak{L}_{n-1})^2 = E(\tilde{X}_n - X_n|\mathfrak{L}_{n-1})^2 \leq \left(2\int_{x^2>f(V_n)} |x| d \Pr \{X_n \leq x|\mathfrak{L}_{n-1}\}\right)^2 \leq 4 \int_{x^2>f(V_n)} x^2 d \Pr \{X_n \leq x|\mathfrak{L}_{n-1}\};$$

thus

$$(147) \qquad |E(Y_i^2|\mathfrak{L}_{i-1}) - E(X_i^2|\mathfrak{L}_{i-1})| = |E(\tilde{X}_i^2|\mathfrak{L}_{i-1}) - E(\tilde{X}_i|\mathfrak{L}_{i-1})^2 - E(X_i^2|\mathfrak{L}_{i-1})| \leq E(\tilde{X}_i|\mathfrak{L}_{i-1})^2 + |E(\tilde{X}_i^2 - X_i^2|\mathfrak{L}_{i-1})| \leq 6 \int_{x^2>f(V_i)} x^2 d \Pr \{X_i \leq x|\mathfrak{L}_{i-1}\}.$$

Summing up and using the fact that $f(V_i)$ is nondecreasing in i, we get

$$(148) \qquad |W_n - V_n| \leq 6Af(V_n), \qquad \text{a.s.},$$

where

$$(149) \qquad W_n = \sum_{i \leq n} E(Y_i^2|\mathfrak{L}_{i-1}),$$

and A is the random variable represented by the series in (138). Now let $g = (\tau f)^{1/2} \log (\tau \vee 2)$ and $h = (\tau f)^{1/4} \log (\tau \vee 2)$. Then by (148), (136) and (137),

$$(150) \qquad W_n - V_n = o(g(V_n)), \qquad \text{a.s. as} \quad n \to \infty.$$

Inequality (138) implies

$$(151) \qquad \sum_{n \geq 1} \Pr \{\tilde{X}_n \neq X_n|\mathfrak{L}_{n-1}\} \leq \sum_{n \geq 1} \Pr \{X_n^2 > f(V_n)|\mathfrak{L}_{n-1}\} < \infty, \qquad \text{a.s.}$$

Thus by P. Lévy's conditional form of the Borel-Cantelli lemma (Lévy ([33], p. 249); Dubins-Freedman [15]; it also follows easily from (122) in the proof of lemma 4.1 above, letting T_n be the indicator of $\{\tilde{X}_n \neq X_n\}$ and $g = \tau$, verifying (117) in a similar way as (156) below),

$$(152) \qquad \Pr \{\tilde{X}_n \neq X_n \text{ infinitely often}\} = 0,$$

in particular, by (137),

$$(153) \qquad |\sum_{i \leq n} \tilde{X}_i - \sum_{i \leq n} X_i| = o(h(V_n)), \qquad \text{a.s.}$$

Moreover,

$$(154)\qquad \begin{aligned} \Big|\sum_{i\le n} Y_i - \sum_{i\le n}\tilde{X}_i\Big| &= \Big|\sum_{i\le n} E(\tilde{X}_i|\mathfrak{L}_{i-1})\Big| \\ &= \Big|\sum_{i\le n} E(\tilde{X}_i - X_i|\mathfrak{L}_{i-1})\Big| \\ &\le 2\sum_{i\le n}\int_{x^2>f(V_i)} |x|d\Pr\{X_i\le x|\mathfrak{L}_{i-1}\} \\ &\le 2\sum_{i\le n} f(V_i)^{-1/2}\int_{x^2>f(V_i)} x^2 d\Pr\{X_i\le x|\mathfrak{L}_{i-1}\} \\ &\le 2f(V_n)^{1/2}A = o(h(V_n)), \qquad \text{a.s.} \end{aligned}$$

by (136) and (137). Together with (153) we have, therefore,

$$(155)\qquad \Big|\sum_{i\le n} Y_i - \sum_{i\le n} X_i\Big| = o(h(V_n)), \qquad \text{a.s.}$$

We now apply theorem 4.3 to $Y_1, Y_2, \cdots$. Equality (143) and the remarks preceding it show that the assumptions of the theorem are satisfied, and moreover that $E(T_n^k|\mathfrak{A}_{n-1})$ is defined for any k, and (135) holds. Remember that the phrase 'without loss of generality' in theorem 4.3 represents the passage to a new probability space, and that only random variables (defined on the original space) which are Borel functions of $Y_1, Y_2, \cdots$ have an immediate meaning in the new space. Fortunately the X_n, and therefore also the other random variables which have occurred thus far in this proof, are Borel functions of the Y_j (the somewhat clumsy kind of truncation in the definition of $\tilde{X}_n$ has been used just to achieve this aim).

