A CONTRIBUTION TO THE MULTIPLICITY THEORY OF STOCHASTIC PROCESSES

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1. Introduction

In a paper [1] read before the Fourth Berkeley Symposium in 1960, I communicated the elements of a theory of spectral multiplicity for stochastic processes. A related theory was given about the same time by Hida [5]. Since then, I have developed the theory in some subsequent papers [2]–[4], the most recent of which contains the text of a lecture given at the Seventh All-Soviet Conference of Probability and Mathematical Statistics in Tbilisi 1963. Further important work in the field has been made by Kallianpur and Mandrekar [6]–[8].

Many interesting problems arising in connection with this theory are still unsolved. The object of this paper is to offer a small contribution to the investigation of one of these problems.

We shall begin by giving in section 2 a brief survey of the results of multiplicity theory so far known for the simplest case of one-dimensional processes. For proofs and further developments we refer to the papers quoted above. A major unsolved problem will be discussed in section 3, whereas section 4 is concerned with some aspects of the well-known particular class of stationary processes, which are relevant for our purpose. Finally, section 5 is concerned with the construction of a class of examples which may be useful in the further study of the problem stated in section 3.

2. Spectral multiplicity of stochastic processes

Consider a stochastic process $x(t)$, where $x(t)$ is a complex-valued random variable defined on a fixed probability space, while t is a real-valued parameter. In general we shall allow t to take any real values, and shall only occasionally consider the case when t is restricted to the integers. We shall always assume that the relations

$$Ex(t) = 0, \qquad E|x(t)|^2 < \infty \tag{2.1}$$

are satisfied for all t.

We denote by $H(x)$ the Hilbert space spanned in a well-known way by the random variables $x(t)$ for all t, while $H(x, t)$ is the subspace of $H(x)$ spanned only by the $x(u)$ with $u \leq t$. The tail space $H(x, -\infty)$ may be regarded as representing the "infinitely remote past" of the process. If $H(x, -\infty)$ only contains the zero element of $H(x)$, the $x(t)$ process is said to be purely nondeterministic.

All processes $x(t)$ considered in the sequel will be assumed to satisfy the following conditions (A) and (B):

(A) the process is purely nondeterministic;

(B) the limits in quadratic mean $x(t + 0)$ and $x(t - 0) = x(t)$ exist for every t. Under these conditions the space $H(x)$ will be separable. We note that, in the case of a parameter t taking only integral values, condition (B) is irrelevant.

Let us now for a moment consider the case when t is restricted to the integers, so that we are concerned with a sequence of random variables x_n, with $n = 0, \pm 1, \cdots$. Then there exists a sequence of mutually orthogonal random variables z_n with

$$(2.2) \qquad \begin{aligned} &Ez_n = 0, \qquad E|z_n|^2 = 1 \text{ or } 0 \text{ for every } n, \\ &Ez_m\overline{z_n} = 0 \qquad \text{for} \quad m \neq n, \end{aligned}$$

such that

$$(2.3) \qquad x_n = \sum_{k=-\infty}^{n} c_{nk} z_k,$$

where the series

$$(2.4) \qquad \sum_{k=-\infty}^{n} |c_{nk}|^2$$

converges for every n, so that the expression for x_n converges in quadratic mean. The variable z_n may then be regarded as a (normalized) *innovation* entering into the process at time $t = n$.

By analogy, we might expect to have in the case of a continuous paramter t a representation of the form

$$(2.5) \qquad x(t) = \int_{-\infty}^{t} g(t, u)\, dz(u)$$

where $z(u)$ would be a process with orthogonal increments, the increment $dz(u)$ representing the *innovation element* entering into the $x(t)$ process during the time element $(u, u + du)$.

However, in general this is not true. The situation in the continuous case turns out to be more complicated than in the discrete case. In general the innovation associated with a given time element must be regarded as a multi-dimensional or even infinite-dimensional random variable, so that the representation (2.5) is definitely too simple.

