ABSTRACT HARMONIC ANALYSIS AND LÉVY'S BROWNIAN MOTION OF SEVERAL PARAMETERS

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1. Introduction

Paul Lévy's studies of the Gaussian process $\{\xi(a), a \in R^d\}$ defined by $E(\xi(a)) = 0$, and

$$E(\xi(a)\xi(b)) = \tfrac{1}{2}(|a| + |b| - |a - b|), \qquad a, b \in R^d, \tag{1.1}$$
$$= f(a, b) \text{ say,}$$

are well known, for example, [12], [13], [14]. He calls this process Brownian motion of several parameters.

Lévy has also studied [15] a Gaussian process $\{\xi(a), a \in S^d\}$ (S^d = the unit sphere in R^{d+1}) defined by $E(\xi(a)) = 0$, and

$$E(\xi(a)\xi(b)) = \tfrac{1}{2}(d(a, o) + d(b, o) - d(a, b)) \tag{1.2}$$
$$= f(a, b) \text{ say,}$$

where $a, b \in S^d$, o is any point (fixed once and for all) of S^d, and $d(x, y)$ stands for the geodesic distance between $x, y \in S^d$, taken along the sphere. This motion may be called Brownian motion with parameter running on S^d [15].

The functions $f(a, b)$ described by (1.1) and (1.2) are both real-valued, symmetric and positive-definite; that is, given $\alpha_1, \cdots, \alpha_n \in R$, $a_1, \cdots, a_n \in R^d$ (or S^d), we have

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \alpha_j f(a_i, a_j) \geq 0. \tag{1.3}$$

For (1.1) this fact is due to a theorem of Schoenberg [19], and Lévy used this fact to establish the existence of the process ξ. On the other hand, for (1.2), no direct proof of the positive-definiteness is known. The process $\{\xi(a), a \in S^d\}$ is constructed by Lévy by means of white noise integrals, and then it is checked by explicit evaluation that its covariance is (1.2). It follows that (1.2) is positive-definite. Here Lévy was adopting an idea of Chentsov [3], where a white noise integral for $(\xi(a), a \in R^d)$ is described.

The processes described above have several interesting properties, and there seems to be several, as yet not completely clear, connections between their study

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and several problems in harmonic analysis and differential equations. Indications of this may be found in McKean's paper [16]. It is therefore natural to ask for a description of kernels like (1.1) and (1.2) on spaces within the domain of harmonic analysis, and to seek to develop a theory for the corresponding Gaussian process.

A question of this sort was raised by Lévy ([15], p. 309) but there matters have stood, largely because of the ad hoc nature of the proofs of the positive-definiteness property of (1.1) and (1.2).

It is the purpose of this note to report some progress in this direction. That is, a class of kernels embodying the main features of (1.1) and (1.2) will be studied on a fair variety of spaces, and a more or less complete description of these kernels will be obtained. It turns out that the problem has a natural formulation in the context of abstract harmonic analysis.

Having obtained the description of kernels "like" (1.1) and (1.2) above, the problem of studying the corresponding Gaussian process arises naturally. Some partial results of this kind will also be described below.

The present note is confined to a description of the notions, methods, and results obtained so far by the methods. Proofs, along with results of further investigation, will be presented elsewhere in the future.

2. Lévy-Schoenberg kernels

Given a topological space S, a *kernel defined on* S is a continuous real-valued function f on $S \times S$. A kernel is to be called *positive-definite* if it is symmetric, that is, $f(a, b) = f(b, a)$, and if given $\alpha_1, \cdots, \alpha_n \in R$, $a_1, \cdots, a_n \in S$, we have

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \alpha_j f(a_i, a_j) \geq 0. \tag{2.1}$$

Let G be a locally compact topological group satisfying the second axiom of countability and K be a closed subgroup of G. Then $M = G/K$ will denote the space of cosets xK, $x \in G$, with the quotient topology. The group G acts on G/K by left translation as follows: $x(yK) = (xy)K$.

DEFINITION 2.1. *A kernel f on G/K is said to be a Lévy-Schoenberg kernel if it has the following properties:*

(i) $f(a, b) = f(b, a)$, $\qquad a, b \in G/K;$

(ii) *there exists a point* $o \in G/K$ *such that*

$$f(a, o) = 0, \qquad a \in G/K; \tag{2.2}$$

(iii) *if* $r(a, b) = f(a, a) + f(b, b) - 2f(a, b)$, *then the kernel r is invariant under G, namely*

$$r(xa, xb) = r(a, b), \qquad a, b \in G/K, \quad x \in G; \tag{2.3}$$

(iv) f *is positive-definite.*

It is to be noted that (1.1) and (1.2) are Lévy-Schoenberg kernels. Indeed, R^d may be viewed as the homogeneous space of the group G of all proper rigid motions of R^d, modulo the subgroup $K = SO(d)$ of all proper rotations about the origin. The kernel f of (1.1) thus lives on G/K and (i), (ii) are easily checked, the origin in R^d serving as the point 0 envisaged in (ii). As to (iii), the kernel $r(a, b)$ is just $|a - b|$ in this case, and this is certainly invariant under all rigid motions of R^d. Condition (iv) is just Schoenberg's theorem. As for (1.2), here S^d may be viewed as the homogeneous space of $G = SO(d + 1)$ modulo the subgroup K consisting of those rotations which leave the point o fixed. $K \cong SO(d)$. In this case $r(a, b) = d(a, b)$ and clearly fulfills (iii) since rotations are isometries of S^d.

Note that r is just the "polarization" of f, and indeed, r determines f, because in view of (ii),

$$f(a, b) = \tfrac{1}{2}(r(a, o) + r(b, o) - r(a, b)). \tag{2.4}$$

It follows that all properties of f have equivalents in properties of r. Our theory steps off with the following observation.