Next we are going to apply lemma 4.1 to the T_n and $\mathfrak{A}_n$ provided by theorem 4.3 and to the V_n and g of the present proof. All assumptions of the lemma except (117) are easily checked, if one notices that by (134) the definition of W_n in the lemma and its definition (149) in the present proof are equivalent, so that (150) takes care of (119). To prove (117), use (135) and the definition of g, and then (145) to get

$$(156)\qquad \begin{aligned} &\sum_{n\ge 1} g(V_n)^{-2}E(T_n^2|\mathfrak{A}_{n-1}) \\ &\quad\le \sum_{n\ge 1}(V_nf(V_n))^{-1}(\log(V_n\vee 2))^{-2}L_2E(Y_n^4|Y_1,\cdots,Y_{n-1}) \\ &\quad\le \text{const.}\sum_{n\ge 1}(V_n - V_{n-1})V_n^{-1}(\log(V_n\vee 2))^{-2} \\ &\quad\le \text{const.}\left((\log(V_1\vee 2))^{-2} + \int_{V_1}^{\infty}\frac{dt}{t(\log(t\vee 2))^2}\right), \qquad \text{a.s.,} \end{aligned}$$

which is a finite constant. This proves (117). Now the conclusion (120) of the lemma together with (137) yields

$$(157)\qquad \sum_{i\le n} T_i - V_n = o(g(V_n)), \qquad \text{a.s.;}$$

thus by lemma 4.2,

$$(158)\qquad \xi\Big(\sum_{i\le n} T_i\Big) - \xi(V_n) = o(h(V_n)), \qquad \text{a.s.}$$

Together with (133) and (155), we obtain

$$S_n - \xi(V_n) = o(h(V_n)), \qquad \text{a.s. as} \quad n \to \infty. \tag{159}$$

Now by (134) and (156),

$$(W_n - W_{n-1})^2 = (E(T_n|\mathfrak{A}_{n-1}))^2 \leq E(T_n^2|\mathfrak{A}_{n-1}) = o(g(V_n)^2), \qquad \text{a.s.,} \tag{160}$$

so that by (150),

$$V_n - V_{n-1} = o(g(V_n)), \qquad \text{a.s.,} \tag{161}$$

using that $g(V_n)$ is a.s. nondecreasing in n. We are now ready to prove (139). Given $t > 0$, let $n = n(t)$ be the largest integer such that $V_{n-1} \leq t \leq V_n$. Then (159) implies

$$S_n - \xi(V_n) = o(h(t)), \qquad \text{a.s. as} \quad t \to \infty, \tag{162}$$

and (161) and lemma 4.2 together imply

$$\xi(V_n) - \xi(V_{n-1}) = o(h(t)), \qquad \text{a.s.} \tag{163}$$

and

$$\xi(V_n) - \xi(t) = o(h(t)), \qquad \text{a.s.} \tag{164}$$

Bearing this and the definition of $S(t)$ in mind, we get

$$\begin{aligned} |S(t) - \xi(t)| &\leq |S(t) - S_n| + |S_n - \xi(V_n)| + |\xi(V_n) - \xi(t)| \\ &\leq |S_{n-1} - S_n| + o(h(t)) \\ &\leq |S_{n-1} - \xi(V_{n-1})| + |\xi(V_{n-1}) - \xi(V_n)| + |\xi(V_n) - S_n| + o(h(t)) \\ &= o(h(t)) \text{ a.s. as } t \to \infty. \end{aligned} \tag{165}$$

This proves the theorem.