In order to present the representation formula which in the general case takes the place of (2.5), we must first consider the class C of all real-valued and never decreasing, not identically constant functions $F(t)$ which are continuous to the left for all t. A subclass D of C is called an *equivalence class* if any two functions F_1 and F_2 in D are mutually absolutely continuous. If D_1 and D_2 are equivalence classes, D_1 is said to be *superior* to D_2, and we write $D_1 > D_2$, if any $F_2 \in D_2$ is absolutely continuous relative to any $F_1 \in D_1$. Evidently, the relation $D_1 > D_2$ does not exclude the case that the two classes are identical.

Consider now a finite or infinite never increasing sequence of equivalence classes

$$D_1 > D_2 > \cdots > D_N, \tag{2.6}$$

where N may have any of the values $1, 2, \cdots, \infty$. Then N will be called the *total multiplicity* of the sequence. Further, let $N(t)$ for every t denote the number of those classes in (2.6) for which t is a point of increase of the corresponding functions F. Then $N(t)$ is called the *multiplicity function* of the sequence (2.6). Like N, $N(t)$ may be finite or infinite, and we have

$$N = \sup N(t), \tag{2.7}$$

where t runs through all real values.

The fundamental proposition of multiplicity theory for stochastic processes is the following. To any $x(t)$ stochastic process satisfying conditions (A) and (B), there is a uniquely determined sequence of the form (2.6) such that the following properties hold. For every $n = 1, 2, \cdots, N$, there is a process $z_n(t)$ of orthogonal increments, such that

$$\begin{aligned} &Ez_n(t) = 0, \qquad E|z_n(t)|^2 = F_n(t) \in D_n, \\ &Ez_m(t)\overline{z_n(u)} = 0 \qquad \text{for } m \neq n \text{ and all } t, u, \\ &H(x, t) = \sum_1^N H(z_n, t) \qquad \text{for all } t, \end{aligned} \tag{2.8}$$

where the last sum denotes the vector sum of the orthogonal subspaces $H(z_n, t)$. We then have for every t the representation

$$x(t) = \sum_1^N \int_{-\infty}^t g_n(t, u)\, dz_n(u), \tag{2.9}$$

where the g_n are nonrandom functions such that

$$\sum_1^N \int_{-\infty}^t |g_n(t, u)|^2\, dF_n(u) < \infty. \tag{2.10}$$

It is important to observe that the D_n sequence (2.6) is uniquely determined by the $x(t)$ process. Thus, in particular, the multiplicity function $N(t)$ and the total multiplicity N are also uniquely determined by $x(t)$. Accordingly, we shall say that the D_n sequence, as well as $N(t)$ and N, are spectral multiplicity characteristics of the stochastic process $x(t)$.

On the other hand, the $g_n(t, u)$ and $z_n(u)$ occurring in the representation (2.9) are not uniquely determined by the $x(t)$ process. Thus, for a given $x(t)$ we may have different representations of the form (2.9), all satisfying the relations (2.8). However, the D_n sequence (2.6), as well as the multiplicity characteristics $N(t)$ and N, will be identical for all these representations.

According to the representation (2.9), we may say that the multiplicity function $N(t)$ determines the dimensionality of the innovation element $[dz_1(u), dz_2(u), \cdots]$ entering into the process during the time element $(u, u + du)$.

It has been shown that the multiplicity characteristics of a given stochastic process $x(t)$ are uniquely determined by the covariance function of the process

$$r(t, u) = Ex(t)\overline{x(u)}. \tag{2.11}$$

We finally remark that the multiplicity theory as outlined above can be directly generalized to stochastic vector processes of a very general kind. We shall, however, not deal with these generalizations in the present paper.

3. Processes of total multiplicity $N = 1$

According to the above, we know that any given stochastic process $x(t)$ satisfying (A) and (B) has multiplicity characteristics which are uniquely determined by the process, and even by the covariance function $r(t, u)$ of the process.

On the other hand, so far we know very little about those properties of the process, or of the corresponding covariance function, which determine the actual values of multiplicity characteristics like $N(t)$ and N.

In the discrete case it follows from the above that, by analogy, it can be said that the total multiplicity is always $N = 1$. In the continuous case, the important class of (second-order) stationary processes has even $N(t) = 1$ for all t, and consequently $N = 1$, as follows from well-known properties of these processes to be presently recalled.

