ON PREDICTION THEORY FOR NONSTATIONARY SEQUENCES

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1. Introduction

Let H be a Hilbert space with inner product $(,)$ and let x_n, $n = 0, \pm 1, \cdots$, be a (two-sided) sequence of elements of H. Let $B(m, n) = (x_m, x_n)$, and for each n let $H_n(x)$ be the smallest closed (linear) subspace of H containing all the x_m for $m \leq n$. Let $H_{-\infty}(x)$ be the intersection of all the $H_n(x)$, and let $H_{+\infty}(x)$ be the closure of the union of all $H_n(x)$. We call $\{x_n\}$ a *Hilbert sequence.*

A Hilbert sequence $\{x_n\}$ is called *deterministic* if $H_{-\infty}(x) = H_{+\infty}(x)$ and will be called *linearly free* if $H_{-\infty}(x) = 0$. (Some authors call sequences satisfying the latter condition "completely nondeterministic.") Cramér [1] has shown that for any Hilbert sequence $\{x_n\}$ there exist Hilbert sequences $\{u_n\}$ and $\{v_n\}$ with $x_n = u_n + v_n$, $u_n \in H_n(x)$ and $v_n \in H_n(x)$ for all n, $u_m \perp v_n$ for all m and n, $\{u_n\}$ linearly free, and $\{v_n\}$ deterministic ("Wold decomposition").

There is a well worked out theory for stationary sequences, where $B(m, n)$ depends only on $m - n$. Cramér [1] has proved some results for certain classes of nonstationary sequences. Most of these results involve Fourier series or transforms. In recent years, great progress has been made in Fourier analysis by way of L. Schwartz's theory of distributions. However, it appears that distributions have not yet been used much in prediction theory. One paper by Rozanov [5] extends classical results in prediction theory to stationary "random distributions."

In this paper, we find a necessary and sufficient condition that a Hilbert sequence be deterministic in terms of the Fourier transform of its covariance $B(m, n)$, assuming only the following:

(1) $\qquad$ for some polynomial P, $B(n, n) \leq P(n)$ $\qquad$ for all n.

The Fourier transform of $B(m, n)$ will be a Schwartz distribution on the two-dimensional torus (product of two circles).

For stationary sequences, there is a classical criterion for determinacy. Our criterion for nonstationary Hilbert sequences is less satisfactory, since it involves an existence assertion, but it *is* a criterion, and the partial results of Cramér [1] proved under more restrictive hypotheses on B follow fairly easily from it. We also obtain a characterization of the covariances of linearly free sequences.

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As in the stationary case, the results of this paper can be extended to the case of functions x_t where t runs through all real values and/or x_t takes values in a finite product of Hilbert spaces. Such extensions have been obtained by my student C. M. Deo [2]. The results also have natural generalizations to arbitrary Hilbert sequences (not necessarily satisfying hypothesis (1)). These generalizations, not involving Fourier analysis, are also proved by Deo in [2].

2. Distributions on tori and Fourier series

Most of the definitions and results in this section are taken from L. Schwartz ([6], chapter 7, section 1).

Let T^k be the Cartesian product of the k circles

$$\{\exp(2\pi i s_j): -\infty < s_j < \infty\}, \qquad j = 1, \cdots, k. \tag{2}$$

Let $\mathfrak{D}(T^k) = \mathfrak{D}_k$ be the linear space of all complex-valued infinitely differentiable functions on T^k (or C^∞ functions of the s_j, periodic of period 1 in each variable). Each $f \in \mathfrak{D}_k$ has a Fourier series

$$f(s_1, \cdots, s_k) = \sum_{m_1, \cdots, m_k = -\infty}^{\infty} a_{m_1, \cdots, m_k} \exp\left(2\pi i \sum_{j=1}^{k} m_j s_j\right) \tag{3}$$

where for any $r > 0$ there is a $K > 0$ such that for all $m_1, \cdots, m_k$,

$$|a_{m_1, \cdots, m_k}| \leq K/(1 + m_1^2 + \cdots + m_k^2)^r, \tag{4}$$

so that the series in (3) is uniformly absolutely convergent. Conversely, given any set $\{a_{m_1, \cdots, m_k}\}$ of complex numbers such that for any $r > 0$, (4) holds for some K, the series (3) defines an $f \in \mathfrak{D}_k$.