LEMMA 2.2. *Suppose r is a symmetric kernel on a topological space S, and suppose $o \in S$. Next, suppose $r(o, o) = 0$, and let f be the kernel defined in terms of r by (2.4); then f is positive-definite if and only if for each $t \geq 0$, the kernel θ_t defined by $\theta_t(a, b) = \exp - tr(a, b)$ is positive-definite.*

The proof is elementary. One half of it is similar to Lévy's proof [12].

COROLLARY 2.3. *If f is a Lévy-Schoenberg kernel on G/K, then the kernel r defined by*

$$r(a, b) = f(a, a) + f(b, b) - 2f(a, b) \tag{2.5}$$

has the following properties:

$$r(a, b) = r(b, a), \tag{2.6}$$

$$r(a, a) = 0, \tag{2.7}$$

$$r(xa, xb) = r(a, b), \qquad x \in G. \tag{2.8}$$

$$\text{For each} \quad t \geq 0, \qquad \theta_t = \exp(-tr) \quad \text{is positive-definite.} \tag{2.9}$$

Conversely, if r is any kernel on G/K satisfying (2.6)–(2.9) and for any point $o \in G/K$, we define f by

$$f(a, b) = \tfrac{1}{2}(r(a, o) + r(b, o) - r(a, b)), \qquad a, b \in G/K. \tag{2.10}$$

Then f is a Lévy-Schoenberg kernel on G/K.

The point o might as well be taken to be the coset eK, where e is the identity of G.

What kernels r have properties (2.6)–(2.9)? It is clear that a satisfactory answer to this question would lead to a classification of all Lévy-Schoenberg kernels. Suppose r is a kernel satisfying (2.6)–(2.9), and let $\theta(a, b) = \exp - r(a, b)$. Then $\theta(xa, xb) = \theta(a, b)$ for $x \in G$. This enables us to lift θ to a function on G. Namely, if $a = yK$, $b = zK$ with $y, z \in G$, then $\theta(a, b) = \theta(yK, zK) = \theta(z^{-1}yK, K)$; hence if the function Φ is defined on G by $\Phi(x) = \theta(xK, K)$, we see that $\theta(yK, zK) = \Phi(z^{-1}y)$. The function Φ is K-spherical, that is, $\Phi(k_1xk_2) = \Phi(x)$ for all $x \in G$, $k_1, k_2 \in K$. It is trivial to check that the properties (2.6)–(2.9) of r imply the following properties of Φ:

(2.11) $$\Phi(x^{-1}) = \Phi(x), \text{ and } \Phi \text{ is continuous;}$$

(2.12) $$\Phi(k_1xk_2) = \Phi(x), \qquad x \in G, \quad k_1, k_2 \in K;$$

(2.13) $$\Phi(e) = 1;$$

for each $t \geq 0$, Φ^t is positive-definite on G in the sense: given any $\alpha_1, \cdots, \alpha_n \in \mathbf{C}$ $x_1, \cdots, x_n \in G$, we have

(2.14) $$\sum_{i,j} \alpha_i\bar{\alpha}_j\Phi(x_j^{-1}x_i) \geq 0;$$

(2.15) $$\Phi^t(x) \to 1 \quad \text{as} \quad t \to 0, \qquad x \in G.$$

Conversely, if a function Φ on G has these properties, then, provided its logarithm is well defined, it can be shown very easily that the function $r(a, b)$ defined on G/K by

(2.16) $$r(a, b) = -\log \Phi(z^{-1}y), \qquad a = yK, \qquad b = zK,$$

enjoys properties (2.6)–(2.9). One must therefore look for functions Φ on G satisfying (2.11)–(2.15). We shall initially ignore the requirement $\Phi(x^{-1}) = \Phi(x)$ which, in the class of positive-definite functions, merely means that Φ is real-valued.

A continuous complex-valued function Φ on G is said to be (a) *K-spherical* if $\Phi(k_1xk_2) = \Phi(x)$ for all $x \in G$, k_1, $k_2 \in K$; (b) *normalized* if $\Phi(e) = 1$; and (c) *imbeddable* if Φ^t is positive-definite for each $t \geq 0$ and $\Phi^t \to 1$ as $t \to 0$. This last terminology comes about from the fact that Φ can then be imbedded in a continuous one-parameter semigroup (under pointwise multiplication), namely $\{\Phi^t, t \geq 0\}$, of positive-definite functions.

Apart from the question of existence of the logarithm of Φ, we have thus reduced the problem of describing all Lévy-Schoenberg kernels to the problem of finding real-valued, normalized, K-spherical, continuous, imbeddable, positive-definite functions on G.

A closely related notion to that of imbeddability is the notion of *infinite divisibility*. A positive-definite function Φ on G is said to be *infinitely divisible* if for each positive integer n, there exists a continuous positive-definite function Φ_n on G such that $\Phi_n^n = \Phi$.

A continuous imbeddable Φ is clearly infinitely divisible. At this point we may therefore study *the class of complex-valued, continuous, K-spherical, normalized, infinitely divisible, positive-definite* functions on G. Let $\mathfrak{D}$ denote this class.

In the next section some results obtained in this connection are described, for the following four cases.

Case I: G = A connected locally compact Abelian group.
K = Any closed subgroup of G.

Case II: G = The group of all proper (that is, orientation preserving) motions of Euclidean space R^d.
K = The subgroup of G consisting of rotations about 0.
$= SO(d)$.

Case III: G = Any compact connected semisimple Lie group.
K = A closed subgroup of G such that G/K is a (compact) symmetric space.

Case IV: G = A noncompact, connected, semisimple Lie group with finite center.
K = A maximal compact subgroup of G.

In this case G/K is a symmetric space of the noncompact type.