In view of these examples, it might well be asked if there exist any stochastic processes with a total multiplicity exceeding unity. The answer to this question is that such processes do, in fact, exist. It can even be shown that, as soon as we proceed from the class of stationary processes to the more general class of harmonizable processes introduced by Loève, any prescribed multiplicity properties may occur. In fact, it has been shown in [4] that, given any D_n sequence (2.6), there exists a harmonizable process $x(t)$ associated with this given D_n sequence. However, the example of such a process given in [4] is of a very special kind, and the corresponding representation (2.9) contains functions $g_n(t, u)$ having rather pathological properties, not likely to occur in applications to any physical problems.

Accordingly, it seems to be a problem of some interest to study more closely those properties of a stochastic process which determine the actual values of the multiplicity characteristics. In particular, it would be interesting to be able to define some fairly general class of processes having total multiplicity $N = 1$.

A natural approach to this last problem might be to start from the class of stationary processes, which always have $N = 1$, and then try to generalize the definition, still keeping sufficiently near the property of stationarity to conserve the multiplicity characteristic $N = 1$. We propose to give in the sequel an example of a generalization of this type. In order to do this, we must first recall some of the relevant properties of stationary processes.

4. Stationary processes

Let $x(t)$ be a (second-order) stationary process, satisfying (A) and (B). It then follows that the covariance function

$$r(t) = Ex(t+h)\overline{x(h)} \tag{4.1}$$

is everywhere continuous, and has the spectral representation

$$r(t) = \int_{-\infty}^{\infty} e^{it\lambda} f(\lambda)\, d\lambda \tag{4.2}$$

with a spectral density $f(\lambda) > 0$ for almost all λ (Lebesgue measure), such that $f(\lambda) \in L_1(-\infty, \infty)$, and

$$\int_{-\infty}^{\infty} \frac{\log f(\lambda)}{1+\lambda^2}\, d\lambda > -\infty. \tag{4.3}$$

The random variable $x(t)$ has the corresponding spectral representation

$$x(t) = \int_{-\infty}^{\infty} e^{it\lambda}\, dw(\lambda), \tag{4.4}$$

where $w(\lambda)$ is a process with orthogonal increments such that

$$E\, dw(\lambda) = 0, \qquad E|\, dw(\lambda)\,|^2 = f(\lambda)\, d\lambda. \tag{4.5}$$

Further, there exists a complex-valued function $h(\lambda) \in L_2(-\infty, \infty)$ and a process $z(t)$ of orthogonal increments such that

$$E\, dz(t) = 0, \qquad E|\, dz(t)\,|^2 = dt, \qquad |h(\lambda)|^2 = f(\lambda), \tag{4.6}$$

while the Fourier transform $g(t)$ of $h(\lambda)$ reduces to zero for $t < 0$, and we have the representation

$$x(t) = \int_{-\infty}^{t} g(t-u)\, dz(u) \tag{4.7}$$

with

$$H(x, t) = H(z, t) \tag{4.8}$$

for all t. The functions $h(\lambda)$ and $g(t)$ are uniquely determined, up to a constant factor of absolute value 1. Comparing this with the general representation formula (2.9), it is seen that the stationary process $x(t)$ has the multiplicity characteristics $N = 1$ and $N(t) = 1$ for all t.

5. A class of harmonizable processes with $N = 1$

We shall now define a class of harmonizable processes containing the stationary process $x(t)$ given by (4.4) or (4.7) as a particular case, and such that the multiplicity characteristics are the same as for $x(t)$, that is $N = 1$ and $N(t) = 1$ for all t.

Let $Q(\rho)$ be a never-decreasing function of the real variable ρ such that Q has a jump of size 1 at $\rho = 0$, whereas $Q(-\infty) = 0$, $Q(+\infty) < 2$, and

$$Q(\rho) + Q(-\rho) = Q(+\infty) \tag{5.1}$$

in all continuity points ρ of Q. The Fourier-Stieltjes transform of Q will then be real and positive, so that we may define an everywhere positive, continuous and bounded function $q(u)$ by the relation

$$[q(u)]^2 = \int_{-\infty}^{\infty} e^{-i\rho u} \, dQ(\rho). \tag{5.2}$$

We now define a stochastic process $X(t)$ by writing

$$X(t) = \int_{-\infty}^{t} g(t - u) q(u) \, dz(u), \tag{5.3}$$

where $g(t)$ and $z(u)$ are the same as in (4.7). As $q(u)$ is bounded, and $g(t) \in L_2(0, \infty)$, the integral in (5.3) exists as a quadratic mean integral. When Q is identically constant except for the jump at $\rho = 0$, it is seen that $X(t)$ reduces to the stationary process $x(t)$ given by (4.7).