COROLLARY 4.5. *Let $X_1, X_2, \cdots$ be random variables such that*

$$E(X_n^2|X_1, \cdots, X_{n-1})$$

is defined and $E(X_n|X_1, \cdots, X_{n-1}) = 0$, a.s. *for all n. Put $S_n = \sum_{i \leq n} X_i$ and $V_n = \sum_{i \leq n} E(X_i^2|X_1, \cdots, X_{i-1})$, and assume $V_1 = EX_1^2 > 0$ and $V_n \to \infty$*, a.s. *and*

$$\sum_{n \geq 1} V_n^{-1}(\log V_n re)^5 \int_{x^2 > V_n(\log V_n re)^{-5}} x^2 d \Pr\{X_n \leq x|X_1, \cdots, X_{n-1}\} < \infty, \text{ a.s.} \tag{166}$$

Let $\varphi \in \Pi$ such that

$$\tau^{-1/2}\varphi \in \uparrow. \tag{167}$$

Then

$$\Pr\{S_n < \varphi(V_n) \text{ eventually as } n \to \infty\} = 1 \text{ or } 0 \tag{168}$$

according to

$$\int_1^\infty t^{-1}\varphi(t)G_\varphi(t)\, dt < \infty \text{ or } = \infty. \tag{169}$$

PROOF. The easy half of the law of the iterated logarithm for Brownian motion states that convergence in (169) implies

(170) $$\Pr\{\xi(t) < \varphi(t) \text{ eventually as } t \to \infty\} = 1$$

(see the end of the proof of corollary 3.8). Apply now theorem 4.4 with $f = \tau(\log(\tau \vee e)^{-5} ve$ to get (without loss of generality) a Brownian motion ξ such that

(171) $$S(t) = \xi(t) + o(t^{1/2}(\log t)^{-1/4}), \qquad \text{a.s.}$$

as $t \to \infty$, where we choose S to be constant in each $\langle V_{n-1}, V_n)$. It is easy to see that (167) and convergence in (169) remain true for $\varphi_1 = \varphi \wedge \tau^{(1/2)} (\log(\tau \vee e))^{(1/4)}$ and also for $\varphi_2 = \varphi_1 - \tau^{(1/2)} (\log(\tau \vee e))^{-(1/4)}$. Thus by the remark at the beginning of the proof,

(172) $$\Pr\{\xi(t) < \varphi_1(t) - t^{(1/2)}(\log t)^{-(1/4)} \text{ eventually as } t \to \infty\} = 1.$$

Using (171), we get

(173) $$\Pr\{S(t) < \varphi(t) \text{ eventually as } t \to \infty\} = 1,$$

and hence the corresponding case in (168).

Conversely, assume $\Pr\{S_n < \varphi(V_n) \text{ eventually}\} > 0$. Because $\varphi \in \uparrow$, this implies

(174) $$\Pr\{S(t) < \varphi(t) \text{ eventually as } t \to \infty\} > 0,$$

and by (171) and a zero-one law for ξ

(175) $$\Pr\{\xi(t) < \varphi(t) + t^{1/2}(\log t)^{-1/4} \text{ eventually as } t \to \infty\} = 1.$$

From the remark at the beginning of the proof, it follows that

(176) $$\Pr\{\xi(t) < t^{1/2}(\log t)^{1/4} \text{ eventually as } t \to \infty\} = 1.$$

Thus,

(177) $$\Pr\{\xi(t) < \varphi_3(t) \text{ eventually as } t \to \infty\} = 1,$$

where

(178) $$\begin{aligned} \varphi_3 &= \varphi_1 + \tau^{1/2}(\log(\tau \vee e))^{-1/4} \\ &\geq (\varphi + \tau^{1/2}(\log(\tau \vee e))^{-1/4}) \wedge \tau^{1/2}(\log(\tau \vee e))^{1/4}. \end{aligned}$$

Now $\tau^{-1/4}\varphi_3 \in \uparrow$, so that corollary 3.7 yields $\int_1^\infty t^{-1}\varphi_3(t)G_{\varphi_3}(t)\,dt < \infty$, which implies convergence in (169).

THEOREM 4.6. *Let $X_1, X_2, \cdots$ be a sequence of independent identically distributed random variables such that $EX_1 = 0$ and $EX_1^2 = 1$. Let $r > 0$ and $z > 2(r + 1)$ and $E|X_1|^z < \infty$. Then, without loss of generality, there is a Brownian motion ξ such that if S takes the constant value $S_n = \sum_{i \leq n} X_i$ in the interval $\langle n, n + 1)$ for each n, then*

(179) $$E(\sup\{t: |S(t) - \xi(t)| \geq t^{(1/2)-\delta}\})^r < \infty$$

for some $0 < \delta < \frac{1}{2}$.