It turns out that in each of these cases, if $\Phi \in \mathfrak{D}$, then Φ *is necessarily imbeddable* and Φ *never vanishes.* Indeed, one can get more or less explicit representation formulas for Φ as $\Phi(x) = \exp - \Psi(x)$, and Ψ can be described precisely. Further, it is possible also to isolate those Ψ for which Φ is real-valued. It follows that if $r(a, b) = \Psi(z^{-1}y)$, $a = yK$, $b = zK$, for such Ψ, then r enjoys properties (2.7)–(2.10). Thus one gets a complete description of all Lévy-Schoenberg kernels, in the cases described above.

We might point out that cases I and II may be regarded as essentially known. Our results are new in cases III and IV. Equation (1.2) is a special case of results of III, and one gets in this way a new proof, independent of white noise integrals, of the fact that (1.2) is positive-definite. In case IV, so far as we know, no analogues of our results have been hitherto found. When properly interpreted, they give rise to Gaussian processes $\{\xi(a); a \in G/K\}$ where G/K is a noncompact symmetric space. In each of these situations (1.1) and (1.2) have analogues which will be pointed out below.

3. The class $\mathfrak{D}$

Case I. Here G is a connected, locally compact, separable Abelian group and K a closed subgroup. A function Φ on G is K-spherical if and only if Φ is constant on each K-coset, and thus Φ can be lifted to a function Φ^* on the factor group $H = G/K$, by setting $\Phi^*(xK) = \Phi(x)$. It is trivial to check that $\Phi \in \mathfrak{D}$ on G if and only if $\Phi^* \in \mathfrak{D}$ on H. (The class $\mathfrak{D}$ was defined in the context of a homogeneous space G/K. Here we are thinking of H as the homogeneous space $H/\{e\}$, and the class $\mathfrak{D}$ on H means the class $\mathfrak{D}$ for this homogeneous space.) By Bochner's theorem, the problem of describing the class $\mathfrak{D}$ on H is equivalent to characterizing probability measures on $\hat{H}$ (= the character group of H) which are infinitely divisible under convolution. The results of Parthasarathy, Ranga Rao, and Varadhan [18] may now be applied more or less directly to our situation to yield the following results.

THEOREM 3.1. *A function Φ^* on H is in the class $\mathfrak{D}$ and is real-valued if and only if Φ^* has a representation*

$$\Phi^*(a) = \exp - \left\{g(a) + \int_{\hat{H} - \{e\}} (1 - \chi(a))\, dL\,(\chi)\right\} \tag{3.1}$$

where g is a continuous nonnegative solution of

$$\tfrac{1}{2}(g(a_1 + a_2) + g(a_1 - a_2)) = g(a_1) + g(a_2), \tag{3.2}$$

and where L is a nonnegative measure on $\hat{H}$ such that $L(-A) = L(A)$, and such that L gives finite mass to the complement of any neighborhood of e, and

$$\int_{\hat{H}-\{e\}} (1 - \mathrm{Re}\chi(a))\, dL(\chi) < \infty, \qquad a \in H. \tag{3.3}$$

Further, g, L are uniquely determined by Φ^*.

The proof uses the connectedness of G in an essential way. Note that as a consequence of theorem 3.1, if $\Phi \in \mathfrak{D}$, then Φ is imbeddable.

THEOREM 3.2. *A kernel f on $H = G/K$ is a Lévy-Schoenberg kernel if and only if*

$$f(a, b) = \tfrac{1}{2}(\Psi(a) + \Psi(b) - \Psi(a - b)), \qquad a, b \in H, \tag{3.4}$$

where

$$\Psi(a) = g(a) + \int_{\hat{H}-\{e\}} (1 - \chi(a))\, dL(\chi), \tag{3.5}$$

the function g and measure L having the meanings described above.

(When H is a vector group R^d, the solutions of (3.2) are the positive semidefinite quadratic forms $g(a) = (\mathbf{G}a, a)$ where $\mathbf{G}$ is any positive semidefinite linear operator on R^d and (,) is an inner product on R^d. The classical Gauss kernel arises from this point of view when $g(a) = |a|^2$, namely when $\mathbf{G}$ = Identity.)

It is worthwhile to point out how (1.1) and its analogues can be arrived at now. Theorem 3.1 guarantees that for each $t \geq 0$ the function $a \to \exp - tg(a)$ is also in the class $\mathfrak{D}$. It can be shown easily that the class of positive-definite continuous functions is closed under the operations of taking pointwise products and uniform limits. Thus if $\Phi \in \mathfrak{D}$, then so does the function $a \to \exp(\Phi(a) - 1)$. It follows that for each $t \geq 0$ the function $a \to \exp(-(1 - \exp - tg(a)))$ is in $\mathfrak{D}$, and that for any nonnegative measure ν on $[0, \infty)$ such that

$$\int_{0+}^{\infty} (1 - \exp - tg(a))\, d\nu(t) < \infty, \qquad a \in H, \tag{3.6}$$

we have that the function Φ defined by

$$\Phi(a) = \exp - \left\{\int_0^{\infty} (1 - \exp - tg(a))\, d\nu(t)\right\} \tag{3.7}$$

is again in $\mathfrak{D}$. Thus for such ν, if we write

$$\Psi(a) = \int_0^{\infty} (1 - \exp - tg(a))\, d\nu(t), \tag{3.8}$$

then

$$f(a, b) = \tfrac{1}{2}(\Psi(a) + \Psi(b) - \Psi(a - b)) \tag{3.9}$$

is a Lévy-Schoenberg kernel.

Various choices for ν may now be made. For example it is well known that for each $0 < \alpha < 2$, there exists a nonnegative measure ν_α on $[0, \infty)$ such that

$$\int_0^{\infty} (1 - \exp - t\ell^2)\, d\nu_\alpha(t) = \ell^\alpha, \qquad \ell \geq 0. \tag{3.10}$$

Using these measures ν_α in (3.8), we see that

$$f(a, b) = \tfrac{1}{2}(g(a)^{\alpha/2} + g(b)^{\alpha/2} - g(a - b)^{\alpha/2}) \tag{3.11}$$

is a Lévy-Schoenberg kernel on H. In particular, if $G = R^d$, $K = \{0\}$, and if $\alpha = 1$ and $g(a) = |a|^2$, one recovers (1.1). The idea behind this approach is essentially due to Bochner [2], who calls this procedure subordination.