We shall now first show that $X(t)$ has the required multiplicity characteristics. According to (2.8) and (2.9), we have to show that $H(X, t) = H(z, t)$ for all t. As it evidently follows from (5.3) that $H(X, t) \subset H(z, t)$, it will be sufficient to show that the opposite inclusion relation is also true. If, for some t, this were not so, there would be a nonzero element in $H(z, t)$ orthogonal to $X(u)$ for all $u \leq t$. Now every nonzero element in $H(z, t)$ is of the form

$$\int_{-\infty}^{t} m(v) \, dz(v), \tag{5.4}$$

with a quadratically integrable $m(v)$ not almost everywhere equal to zero. If this is orthogonal to $X(u)$ for $u \leq t$, we have

$$\int_{-\infty}^{u} g(u - v) q(v) \overline{m(v)} \, dv = 0 \tag{5.5}$$

for all $u \leq t$. However, since $q(v)$ is bounded and positive, it would follow that there is a nonzero element in $H(z, t)$ orthogonal to $x(u)$ for all $u \leq t$, in contradiction with the relation (4.8). Thus our assertion is proved.

We now proceed to prove that $X(t)$ as defined by (5.3) is a harmonizable process, and to deduce an expression for its spectral distribution. From (5.3) we obtain for the covariance function $R(s, t)$ of $X(t)$ the expression

$$\begin{aligned} R(s, t) = EX(s)\overline{X(t)} &= \int_{-\infty}^{\infty} g(s - u) \overline{g(t - u)} \, [q(u)]^2 \, du \\ &= \int_{-\infty}^{\infty} g(s - u) \, \overline{g(t - u)} \, du \int_{-\infty}^{\infty} e^{-i\rho u} \, dQ(\rho). \end{aligned} \tag{5.6}$$

As $g(t) = 0$ for $t < 0$, and $g(t) \in L_2(0, \infty)$, it follows that the double integral is absolutely convergent, so that

$$R(s, t) = \int_{-\infty}^{\infty} dQ(\rho) \int_{-\infty}^{\infty} e^{-i\rho u} g(s - u) \overline{g(t - u)} \, du. \tag{5.7}$$

By the Parseval formula, this gives

$$R(s, t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i[s\lambda - t(\lambda+\rho)]} h(\lambda) \overline{h(\lambda + \rho)} \, d\lambda \, dQ(\rho). \tag{5.8}$$

Substituting here μ for $\lambda + \rho$, it will be seen that this is the expression of a harmonizable covariance function. The corresponding spectral mass is distributed over the (λ, μ)-plane so that the infinitesimal strip between the lines $\mu = \lambda + \rho$ and $\mu = \lambda + \rho + d\rho$ contains the mass $dQ(\rho)$, whereas the distribution within the strip has the relative density $h(\lambda)\overline{h(\mu)}$. Again we see that, in the particular case when $Q(\rho)$ is identically constant except for the jump at $\rho = 0$, the whole spectral mass is situated on the diagonal $\lambda = \mu$, so that we have the covariance function of a stationary process with spectral density $|h(\lambda)|^2 = f(\lambda)$. As soon as $Q(\rho)$ has some variation outside the point $\rho = 0$, we have the two-dimensional spectral distribution of a harmonizable covariance.

Thus the covariance function of the $X(t)$ process, given by (5.3), is harmonizable, and it then follows from known properties of harmonizable processes that $X(t)$ itself is harmonizable; that is, we have

$$X(t) = \int_{-\infty}^{\infty} e^{it\lambda}\, dZ(\lambda), \tag{5.9}$$

where the covariance function $EZ(\lambda)\overline{Z(\mu)}$ is obtained from the expression (5.8) with $\mu = \lambda + \rho$. At the same time, we have seen that the harmonizable process $X(t)$ has the multiplicity characteristics $N = 1$ and $N(t) = 1$ for all t.

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