PROOF. Apply theorem 4.3 to the X_n. Then by (135) and (134), $E(T_1^{z/2}) < \infty$ and $ET_1 = 1$. We may assume that $z < 4(r + 1)$. Apply theorem 3 of L. E. Baum and M. Katz [2] to get

$$(180)\qquad \sum_{m\geq 1} m^{r-1}\Pr\left\{\sup_{n\geq m}\frac{1}{n^{1-\delta'}}\left|\sum_{i\leq n} T_i - n\right| \geq 1\right\} < \infty$$

for some $0 < \delta' < 1$. Now if $\delta < \delta'/2$, we have

$$\begin{aligned}(181)\qquad \Pr\{|S(t) - \xi(t)| > t^{1/2-\delta} \text{ for some } t \geq m\} &\\ \leq \Pr\{|S(t) - \xi(t)| \geq t^{1/2-\delta} &\text{ for some } t \geq m,\\ \left|\sum_{i\leq n} T_i - n\right| < n^{1-\delta'} &\text{ for all } n \geq m\}\\ + \Pr\left\{\sup_{n\geq m}\frac{1}{n^{1-\delta'}}\left|\sum_{i\leq n} T_i - n\right| \geq 1\right\} &= P_m^1 + P_m^2 \quad \text{(say)}.\end{aligned}$$

Using (133) and the definition of $S(t)$, we obtain

$$\begin{aligned}(182)\qquad P_m^1 &\leq \sum_{n\geq m}\Pr\left\{\sup_{|t-n|<n^{1-\delta'}}|\xi(t) - \xi(n - n^{1-\delta'})| \geq \frac{n^{1/2-\delta}}{2}\right\}\\ &\leq \text{const.}\sum_{n\geq m} n^{-\epsilon}\exp\left\{-\frac{n^{2\epsilon}}{16}\right\}\end{aligned}$$

by lemma 3.2, (39), where we put $\epsilon = (\delta'/2) - \delta > 0$. Thus, $\sum_{m\geq 1} m^{r-1}P_m^1 < \infty$. By (180) we have $\sum_{m\geq 1} m^{r-1}P_m^2 < \infty$, so that $\sum_{m\geq 1} m^{r-1}\Pr\{A \geq m\} < \infty$, where $A = \sup\{t\colon |S(t) - \xi(t)| \geq t^{1/2-\delta}\}$. But this implies $EA^r < \infty$.

COROLLARY 4.7. *Let the assumptions of theorem* 4.6 *be satisfied. Put* $S_n = \sum_{i\leq n} X_i$, *and let* $\varphi \in \Pi$ *be such that*

$$(183)\qquad \tau^{-1/2}\varphi \in \uparrow,$$

$$(184)\qquad \tau^{-q}\varphi \in \downarrow$$

for some $q > \frac{1}{2}$. *Moreover, let* $\hat{T} = \sup\{n\colon S_n \geq \varphi(n)\}$. *Then*

$$(185)\qquad E\hat{T}^r < \infty$$

if and only if

$$(186)\qquad \int_1^\infty t^{r-1}\varphi(t)G_\varphi(t)\,dt < \infty.$$

PROOF. Let ξ be such that (179) holds, and put for any continuous function ψ on R^+,

$$(187)\qquad \begin{aligned}\hat{T}_\psi &= \sup\{n\colon S_n \geq \psi(n)\},\\ \tilde{T}_\psi &= \sup\{t\colon S(t) \geq \psi(t)\}.\end{aligned}$$

The following considerations are somewhat analogous to those used in the proof of corollary 4.5. If $E\hat{T}^r = E\hat{T}_\varphi^r < \infty$, then $E\tilde{T}_\varphi^r < \infty$, which by (179) implies

$$(188)\qquad ET^r_{\varphi+\tau^{1/2-\delta}} < \infty.$$

This together with the second part of corollary 3.8 applied to $\tau^{(1/2)}\log(\tau \vee 2)$ yields

$$(189)\qquad ET^r_{\varphi_0+\tau^{1/2-\delta}} < \infty,$$

where $\varphi_0 = \varphi \wedge \tau^{1/2}\log(\tau \vee 2)$. The first part of corollary 3.8 with $\epsilon = \frac{1}{2} - \delta$ now gives

$$\int_1^\infty t^{r-1}(\varphi_0(t) + t^{1/2-\delta})G_{\varphi_0+\tau^{1/2-\delta}}(t)\,dt < \infty, \tag{190}$$

which implies (186).