We say $f_n \to f$ in $\mathfrak{D}_k$ if for any k-tuple $p = (p_1, \cdots, p_k)$ of nonnegative integers,

$$D^p(f_n - f) = \partial^{p_1 + \cdots + p_k}(f_n - f)/\partial s_1^{p_1} \cdots \partial s_k^{p_k} \to 0 \tag{5}$$

uniformly on T^k.

If f has Fourier series (3) and f_n has Fourier coefficients $a_m^{(n)}$, $m = (m_1, \cdots, m_k)$, then $f_n \to f$ in $\mathfrak{D}_k$ if and only if for every polynomial P in k variables,

$$P(m)(a_m^{(n)} - a_m) \to 0 \qquad \text{as} \quad n \to \infty, \tag{6}$$

uniformly in m.

Convergence in $\mathfrak{D}_k$ is equivalent to simultaneous convergence with respect to a countable family of pseudo-norms $\| \quad \|_p$, where

$$\|f\|_p = \sup_{x \in T^k} |D^p f(x)|. \tag{7}$$

Thus $\mathfrak{D}_k$ is a Fréchet space: the convergence is convergence with respect to a locally convex, metrizable topology which is easily seen to be separable and complete (but *not* normizable).

A set B in $\mathfrak{D}_k$ is bounded (that is, included in some scalar multiple of each neighborhood of zero) if and only if each pseudo-norm $\| \quad \|_p$ is bounded on B.

Given any sequence f_n of functions in such a set B, we can find a subsequence g_m such that $D^p g_m$ converges as $m \to \infty$ at each point of a countable dense subset of T^k, for each p. Equicontinuity of the $D^p g_m$ (because of uniform boundedness of higher-order derivatives) then gives pointwise convergence of $D^p g_m$ on T^k for each p. Finally, pointwise bounded convergence of higher-order derivatives gives uniform convergence of $D^p g_m$ for each p to $D^p g$ for some $g \in \mathfrak{D}_k$. Thus B is sequentially relatively compact. By metrizability and completeness, closed bounded sets in $\mathfrak{D}_k$ are compact.

A set M in a topological linear space S is called a *barrel* if it is closed, symmetric, and convex, and for each $f \in S$, $cf \in M$ for some $c > 0$. The space S is called *barreled* if each barrel is a neighborhood of 0. A simple Baire category argument shows that any complete metric linear space, for instance $\mathfrak{D}_k$, is barreled. A barreled space in which closed, bounded sets are compact is called a *Montel* space.

The space $\mathfrak{D}'(T^k) = \mathfrak{D}_k'$ of distributions on T^k is the linear space of all complex linear functionals on $\mathfrak{D}_k$, continuous for the given convergence. For each element A of $\mathfrak{D}_k'$, we let

$$A_{m_1,\cdots,m_k} = A\,(\exp\,(-2\pi i \sum_j m_j s_j)). \tag{8}$$

Continuity implies that there is a polynomial P (depending on A) such that

$$|A_{m_1,\cdots,m_k}| \leq P(m_1, \cdots, m_k) \qquad \text{for all} \quad m_1, \cdots, m_k. \tag{9}$$

Then we have

$$A = \sum_{m_1,\cdots,m_k} A_{m_1,\cdots,m_k} \exp\,(2\pi i \sum_j m_j s_j), \tag{10}$$

where the series of distributions on the right converges for any $f \in \mathfrak{D}_k$. This agrees with the classical assignment of Fourier series if A is defined by a finite Borel measure μ, for example by an integrable function h,

$$A(g) = \int_{T^k} g\, d\mu = \int_{T^k} gh\, ds_1 \cdots ds_k. \tag{11}$$

Again, conversely, for any numbers $A_{m_1,\cdots,m_k}$ satisfying (9), there is an $A \in \mathfrak{D}_k'$ satisfying (8) and hence (10).

Given that a sequence S_n converges to S in $\mathfrak{D}_k'$ pointwise on $\mathfrak{D}_k$, the set of all $f \in B_k$ such that $|S_n(f)| \leq 1$ for all n is a barrel. Hence the S_n are equicontinuous, and converge uniformly on compact sets, thus uniformly on bounded sets. In other words, a weak* convergent sequence in $\mathfrak{D}_k'$ is strongly convergent.