Case II. Now let G be the group of all proper rigid motions of R^d, $(d \geq 2)$, and let K be the subgroup consisting of rotations about $0 \in R^d$. Then G is a connected Lie group and K is a compact normal subgroup of G. Indeed, $K = SO(d)$, the proper orthogonal group, and G/K is topologically isomorphic to R^d. If Φ is a function on G which is constant on right K-cosets, then one may lift Φ to a function Φ^* on $R^d = G/K$, by letting $\Phi^*(a) = \Phi(x)$ where $a = xK$, $x \in G$. $\Phi \in \mathfrak{D}$ iff Φ^* has the properties (i) Φ^* is continuous, (ii) $\Phi^*(0) = 1$, (iii) Φ^* is positive-definite on R^d in the usual sense, (iv) Φ^* is infinitely divisible in the obvious sense, and (v) Φ^* is invariant under the left action of K on R^d. This means that Φ^* is a *radial* function on R^d.

Leaving aside (v) for the moment, such functions Φ^* are characterized by the classical formula of Lévy-Khinchine for fourier transforms of infinitely divisible probability measures on R^d. If one then takes into account the condition (v), the high degree of transitivity of $SO(d)$ on the unit sphere of R^d implies that Φ^* is necessarily real-valued, and indeed, one gets the following theorem.

THEOREM 3.3. *A function Φ on G is in $\mathfrak{D}$ if and only if*

$$\Phi^*(a) = \exp - \left\{c|a|^2 + \int_{0+}^{\infty} (1 - Y_d(\lambda|a|))\, dL\,(\lambda)\right\}, \tag{3.12}$$

with $a \in R^d = G/K$ where *$d \geq 2$ and where $c \geq 0$, $|a|$ is the Euclidean length of a, and Y_d is the Bessel function defined by*

$$Y_d(t) = \frac{\Gamma \frac{d}{(2)}}{\sqrt{\pi}\Gamma\left(\frac{d-1}{2}\right)} \int_0^{\pi} e^{it\cos\theta} \sin^{d-2}\theta\, d\theta. \tag{3.13}$$

$$= \Gamma\left(\frac{d}{2}\right) (2t^{-1})^{(d-2)/2} J_{(d-2)/2}(t),$$

and L is a nonnegative measure on $[0, \infty)$ such that $\int_{0+}^{\infty} (\lambda^2/1 + \lambda^2)\, dL\,(\lambda) < \infty$. The constant c and the measure L are determined uniquely by Φ^*.

Note in particular that if $\Phi \in \mathfrak{D}$, then Φ is imbeddable.

THEOREM 3.4. *A kernel f on R^d, $d \geq 2$ (identified with G/K here) is a Lévy-Schoenberg kernel if and only if*

$$f(a, b) = \tfrac{1}{2}(\Psi(a) + \Psi(b) - \Psi(a - b)), \qquad a, b \in R^d, \tag{3.14}$$

where

$$\Psi(a) = c|a|^2 + \int_{0+}^{\infty} (1 - Y_d(\lambda|a|))\, dL\,(\lambda), \tag{3.15}$$

and c, L have the meanings described above.

By choosing $c = 0$ and $dL(\lambda) = d\lambda/\lambda^{\alpha+1}$ with $0 < \alpha < 2$, one is led back $\Psi(a) = |a|^\alpha$, and

$$f(a, b) = \tfrac{1}{2}(|a|^\alpha + |b|^\alpha - |a - b|^\alpha), \qquad a, b \in R^d, \tag{3.16}$$

of which (1.1) is again a special case with $\alpha = 1$.

That these kernels (3.16) are Lévy-Schoenberg kernels is probably known to many people, though we have not seen explicit proof of it in the literature. It would be of interest to study the Gaussian process $\{\xi(a), a \in R^d\}$ which have (3.16) as their covariance, with $0 < \alpha < 2$. For $\alpha = 1$, this is done in McKean's work [16]. Of course, a proof of the fact that (3.16) are Lévy-Schoenberg kernels could have also been based on the method of subordination described in case I, instead of on explicit computation.

The restriction $d \geq 2$ can of course be dropped. A formula similar to (3.12) then results, the function Y_d being replaced by the cosine function. We do not bother with this detail.

Case III. Here G is to be a connected, *compact* semisimple Lie group, and K a closed subgroup such that G/K is a symmetric space of compact type. Élie Cartan's famous classification of such pairs G, K is well known (see, for example, Helgason [9]).

Our problem is to describe the class $\mathfrak{D}$ in this situation. The tools necessary pertain to harmonic analysis on compact groups, especially the Peter-Weyl theorem, and the theory of spherical functions, as outlined for example in [9].

If T is an irreducible unitary representation of G, T is said to be of class 1 with respect to K, if the reduction of the restriction of T to K contains the trivial representation $k \to 1$ of K. Let χ_T be the character of T, that is, χ_T is the function on G defined by $\chi_T(x) = \text{Trace } T_x$. Then the function

$$\phi_T(x) = \int_K \chi_T(x^{-1}k)\, dk, \qquad x \in G \tag{3.17}$$

is called the *elementary K-spherical function associated with T*. It is nonzero if and only if T is of class 1. The function ϕ_T satisfies the following three relations:

$$\int_K \phi_T(xky)\, dk = \phi_T(x)\cdot\phi_T(y), \qquad x, y \in G; \tag{3.18}$$

$$\phi_T(e) = 1; \tag{3.19}$$

$$\phi_T \text{ is positive-definite as a function on } G. \tag{3.20}$$

Given a continuous positive-definite function on G, there is a well-known construction, due to Gelfand and Raikov, which associates with such a function, a continuous unitary representation of G (cf. Naimark [17]). It can be shown that when this construction is applied to ϕ_T, the representation produced is equivalent to the representation T (provided, of course, $\phi_T \not\equiv 0$). Indeed, any function ϕ on G satisfying (3.18)–(3.20) gives rise by that construction to exactly one equivalence class of irreducible representations of G. We may therefore identify the set of all equivalence classes of (class 1) irreducible unitary representations of G with the set of solutions of (3.18)–(3.20), which set shall be denoted by $\mathfrak{M}$. It is countable, and will be equipped with the discrete topology, and the elements of $\mathfrak{M}$ will be labelled $\phi_0, \phi_1, \cdots, \phi_n, \cdots$, where tacitly, $\phi_0(x) \equiv 1$, $x \in G$.