Conversely, if (186) holds, then one may replace φ by φ_0 and then by $\varphi_0 - \tau^{(1/2)-\delta}$, so that by the second part of corollary 3.8,

$$ET^r_{\varphi_0-\tau^{1/2-\delta}} < \infty. \tag{191}$$

Here we have used the remark after the proof of theorem 3.6. Inequality (191) implies

$$ET^r_{\varphi-\tau^{1/2-\delta}} < \infty. \tag{192}$$

Thus by (179), $E\tilde{T}^r_\varphi < \infty$, which implies (185).

THEOREM 4.8. *Let $X_1, X_2, \cdots$ be independent identically distributed random variables such that*

$$\begin{aligned} EX_1 &= 0,\\ EX_1^2 &= 1,\\ Ee^{uX_1} &< \infty \quad \text{for} \quad |u| < \epsilon, \end{aligned} \tag{193}$$

where $\epsilon > 0$. Then without loss of generality there is a Brownian motion ξ such that if S is obtained as in theorem 4.6, and if $\frac{1}{4} < b < \frac{1}{2}$, $0 < 3a < 4b - 1$, then

$$\Pr\{|S(t) - \xi(t)| > t^b \text{ for some } t > s\} = o(e^{-s^a}). \tag{194}$$

PROOF. Apply theorem 4.3 to the X_n. Then $T_1, T_2, \cdots$ are independent, identically distributed, and one has

$$ET_1 = 1, \tag{195}$$

$$\beta(u) = E\exp\{u(T_1 - 1)\} < \infty \qquad \text{for} \quad |u| < \epsilon. \tag{196}$$

Determine a' and c such that $a < a'$, $1 + a' < 2c$, $c < 2b - a$. For $u > 0$ we have

$$\Pr\{\sum_{i\le n}(T_i - 1) > n^c\} \le E\exp\{u\sum_{i\le n}(T_i - 1)\}e^{-un^c} = \beta(u)^n e^{-un^c}. \tag{197}$$

Now by (195) there is a $d > 0$ such that for u sufficiently small $\beta(u) \le 1 + du^2$. The inequalities $1 + a' < 2c$ and $c < 2b < 1$ imply $a' < c$; thus for large n and for $u = n^{a'-c}$,

$$\begin{aligned} \beta(n^{a'-c})^n &= \left[\left(1 + \frac{d}{n^{2(c-a')}}\right)^{n^{2(c-a')}}\right]^{n^{1-2(c-a')}}\\ &\le \exp\{2dn^{1-2(c-a')}\}. \end{aligned} \tag{198}$$

Hence by $1 + a' < 2c$, $\beta(n^{a'-c})^n \exp\{-n^{a'-c}n^c\} \le \exp\{-\frac{1}{2}n^{a'}\}$ for large n. Together with (197) and a similar consideration for $\Pr\{\sum_{i\le n}(T_i - 1) < -n^c\}$, we get

$$\Pr\{|\sum_{i\le n} T_i - n| > n^c\} \le 2\exp\{-\tfrac{1}{2}n^{a'}\} \tag{199}$$

for large n. Because $a < 1$, it is sufficient to prove (194) for integers $s = m$:

$$\text{(200)} \quad \begin{aligned} &\Pr\{|S(t) - \xi(t)| > t^b \text{ for some } t > m\} \\ &\leq \Pr\{|S(t) - \xi(t)| > t^b \text{ for some } t > m, |\sum_{i \leq n} T_i - n| \leq n^c \text{ for all } n \geq m\} \\ &+ \sum_{n \geq m} \Pr\{|\sum_{i \leq n} T_i - n| > n^c\} \\ &= P_m^1 + P_m^2 \quad \text{(say)}. \end{aligned}$$

By (199), $P_m^2 = o(e^{-m^a})$. By the definition of $S(t)$, we get

$$\text{(201)} \quad \begin{aligned} P_m^1 &\leq \sum_{n \geq m} \Pr\left\{\sup_{|t-n| \leq n^c} |\xi(t) - \xi(n)| > \frac{n^b}{2}\right\} \\ &\leq \sum_{n \geq m} \text{const. } n^{-b+c/2} \exp\left\{-\frac{n^{2b-c}}{16}\right\} \end{aligned}$$

using lemma 3.2, (39). Thus, because $a < 2b - c$, $P_m^1 = o(e^{-m^a})$, proving (194).