3. Positive kernels

Let S be a vector space over the field of complex numbers. We assume that a *conjugation* of S is given, that is, a conjugate-linear map $x \to \bar{x}$ of S into itself such that $\bar{\bar{x}} = x$ for all x. (If S is a suitable function space, for instance $S = \mathfrak{D}_1$, and $f \in S$, we let $\bar{f}(t) = f(t)^-$.) For any linear functional U on S, we let $\bar{U}(x) =$

$U(\bar{x})^-$. Then clearly $U \to \bar{U}$ is a conjugation of the dual linear space S' of all linear functionals on S.

Given complex linear spaces S and T, let $B(S, T)$ be the linear space of all bilinear functions from $S \times T$ to the complex numbers, and let $B'(S, T)$ be its dual space. Given $U \in S'$ and $V \in T'$, we let

$$(U \otimes V)(x, y) = U(x)V(y). \tag{12}$$

Then $\otimes$ is a bilinear map of $S' \times T'$ into $B(S, T)$.

For any $s \in S$, $t \in T$, we have an element

$$s \otimes t\colon B \to B(s, t) \tag{13}$$

of $B'(S, T)$. The tensor product of S and T, $S \otimes T$, is defined as the linear subspace of $B'(S, T)$ generated by all such $s \otimes t$. For $s \in S$, $t \in T$, $U \in S'$, $V \in T'$ we have

$$(U \otimes V)(s \otimes t) = U(s)V(t). \tag{14}$$

If S has a conjugation and $A, B \in B(S, S)$ we write $A \gg B$ if for all $x \in S$,

$$A(x, \bar{x}) \geq B(x, \bar{x}). \tag{15}$$

If $A \gg 0$, then $A(x, \bar{y}) = A(y, \bar{x})^-$ for all x, y, and $(x, y) = A(x, \bar{y})$ is an inner product except that $A(x, \bar{x})$ may vanish for some $x \neq 0$. Note that for any $U \in S'$, $\bar{U} \otimes U \gg 0$.

PROPOSITION 1. *If U, $V \in S'$, then $\bar{U} \otimes U \gg \bar{V} \otimes V$ if and only if $V = cU$ for some complex number c with $|c| \leq 1$.*

PROOF. If $V = 0$, let $c = 0$. If not, then since

$$|U(x)|^2 \geq |V(x)|^2 \tag{16}$$

for all $x \in S$, the null space of U is included in that of V. Since both null spaces have codimension 1, they are equal. Thus $V = cU$ for some $c \neq 0$, and clearly $|c| \leq 1$, q.e.d.

For $f, g \in \mathfrak{D}_1$ let $(f \times g)(s, t) = f(s)g(t)$. Then $\times$ is a bilinear map of $\mathfrak{D}_1 \times \mathfrak{D}_1$ into $\mathfrak{D}_2$. This yields of map of $\mathfrak{D}_2'$ into $B(\mathfrak{D}_1, \mathfrak{D}_1)$ which is one-to-one since finite sums $\sum f_i \times g_i$ are dense in $\mathfrak{D}_2$ (by Fourier series). To show that the natural linear map of $\mathfrak{D}_1 \otimes \mathfrak{D}_1$ into $\mathfrak{D}_2$ is one-to-one, suppose

$$\sum_{j=1}^{n} f_j \times g_j = 0, \qquad f_j, g_j \in \mathfrak{D}_1, j = 1, \cdots, n. \tag{17}$$

Let $F_1, \cdots, F_m$ be a basis for the subspace of $\mathfrak{D}_1$ spanned by the f_j. We can then write

$$\sum_{j=1}^{n} f_j \otimes g_j = \sum_{k=1}^{m} F_k \otimes G_k \tag{18}$$

for some $G_k \in \mathfrak{D}_1$. For each fixed t, we then have

$$\sum_{k=1}^{m} F_k(s)G_k(t) = 0 \tag{19}$$

for all s, so $G_k(t) \equiv 0$ for each k. Thus

$$\sum_{j=1}^{n} f_j \otimes g_j = 0. \tag{20}$$

Hence $\mathfrak{D}_1 \otimes \mathfrak{D}_1$ can be regarded as a subspace of $\mathfrak{D}_2$, with $\otimes = \times$.