The simplest members of the class $\mathfrak{D}$ are the functions

$$\pi_n(x) = \exp(\phi_n(x) - 1). \tag{3.21}$$

This may be seen as follows. It is clear by (3.18) or even (3.17) that $\phi_n(k_1xk_2) = \phi_n(x)$, $x \in G$, $k_1, k_2 \in K$. Thus π_n is K-spherical, normalized, and continuous. That π_n is positive-definite follows from (3.20) upon observing that the class of positive-definite continuous functions on G is closed under the operations of (i) multiplication by a nonnegative real constant, (ii) forming sums, (iii) pointwise multiplication, and (iv) taking limits of uniformly convergent sequences. Since $\exp \phi_n(x) = \sum_{j \geq 0} \phi_n(x)^j/j!$ and the convergence is uniform, it follows that $\exp \phi_n(x)$, and hence also $\pi_n(x)$ is positive-definite. Finally, π_n is clearly infinitely divisible, for, given any integer $j > 0$, we have $\pi_n = (\pi_{nj})^j$ where $\pi_{nj}(x) = \exp((\phi_n(x) - 1)/j)$, and this last function is positive-definite for reasons outlined above.

It follows that if $a_n \geq 0$, then the functions π of the form

$$\pi(x) = \exp\left(\sum_{n=1}^{N} a_n(\phi_n(x) - 1)\right), \qquad N \text{ finite} \tag{3.22}$$

are also in the class $\mathfrak{D}$.

Finally, since $\mathfrak{D}$ is closed under uniform limits, it follows that uniform limits of such π also belong to the class $\mathfrak{D}$.

The main theorem in case IV asserts that these functions exhaust the class $\mathfrak{D}$. Indeed, one can say more.

THEOREM 3.5. *A function Φ on G belongs to the class $\mathfrak{D}$ if and only if*

$$\Phi(x) = \exp\left(\sum_{n=0}^{\infty} a_n\,(\phi_n(x) - 1)\right) \tag{3.23}$$

where $a_n \geq 0$ and $\sum_{n=0}^{\infty} a_n < \infty$. The numbers a_n are determined uniquely by Φ.

The proof of theorem 3.5 proceeds first by showing that if Φ is infinitely divisible, it cannot vanish anywhere on G. This uses the connectedness of G. One then shows that the logarithm of Φ has the form $\sum_{n \geq 0} a_n(\phi_n(x) - 1)$, with $a_n \geq 0$ and $\sum_{n=1}^{\infty} a_n < \infty$. Note that by virtue of (3.23), if $\Phi \in \mathfrak{D}$, then Φ is imbeddable.

It can be shown that theorem 3.5 is equivalent to saying that $\Phi \in \mathfrak{D}$ if and only if there exists a continuous K-spherical positive-definite function γ on G such that $\Phi(x) = \exp(\gamma(x) - \gamma(e))$.

It is not hard to isolate those functions in $\mathfrak{D}$ which are real-valued. To do this, note that there is on $\mathfrak{M}$ an involution which sends an element of $\mathfrak{M}$ into its complex conjugate. Thus for each n we may define the integer n^* by $\phi_{n^*} = \bar{\phi}_n$. Clearly $0^* = 0$. In a variety of situations (which can be described fully, but will not be described here) it is actually true that $n^* = n$ for all n. (This will, for example, be true whenever the Weyl group of the symmetric space G/K contains

$-I$.) The real-valued elements in $\mathfrak{D}$ arise precisely from those sequences $\{a_n, n \geq 0\}$ such that $a_n \geq 0$, $\sum_{n \geq 0} a_n < \infty$ and $a_{n^*} = a_n$ for each n. Equivalently, $\Phi \in \mathfrak{D}$ and is real-valued if and only if $\Phi(x) = \exp(\gamma(x) - \gamma(e))$ where γ is a real-valued, continuous, K-spherical, positive-definite function on G.

THEOREM 3.6. *A kernel f on G/K is a Lévy-Schoenberg kernel if and only if*

$$f(a, b) = \tfrac{1}{2}(r(a, o) + r(b, o) - r(a, b)), \qquad a, b \in G/K \tag{3.24}$$

where o is the point $eK \in G/K$, and $r(xK, yK) = \Psi(y^{-1}x)$ where Ψ is a function on G of the form

$$\Psi(x) = \sum_{n \geq 0} a_n(1 - \phi_n(x)), \tag{3.25}$$

with

$$a_n \geq 0, \qquad a_{n^*} = a_n \quad \text{for each} \quad n \geq 0 \quad \text{and} \quad \sum_{n \geq 0} a_n < \infty. \tag{3.26}$$

We thus have in case III, a complete classification of Lévy-Schoenberg kernels.