COROLLARY 4.9. *Let the assumptions of theorem* 4.8 *be satisfied, and let* φ *be a positive function on* R^+ *with a continuous derivative such that*

$$\text{(202)} \qquad \tau^{-1/2}\varphi \in \uparrow \quad \text{and} \quad \varphi \leq \tau^h$$

for some $h < \frac{3}{5}$. *Assume that* $\varphi'(s) \sim \varphi'(t)$ *as* $t \to \infty$, $s/t \to 1$. *Put* $S_n = \sum_{i \leq n} X_i$ *and* $\hat{T}_\varphi = \sup\{n: S_n \geq \varphi(n)\}$. *Then* $\Pr\{\hat{T}_\varphi < \infty\} = 1$ *implies*

$$\text{(203)} \qquad \Pr\{\hat{T}_\varphi > n\} \sim \int_n^\infty \varphi'(t) G_\varphi(t)\, dt \qquad \text{as} \quad n \to \infty.$$

PROOF. Let S and ξ be as in theorem 4.8 and T_ψ, $\tilde{T}_\psi$ as in the proof of corollary 4.7. We may assume $\frac{1}{2} < h < \frac{3}{5}$. Let $\frac{1}{4} < b < \frac{2}{5}$, $a > 2h - 1$, $3a < 4b - 1$. Then

$$\text{(204)} \quad \begin{aligned} \Pr\{\hat{T}_\varphi \geq n\} &= \Pr\{\tilde{T}_\varphi \geq n\} \\ &\leq \Pr\{T_{\varphi - \tau^b} \geq n\} + \Pr\{|S(t) - \xi(t)| > t^b \text{ for some } t \geq n\}. \\ &= \Pr\{T_{\varphi-\tau^b} \geq n\} + o(e^{-n^a}). \end{aligned}$$

Thus by theorem 3.6 applied to $\varphi - \tau^b$ ((91) for $\varphi - \tau^b$ follows from $\Pr\{\hat{T}_\varphi < \infty\} = 1$ and, say, corollary 4.5 together with $\varphi \leq \tau^h$, $h < \frac{3}{5}$, $b < \frac{2}{5}$) we have

$$\text{(205)} \quad \begin{aligned} \Pr\{\hat{T}_\varphi \geq n\} &\leq (1 + o(1)) \int_n^\infty (\varphi'(t) - bt^{b-1}) G_{\varphi - \tau^b}(t)\, dt + o(e^{-n^a}) \\ &= (1 + o(1)) \int_n^\infty \varphi'(t) G_\varphi(t)\, dt + o(e^{-n^a}) \\ &= (1 + o(1)) \int_n^\infty \varphi'(t) G_\varphi(t)\, dt \end{aligned}$$

using first $\varphi \leq \tau^h$, $h < \frac{3}{5}$, $b < \frac{2}{5}$ and then $\varphi' \geq \varphi/2$, $\varphi \leq \tau^h$ and $a > 2h - 1$.

Conversely,

$$\text{(206)} \quad \begin{aligned} \Pr\{\hat{T}_\varphi \geq n\} &= \Pr\{\tilde{T}_\varphi \geq n\} \\ &\geq \Pr\{T_{\varphi + \tau^b} \geq n, |S(t) - \xi(t)| \leq t^b \text{ for all } t \geq n\} \\ &\geq (1 - o(1)) \int_n^\infty (\varphi'(t) + bt^{b-1}) G_{\varphi - \tau^b}(t)\, dt - o(e^{-n^a}) \\ &= (1 - o(1)) \int_n^\infty \varphi'(t) G_\varphi(t)\, dt \end{aligned}$$

as above.

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