If $U, V \in \mathfrak{D}_1'$, then $U \otimes V \in \mathfrak{D}_2'$ since the formal product of the Fourier series of U and V converges in $\mathfrak{D}_2'$.

A covariance $B(m, n)$ satisfying assumption (1) of section 1 yields a $B \in \mathfrak{D}_2'$ with

$$B = \sum B(m, n) \exp (2\pi i(-ms + nt)), \tag{21}$$

that is,

$$B(\exp (-2\pi i(ms + nt)) = B(-m, n). \tag{22}$$

We call B the *covariance distribution* of the given Hilbert sequence.

If $f \in \mathfrak{D}_1$, let

$$f(s) = \sum_n a_n \exp (2\pi ins). \tag{23}$$

Then $\sum_n a_n x_n$ is absolutely convergent in H, and

$$B(f \otimes \bar{f}) = \sum_{n,m} B(m, n)a_m\bar{a}_n = (\sum_m a_m x_m, \sum_n a_n x_n) \geq 0. \tag{24}$$

Thus $B \gg 0$. It is easy to see that, conversely, any $B \in \mathfrak{D}_2'$ with $B \gg 0$ is the covariance distribution of some Hilbert sequence satisfying (1).

A proof of the following was kindly supplied by L. Bungart.

PROPOSITION 2. *If $B \gg 0$, $B \in \mathfrak{D}_2'$, then there exist $B_j \in \mathfrak{D}_1'$, $j = 1, 2, \cdots$, such that*

$$B(f \otimes \bar{g}) = \sum_{j=1}^{\infty} \bar{B}_j(f)B_j(\bar{g}) \tag{25}$$

for all $f, g \in \mathfrak{D}_1$.

PROOF. We have an inner product $\beta(f, g) = B(f \otimes \bar{g})$ on $\mathfrak{D}_1$ (where $\beta(f, f)$ may vanish for some $f \neq 0$). Since $\mathfrak{D}_1$ has a countable dense subset and B is continuous, the inner product space $(\mathfrak{D}_1, \beta)$ is separable. Thus we can choose β-orthonormal $f_j \in \mathfrak{D}_1$, $j = 1, 2, \cdots$, whose linear combinations are dense for β (although $\mathfrak{D}_1$ will not generally be complete for β). Let

$$B_j(f) = B(f_j \otimes f) = \beta(f_j, \bar{f}) = \beta(\bar{f}, f_j)^- \tag{26}$$

for any $f \in \mathfrak{D}_1$. Clearly $B_j \in \mathfrak{D}_1'$ and

$$\bar{B}_j(f) = \beta(f_j, f)^- = \beta(f, f_j). \tag{27}$$

The conclusion of the proposition then holds according to the general expansion of an inner product over an orthonormal basis.

PROPOSITION 3. *If $B \in \mathfrak{D}_2'$ and*

$$B(f \otimes \bar{g}) = \sum_{j=1}^{\infty} \bar{B}_j(f)B_j(\bar{g}) \tag{28}$$

for all $f, g \in \mathfrak{D}_1$, then

$$B = \sum_{j=1}^{\infty} (\bar{B}_j \otimes B_j) \tag{29}$$

where the series converges in $\mathfrak{D}_2'$ *(pointwise on* $\mathfrak{D}_2$*, hence uniformly on bounded sets (strongly)).*

PROOF. Clearly the given series of distributions converges for each F belonging to the dense subspace $\mathfrak{D}_1 \otimes \mathfrak{D}_1$ of $\mathfrak{D}_2$. Now if K is any sum of finitely many of the $\overline{B}_j \otimes B_j$, then $B \gg K \gg 0$, so that for any integers m and n,

$$\begin{aligned}(30)\qquad |K(\exp(2\pi i(ms + nt)))| &\le [K(\exp(2\pi im(s-t)))]^{1/2}[K(\exp(2\pi in(s-t)))]^{1/2}\\ &\le [B(\exp(2\pi im(s-t)))B(\exp(2\pi in(s-t)))]^{1/2} \le P(m, n)\end{aligned}$$

where P is a polynomial independent of K. According to the characterization of convergence in $\mathfrak{D}_2$ expressed by (6) in section 2, the set of all such K is equicontinuous. Hence,

$$B(F) = \sum_{j=1}^{\infty} (\overline{B}_j \otimes B_j)(F) \tag{31}$$

for all $F \in \mathfrak{D}_2$, where the series converges absolutely. As mentioned in section 2, sequential pointwise convergence implies strong convergence since $\mathfrak{D}_2$ is a Montel space.