It is of some interest to specialize the above to various particular cases. For example, let $G = SO(d + 1)$ and $K = SO(d)$. Then $G/K = S^d$, the unit sphere in R^{d+1}. We may choose K to be the subgroup of G which consists of rotations about the point $o = (0, 0, \cdots, 1)$, to be called the north pole. Let $a \in S^d$ and let θ be the colatitude of a; that is, θ is defined by $0 \leq \theta \leq \pi$, $\cos \theta = a \cdot o =$ the scalar product of a and o regarded as unit vectors in R^{d+1}. It can be shown very easily that a function on S^d, when regarded as a function on G, is spherical if and only if it is a function of the colatitude only. The elementary spherical functions can now be identified. They turn out to be the ultraspherical polynomials. Indeed for $x \in G$, $\phi_n(x) = P_n^{(d-1/2)}(\cos \theta)$, where θ is the colatitude of the point $xK \in S^d$. A function Φ on G is of class $\mathfrak{D}$ if and only if

$$\Phi(x) = \exp \sum_{n \geq 0} a_n(P_n^{\frac{d-1}{2}}(\cos \theta) - 1), \tag{3.27}$$

with $a_n \geq 0$, $\sum_{n \geq 0} a_n < \infty$, and is the colatitude of $a = xK \in S^d$.

In this special case all the ϕ_n are real-valued automatically, and therefore $n^* = n$ for each $n \geq 0$.

Lévy's kernel (1.2) arises from this point of view as follows. Let $\Psi(x) = \theta$ where θ is the colatitude of $xK \in S^d$. It can be shown after some computation that

$$\theta = \sum_{n \geq 0} a_n(1 - P_n^{\frac{d-1}{2}}(\cos \theta)), \tag{3.28}$$

with $a_n \geq 0$ and $\sum_{n \geq 0} a_n < \infty$. Thus by theorem 3.6, this choice gives rise to a Lévy-Schoenberg kernel. This kernel is in fact (1.2), since the geodesic distance between a and the north pole is exactly the colatitude of a.

Other choices for G and K lead to other classical polynomials. Details will not be given here, but see [5], [9].

We remark in conclusion, that the class $\mathfrak{D}$ arises in other connections also. As is clear from the work of É. Cartan, there is a duality between compact symmetric spaces and certain lattices of points with nonnegative integral coordinates in Euclidean space. Functions of class $\mathfrak{D}$ can be regarded as Fourier transforms of measures on these lattices, which are infinitely divisible in an appropriate sense. Indeed, it follows from theorem 3.5 that one gets, from each function in the class $\mathfrak{D}$, a Markov process on the corresponding lattice. A very special case of this duality was studied by Kennedy [11], where the case $G = SO(d + 1)$, $K = SO(d)$ is considered. [The existence of this paper was brought to our notice by H. P. McKean, Jr. who, during the symposium, also raised the question of giving the group theoretic meaning of this paper.] Kennedy's methods involve the use of special formulas for the ultraspherical polynomials and do not rely on harmonic analysis explicitly. We shall study these processes in a separate paper in the future.

Case IV. To begin with, let G be a noncompact connected semisimple Lie group with a finite center and let K be a maximal compact subgroup of G. In this case G/K is a symmetric space of the noncompact type [9]. We are, of course, interested in describing functions of the class $\mathfrak{D}$ in this case. The tools necessary for doing this involve the theory of spherical functions developed by Gelfand, Godement, and Harish-Chandra. A K-spherical function ϕ on G is said to be an *elementary positive-definite* spherical function if

$$\int_K \phi(xky)\, dk = \phi(x)\phi(y), \tag{3.29}$$

$$\phi(e) = 1, \tag{3.30}$$

$$\phi \text{ is positive definite.} \tag{3.31}$$

Denote by $\mathfrak{M}$ the set of elementary positive-definite spherical functions on G. We denote by 1 the function constantly 1 on G. Clearly $1 \in \mathfrak{M}$.

The set $\mathfrak{M}$ forms the dual object for the harmonic analysis of spherical functions on G (see, for example, Godement [7]). Just as in the case when G is compact, the construction of Gelfand and Raikov again in this case gives an irreducible continuous unitary representation of G, starting from a continuous positive-definite function on G. The representation so obtained is of class one if and only if ϕ is in $\mathfrak{M}$. Of course, the representation is now infinite dimensional (except when $\phi \equiv 1$). Conversely, given an irreducible continuous unitary representation T of G of class one, there exists a unique unit vector η in its representation space such that $T_k\eta = \eta$ for all $k \in K$. Then the function $\phi_T(x) = (T_x\eta, \eta)$ is an elementary positive-definite spherical function on G, and if one constructs the representation associated with ϕ_T, one gets back to the representation (equivalent to) T. Thus $\mathfrak{M}$ is really the set of equivalence classes of irreducible unitary representations of G which are of class one. It can be topologized by the topology of uniform convergence on compact subsets of G; namely, $\phi_n \to \phi$ in $\mathfrak{M}$ if and only if $\phi_n \to \phi$ uniformly on every compact subset of G. In this topology, $\mathfrak{M}$ is a locally compact Hausdorff space. Let us also note that there is a natural involution on $\mathfrak{M}$ which sends a function ϕ in $\mathfrak{M}$ into its complex conjugate $\bar{\phi}$.

The class of continuous positive definite functions on G is closed under the operations of (i) multiplication by a real nonnegative number, (ii) forming sums, (iii) pointwise multiplication, and (iv) uniform passages to limits. Utilizing these facts, one sees easily that the simplest members of the class $\mathfrak{D}$ are of the form

$$\pi(x) = \exp(\phi(x) - 1), \qquad \phi \in \mathfrak{M} \tag{3.32}$$

The argument leading to this is identical with that in case III. It follows that functions of the form

$$\pi(x) = \exp\left(\sum_{i=1}^{N} a_i(\phi_i(x) - 1)\right), \qquad a_i \geq 0, \qquad \phi_i \in \mathfrak{M} \tag{3.33}$$

are also in the class $\mathfrak{D}$, and therefore, so are uniform limits of such functions. Note that π may be written as

$$\pi_{L_N}(x) = \exp \int_{\mathfrak{M}} (\phi(x) - 1)\, dL_N(\phi) \tag{3.34}$$

where L_N is the nonnegative measure which ascribes the masses a_i to the points $\phi_i \in \mathfrak{M}$, $i = 1, \cdots, N$.