4. Diagonalizing the covariance of a linearly free sequence

DEFINITION. *A distribution* $U \in \mathfrak{D}_1'$ *will be called zero-meromorphic if for some finite integer* m,

$$U = \sum_{n=m}^{\infty} U_n \exp(2\pi ins). \tag{32}$$

If U is zero-meromorphic, then since U_n is of polynomial growth as $n \to \infty$,

$$\sum_{n=m}^{\infty} U_n z^n \tag{33}$$

is the Laurent series of a function analytic in the unit disk $|z| < 1$ except possibly for a pole at 0.

THEOREM 1. *If* $\{u_n\}$ *is a Hilbert sequence satisfying* (1), *with covariance distribution* B, *then* $\{u_n\}$ *is linearly free if and only if there exist zero-meromorphic* U_j *such that*

$$B = \sum_{j=-\infty}^{\infty} \overline{U}_j \otimes U_j, \tag{34}$$

where the series converges in $\mathfrak{D}_2'$.

PROOF. First suppose $\{u_n\}$ is linearly free. For each n let P_n be the orthogonal projection of H onto $H_n(u)$, and let $y_n = u_n - P_{n-1}u_n$. Then for distinct n, the y_n are orthogonal. Let z_n be $y_n/\|y_n\|$ if $\|y_n\| > 0$, otherwise let $z_n = y_n = 0$. According to Cramér ([1], theorem 1), the y_m (or z_m) for $m \le n$ span $H_n(u)$ in this (linearly free) case. Since the z_j are orthonormal, for each n we have $u_n = \sum_j c_{nj} z_j$ for some complex constants c_{nj}, which we take as 0 if $z_j = 0$;

the c_{nj} are then uniquely determined, with $c_{nj} = 0$ for $j > n$. For any j, $|c_{nj}| \leq B(n, n)^{1/2}$, so that $|c_{nj}|$ is bounded by a polynomial in n, uniformly in j.

Now if $f, g \in \mathfrak{D}_1$ and

$$(35) \qquad f(s) = \sum a_n \exp(2\pi ins), \qquad g(s) = \sum b_n \exp(2\pi ins),$$

then $\sum a_n u_n$ and $\sum b_n u_n$ converge to elements F, G of H respectively, and

$$(36) \qquad \begin{aligned} B(f \otimes \bar{g}) &= \sum_{m,n} B(m, n) a_m \bar{b}_n = (F, G) \\ &= \sum_j (F, z_j)(G, z_j)^- \\ &= \sum_j (\sum_m a_m c_{mj})(\sum_n \bar{b}_n \bar{c}_{nj}) \end{aligned}$$

since the nonzero z_j are orthonormal, and $c_{nj} = 0$ if $z_j = 0$. Because of the rapid decrease of a_m and b_n for large indices and at most polynomial growth of c_{nj}, all the above series are absolutely convergent. Now let

$$(37) \qquad U_j = \sum_{n=j}^{\infty} \bar{c}_{nj} \exp(2\pi ins).$$

The series converges in $\mathfrak{D}_1'$ since the $|c_{nj}|$ are bounded by a polynomial in n. Clearly,

$$(38) \qquad B(f \otimes \bar{g}) = \sum_{j=-\infty}^{\infty} \bar{U}_j(f) U_j(\bar{g})$$

for any $f, g \in \mathfrak{D}_1$. It follows from proposition 3 that

$$(39) \qquad B = \sum_{j=-\infty}^{\infty} \bar{U}_j \otimes U_j$$

where the series converges strongly in $\mathfrak{D}_2'$. Of course, each U_j is zero-meromorphic.

For the converse part of the theorem, suppose $B \in \mathfrak{D}_2'$ and

$$(40) \qquad B = \sum_{j=-\infty}^{\infty} \bar{U}_j \otimes U_j$$

where each U_j is zero-meromorphic, and

$$(41) \qquad U_j = \sum_{n=n(j)}^{\infty} c_{nj} \exp(2\pi ins).$$

Note that for fixed n,

$$(42) \qquad \sum_j |c_{nj}|^2 = \sum_j (\bar{U}_j \otimes U_j)(\exp(2\pi in(s - t)) < \infty.$$

Let H be a (complex) Hilbert space with orthonormal basis z_j: $j = 0, \pm 1, \pm 2, \cdots$, and let $x_n = \sum_j \bar{c}_{nj} z_j$ (the series clearly converges in H). We shall show that $\{x_n\}$ is a linearly free sequence with covariance distribution B. First, $(x_n, z_j) = 0$ for $n < n(j)$, so z_j is orthogonal to $H_r(x)$ if $r < n(j)$, and $z_j \perp H_{-\infty}(x)$. Since this holds for all j, $H_{-\infty}(x)$ is the zero subspace, and the sequence $\{x_n\}$ is linearly free. Also,

$$(x_m, x_n) = \sum_j \bar{c}_{mj} c_{nj} = \sum_j (\bar{U}_j \otimes U_j)(\exp(2\pi i(ms - nt))) \tag{43}$$

$$= B(\exp(2\pi i(ms - nt))),$$

$$B = \sum_{m,n} (x_m, x_n) \exp(2\pi i(-ms + nt)). \tag{44}$$

Since the property of being linearly free clearly depends only on the covariance, the proof is complete.

5. Characterization of deterministic covariances

THEOREM 2. *A Hilbert sequence $\{x_n\}$ with covariance $B(m, n)$ satisfying* (1) *is deterministic if and only if there does not exist a zero-meromorphic $U \neq 0$ with $B \gg \bar{U} \otimes U$.*

PROOF. First suppose $\{x_n\}$ is not deterministic. Let $x_n = u_n + v_n$ be its Wold decomposition, u_n linearly free, and v_n deterministic. Let B, C, and D be the covariance distributions of x_n, u_n, and v_n respectively. Then since (u_m, v_n) is zero for all m and n, $B = C + D$ and $B \gg C \neq 0$. We have by theorem 1

$$C = \sum_{j=-\infty}^{\infty} \bar{C}_j \otimes C_j \tag{45}$$

where the C_j are zero-meromorphic. Choosing a nonzero C_j, we have $B \gg \bar{C}_j \otimes C_j$ as desired.

Conversely, suppose $\{x_n\}$ is a Hilbert sequence with covariance distribution B and $B \gg \bar{U} \otimes U$ where U is zero-meromorphic and $U \neq 0$, say

$$U = \sum_{n=k}^{\infty} b_n \exp(2\pi ins), \qquad b_k \neq 0. \tag{46}$$

Then for any finite sequence $a_1, \cdots, a_N$ of complex numbers,

$$\left\| x_k - \sum_{j=1}^{N} a_j x_{k-j} \right\|^2 \geq \left| U\left[\exp(-2\pi iks) - \sum_{j=1}^{N} a_j \exp(-2\pi i(k-j)s)\right] \right|^2 \tag{47}$$

$$= |b_k|^2 > 0.$$

Thus $\|x_k - P_{k-1}x_k\| \geq |b_k|$ and $\{x_n\}$ is not deterministic. This completes the proof of theorem 2.

Note that theorem 7 of Cramér [1] is an immediate consequence of our theorem 2. Cramér's theorem 6 can also be derived, with a little more trouble; this will be done in section 6. A derivation of the known criterion in the stationary case will be given in section 7.

Theorem 2 can be reformulated as follows.

COROLLARY. *A Hilbert sequence with covariance distribution B is nondeterministic if and only if there exists a zero-meromorphic $U \neq 0$ such that $\bar{U}$ is continuous on $\mathfrak{D}_1$ for the inner product $(f, g) = B(f \otimes \bar{g})$.*

PROOF. If U is zero-meromorphic, $U \neq 0$, and $B \gg \bar{U} \otimes U$, then $|\bar{U}(f)|^2 \leq (f, f)$ for all f, so $\bar{U}$ is continuous. Conversely, if $\bar{U}$ is continuous, then $B \gg \bar{V} \otimes V$ for some zero-meromorphic $V = cU$, where c is a nonzero constant.