One can next show that if $\Phi \in \mathfrak{D}$, then Φ never vanishes (this uses the fact that G is connected), and that Φ is a limit, uniformly on compacts, of a sequence of functions of the form (3.34). If π_i is a sequence,

$$\pi_i(x) = \exp\left(\int_{\mathfrak{M}} (\phi(x) - 1)\, dL_i(\phi)\right) \tag{3.35}$$

where L_i are nonnegative finite measures on $\mathfrak{M}$, and if $\pi_i \to \Phi$, then one can characterize Φ completely by obtaining for it a representation formula of the Lévy-Khinchine type. In this note, we do not want to burden the reader with the description of *all* functions in the class $\mathfrak{D}$, but content ourselves with the description of the real-valued members of $\mathfrak{D}$ only, which are the only ones relevant for the description of Lévy-Schoenberg kernels. The description of all members of $\mathfrak{D}$ will be forthcoming in the detailed exposition of this note.

THEOREM 3.7. *A real-valued function Φ on G is in the class $\mathfrak{D}$ if and only if*

$$\Phi(x) = \exp - \left\{g(x) + \int_{\mathfrak{M} - \{1\}} (1 - \phi(x))\, dL(\phi)\right\} \tag{3.36}$$

where the function g and the measure L are subject to the following three requirements (a), (b), (c):

(a) *L is a nonnegative measure on $\mathfrak{M} - \{1\}$ such that if $\bar{A}$ is the image of a Borel set A $(\subset \mathfrak{M} - \{1\})$ under the involution $\phi \to \bar{\phi}$ of $\mathfrak{M}$, then*

$$L(A) = L(\bar{A}); \tag{3.37}$$

(b) *if dx is the Haar measure of G and U is any compact neighborhood of e in G, then*

$$\int_{\mathfrak{M} - \{1\}} Q(\phi)\, dL(\phi) < \infty \tag{3.38}$$

where $Q(\phi) = \int_U (1 - \mathrm{Re}\,\phi(x))\, dx$;

(c) $g(x)$ *is a function on* G *of the following form*

$$g(x) = \lim_{j\to\infty} \int_{U_j} (1 - \mathrm{Re}\phi(x))\, dL_j(\phi); \tag{3.39}$$

where $\{U_j\}_{j=1}^{\infty}$ *is a sequence of compact neighborhoods of* 1 *in* $\mathfrak{M}$ *such that* $U_{j+1} \subset U_j$ *and* $\bigcap_j U_j = \{1\}$, *and* L_j *is a finite measure supported by* U_j.

Further, the correspondence (3.36) *between real-valued functions* $\Phi \in \mathfrak{D}$ *and pairs* (g, L) *satisfying* (a), (b), (c) *is one-to-one.*

Note that theorem 3.7 implies if $\Phi \in \mathfrak{D}$, then Φ is imbeddable.

Several comments on this theorem are now in order. The theorem shows that real-valued elements Φ in $\mathfrak{D}$ are in one-to-one correspondence with pairs (g, L) where g is a function described by (a), and L is a measure described by (b). Thus it is a sort of Lévy-Khinchine formula. A function $g(x)$ of this type is to be called the Gaussian function determined by Φ, and L is called the Lévy measure determined by Φ. The result as formulated here is dual to the main result of the paper [5] of the author. There spherical probability measures on G/K which are infinitely divisible under convolution are discussed, and a representation of Lévy-Khinchine type for their Fourier transforms is obtained. The class of nonnegative, finite, spherical measures under convolution is, in a way, dual to the class positive-definite, spherical functions under pointwise multiplication. It is in this sense that the present result is the 'dual' of theorem 6.2 of [5]. Actually, each continuous normalized positive-definite spherical function on G gives rise to a unique probability measure on $\mathfrak{M}$, and a convolution of measures on $\mathfrak{M}$ can be defined in such a way that under this correspondence, pointwise multiplication of functions on G of the type described corresponds to convolution of the corresponding measures on $\mathfrak{M}$. From this point of view, the functions in the class $\mathfrak{D}$ give rise to probability measures on $\mathfrak{M}$ which are infinitely divisible under this convolution. A more detailed description will not be attempted here.

The reader will realize that theorem 3.7 is somewhat less satisfactory than the classical Lévy-Khinchine formula, insofar as the description of the Gaussian functions g is not very explicit. This is due to the present state of the art in representation theory of the semisimple groups. It turns out that for a fuller description of the functions g, one has to know the fine structure of $\mathfrak{M}$ near the point $1 \in \mathfrak{M}$. It is the absence of this information in the case of general semisimple groups that prevents a more explicit description of $g(x)$. To be more specific, it can be shown under the hypotheses in the present section that $g(x)$ is always a continuous nonnegative solution of the following functional equation:

$$\int_K (g(xky))\, dk = g(x) + g(y). \tag{3.40}$$

This functional equation is analogous to (3.2). Similar functional equations arise in other contexts as well (see, for example, Furstenberg [4]).

However, in the absence of further information about the topology of $\mathfrak{M}$ near the point $1 \in \mathfrak{M}$, we are not able to show that *each* nonnegative solution of (3.40) is a Gaussian function $g(x)$ as envisaged in theorem 3.7. There are reasons to believe that proving this last assertion would involve the knowledge of the continuous supplementary series of representations of G.

However, in special cases, we have been able to identify the Gaussian parts $g(x)$ with nonnegative continuous solutions of (3.40). This is the case, for example, if G is a complex classical simple Lie group. We can also do the same thing for the group $SL(2, R)$ = the group of all 2×2 real matrices with determinant 1. The proofs of these assertions must make use of the results of Gelfand and Naimark [6], and Bargmann [1], respectively.