6. A result of Cramér

Suppose that the covariance distribution B of a Hilbert sequence is a finite, complex-valued, countably additive measure on T^2 (the "harmonizable" case). Let $G(u)$ be the total variation of B in the rectangle $0 \leq s \leq 1$, $0 \leq t \leq u$. Then G is a nondecreasing function on $[0, 1]$, differentiable almost everywhere. Cramér ([1], theorem 6) has shown that if

$$\int_0^1 \log G'(u)\, du = -\infty, \tag{48}$$

then $\{x_n\}$ is deterministic. To infer this from our theorem 2, it suffices to show that if $B \gg \bar{U} \otimes U$ for some zero-meromorphic $U \neq 0$, then

$$\int_0^1 \log G'(u)\, du > -\infty. \tag{49}$$

Now if $f \in \mathfrak{D}(T^1)$ and $|f(s)| \leq 1$, $0 \leq s \leq 1$, then

$$|\bar{U}(f)|^2 \leq B(f \otimes \bar{f}) \leq G(1). \tag{50}$$

Thus U is a countably additive complex-valued Borel measure of total variation at most $G(1)^{1/2}$. By the F. and M. Riesz theorem ([4], p. 47) U is absolutely continuous; there is an integrable function u such that

$$U(f) = \int_0^1 f(s)u(s)\, ds \tag{51}$$

for all $f \in \mathfrak{D}_1$.

Now if $f \in \mathfrak{D}_1$, f vanishes outside an open set V, and $|f(s)| \leq 1$ for $0 \leq s \leq 1$, then

$$\int_V dG \geq B(f \otimes \bar{f}) \geq \left|\int_0^1 f(s)\bar{u}(s)\, ds\right|^2. \tag{52}$$

There is a sequence $\{f_n\}$ of such f's with f_n converging almost everywhere as $n \to \infty$ to the function ϕ with

$$\phi(s) = \begin{cases} |u(s)|[\bar{u}(s)]^{-1}, & u(s) \neq 0, \quad s \in V, \\ 0, & \text{otherwise.} \end{cases} \tag{53}$$

Thus

$$\int_V dG \geq \left(\int_V |u(s)|\, ds\right)^2. \tag{54}$$

Here the arbitrary open set V can be replaced by an arbitrary Borel measurable subset A of the unit interval.

Since the singular part of dG is concentrated in a set N of Lebesgue measure zero, we have

$$\int_A G'(s)\, ds = \int_{A\sim N} G'(s)\, ds \geq \left(\int_{A\sim N} |u(s)|\, ds\right)^2 = \left(\int_A |u(s)|\, ds\right)^2. \tag{55}$$

Thus we have use for the following quaint lemma of pure measure theory.

LEMMA. *Suppose f and g are nonnegative measurable functions on a probability space (Ω, B, P) such that both g and $\log g$ are integrable and for every set $A \in B$,*

(56) $$\int_A f\,dP \geq \left[\int_A g\,dP\right]^2.$$

Then $\int \log f\,dP > -\infty$.

Proof. Let A_{-1} be the set where $f > g$, and for $n = 0, 1, \cdots$, let A_n be the set where $(g/e^{n+1}) < f \leq (g/e^n)$. Since $\log g$ is integrable, g is strictly positive almost everywhere, and hence the set A where f vanishes also has measure zero by assumption. Thus

(57) $$P(A_{-1}) + P(A_0) + \cdots = 1.$$

Since g is integrable, we have for each n

(58) $$\int_{A_n} g/e^n\,dP \geq \left[\int_{A_n} g\,dP\right]^2, \qquad \int_{A_n} g\,dP \leq e^{-n}.$$

Let B_n be the subset of A_n on which $g \leq e^{-n/2}$. Then $P(A_n \sim B_n) \leq 1/e^{n/2}$, and since $\log g \leq -n/2$ on B_n and the B_n are disjoint,

(59) $$\sum_{n=1}^{\infty} nP(B_n) < \infty.$$

It follows that $\sum_n nP(A_n)$ also converges, and

(60) $$\int \log f\,dP = \sum_{n=-1}^{\infty} \int_{A_n} \log f\,dP \geq \sum_{n=-1}^{\infty} \int_{A_n} (\log g - n - 1)\,dP$$
$$= \int \log g\,dP - \sum_{n=-1}^{\infty} (n+1)P(A_n) > -\infty, \qquad \text{q.e.d.}$$

Now, the function u mentioned earlier in this section is zero-meromorphic, and hence of the form $\exp(-2\pi iks)h(s)$ for some integer k and some function h in the Hardy class H^1 having nonzero integral (zero-th Fourier coefficient). Thus the function $\log|u| = \log|h|$ is integrable (see, for example, [4], p. 53). Applying the lemma with $f = G'$, $g = |u|$, we obtain

(61) $$\int_0^1 \log(G'(s))\,ds > -\infty,$$

which proves Cramér's assertion.

It is worth noting that Cramér's sufficient condition for determinacy is not necessary. Let h be a real-valued function on T^1 which is not zero-meromorphic, such that $h(s) \geq 1$ for all s. We can even take $h \in \mathfrak{D}_1$. Let $B = h \otimes h$; then $B \gg 0$. If $h \otimes h \gg \bar{U} \otimes U$ for some zero-meromorphic $U \neq 0$, then $h = \lambda U$ with $|\lambda| \geq 1$ by proposition 1, contradicting the fact that h is not zero-meromorphic. Thus B is deterministic. On the other hand, B is clearly harmonizable with

(62) $$G'(s) = h(s)\int_0^1 h(t)\,dt, \qquad \int_0^1 \log G'(s)\,ds \geq 0 > -\infty.$$

Thus we have obtained a class of deterministic covariances which do not have the "thinness" property which had characterized the previously known classes. Whether theorem 2 can be improved to provide a more explicit analytic criterion seems to be an open problem. It is well known, from the stationary case, that unlike linearly free covariances, deterministic covariances do not form an additive class.

We note here that in the harmonizable case, the zero-meromorphic distributions U_j in theorem 1 and U in theorem 2 may be taken as integrable functions for Lebesgue measure on T^1, by the argument given at the beginning of this section.

7. The stationary case

If a covariance $B(m, n)$ satisfies $B(m, n) = \beta(m - n)$ for some function β, then B is the covariance of a stationary Hilbert sequence $\{x_n\}$. The covariance distribution is then equal to a finite nonnegative measure μ concentrated in the "diagonal" $s = t$ in T^2. Letting

$$\nu(A) = \mu(\langle s, t\rangle: s \in A), \tag{63}$$

we obtain a finite measure ν on T^1 with a Fourier series

$$\nu = \sum_n c_n \exp(2\pi i n s). \tag{64}$$

The standard result is that $\{x_n\}$ is deterministic if and only if the logarithm of the density f of the absolutely continuous part of ν is integrable (equivalently, has integral greater than $-\infty$). Let us infer this from our results. For f integrable on T^1, $\log f$ is integrable if and only if $f = |g|^2$ for some nonzero g in the Hardy class H^2, that is

$$g(s) \sim \sum_{n=0}^{\infty} g_n \exp(2\pi i n s), \tag{65}$$

where $\sum_{n=0}^{\infty} |g_n|^2 < \infty$ (see [4], p. 53). Thus if $\log f$ is integrable, and h is any element of $\mathfrak{D}_1$, then

$$\begin{aligned} \int h \otimes \bar{h}\, d\mu &= \int_0^1 |h(s)|^2\, d\nu(s) \geq \int_0^1 |\bar{g}(s)h(s)|^2\, ds \\ &\geq \left|\int_0^1 \bar{g}(s)h(s)\, ds\right|^2 = (\bar{g} \otimes g)(h \otimes \bar{h}), \end{aligned} \tag{66}$$

so that $\mu \gg \bar{g} \otimes g$. Since g is zero-meromorphic and not identically zero, the given Hilbert sequence is not deterministic.

The converse assertion, that if the sequence is not deterministic then $\log f$ is integrable, follows from Cramér's result proved in section 6. (For a direct treatment of the stationary case, see Doob ([3], chapter 12).

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