The case $G = SL(2, R)$ can be studied in more detail. In this case G/K is just the Lobachevsky plane. See Helgason [9] for details. If we use the unit disc with the hyperbolic metric as a model for this plane, the function $g(x)$ can be described explicitly as follows: let the point $0 \in \{z, |z| \leq 1\}$ be taken as the identity coset eK of G/K. Then $K \cong SO(2)$ = the group of all rotations around 0. For $x \in G$, let xK be the point z into which x sends 0. Let ζ be the hyperbolic distance of xK from $eK(=0)$. Then,

$$g(x) = c \log \cosh \zeta/2, \qquad c \geq 0 \tag{3.41}$$

are all the Gaussian functions that arise.

Theorem 3.7 leads at once to the description of Lévy-Schoenberg kernels on the spaces G/K of case IV.

THEOREM 3.8. *A kernel f on G/K is a Lévy-Schoenberg kernel if and only if*

$$f(a, b) = \tfrac{1}{2}(r(a, o) + r(b, o) - r(a, b)), \qquad a, b \in G/K \tag{3.42}$$

where

$$r(xK, yK) = \Psi(y^{-1}x), \qquad x, y \in G \tag{3.43}$$

with

$$\Psi(x) = g(x) + \int_{\mathfrak{M}-\{1\}} (1 - \phi(x))\, dL\, (\phi), \tag{3.44}$$

for a pair (g, L) as described in theorem 3.7.

The process of subordination described in earlier cases leads in this case to the conclusion, among others, that if $0 \leq \alpha \leq 2$, then the kernels

$$f(xK, yK) = \tfrac{1}{2}(g(x)^{\alpha/2} + g(y)^{\alpha/2} - g(y^{-1}x)^{\alpha/2}) \tag{3.45}$$

are all Lévy-Schoenberg kernels, whenever g is as in theorem 3.7. The case $\alpha = 1$ would lead to the analogues of (1.1) in this case.

For the Lobachevsky plane, therefore, the analogue of (1.1) is

$$f(a, b) = \tfrac{1}{2}\Big(\sqrt{\log \cosh \zeta(a, 0)/2} + \sqrt{\log \cosh \zeta(b, 0)/2} - \sqrt{\log \cosh \zeta(a, b)/2}\Big) \tag{3.46}$$

where $\zeta(a, b)$ means the hyperbolic distance between a and $b \in G/K = \{z, |z| < 1\}$. This answers the question raised by Lévy in [15]. It goes without saying that the connections between the functions of class $\mathfrak{D}$ described by

$$\Phi_\alpha(x) = \exp - g(x)^{\alpha/2}, \qquad 0 < \alpha \leq 2, \tag{3.47}$$

and the fractional Riesz potential and "stable" processes to which they must give rise is a subject which is left fascinatingly open for study.

4. A discussion of the corresponding process

The positive definiteness of a Lévy-Schoenberg kernel f enables us to construct a Gaussian process $\{\xi(a), a \in G/K\}$ such that $E(\xi(a)) = 0$ and $E(\xi(a), \xi(b)) = f(a, b); a, b \in G/K$. For $a \in G/K$, $\xi(a)$ is a random variable in $L_2(\Omega, \mathcal{S}, P)$, where $(\Omega, \mathcal{S}, P)$ is some probability space, and E is the expectation. It is natural to seek a unified treatment of this class of processes. We have at present no such extensive theory of *all* the processes which arise. This section will be devoted to a brief description of some of the directions which can be pursued. Details of such results as have been obtained will be included in a fuller account of the subject of this note which is now under preparation.

Fix a Lévy-Schoenberg kernel f and let $\{\xi(a), a \in G/K\}$ be the corresponding centered Gaussian process. One of the first problems that arises is the discussion of the sample functions of these processes. While something can always be said in general, that is, applicable to all four cases I–IV, results that are interesting to us can be obtained only when the underlying space G/K has some differential structure. We shall therefore exclude case I from the considerations of this section.

In each of the cases II–IV, let f be a Lévy-Schoenberg kernel, and let Ψ be the corresponding function on G as envisaged in theorems 3.2, 3.4, and 3.6. It is possible to give a very simple condition on Ψ which guarantees that the Gaussian process which arises from f has almost surely continuous sample functions. The condition is on the behavior of Ψ near $e \in G$. It is enough, for example, that in a sufficiently small neighborhood of e in G, one has $|\Psi(x)| \leq C|x|^\beta$ for some $\beta > 0$, where $|x|$ stands for the distance of xK from eK in the natural Riemannian metric on G/K. Actually, an even weaker condition already suffices.

In cases II and IV, $\Psi(x)$ will satisfy this condition whenever $\Psi(x) = g(x)^\alpha$ $0 \leq \alpha \leq 2$. It will also satisfy this condition if the measure L in its representation decreases rapidly enough at infinity. For example, in case II, if we have $\int_{0+}^{\infty} \lambda^2 \, dL(\lambda) < \infty$, then Ψ will satisfy the above Hölder condition. In case IV, if Δ is the Laplace operator of G/K and if $\lambda(\phi)$ is the eigenvalue of the eigenfunction ϕ of Δ, that is, $\Delta\phi = \lambda(\phi)\phi$, $\phi \in \mathfrak{M}$, then the condition $\int_{\mathfrak{M}-\{1\}} |\lambda(\phi)| \, dL(\phi) < \infty$ on L can be shown to imply that Ψ satisfies the above mentioned Hölder condition.

Finally, in case III, one can similarly show that if the sequence a_n of theorem 3.4 decreases rapidly enough as $n \to \infty$, then the corresponding Ψ will have the described behavior $|\Psi(x)| < C|x|^\beta$ near $e \in G$. For example, if Δ is the Laplace operator of G/K in case III and if λ_n is the eigenvalue of ϕ_n, namely, $\Delta\phi_n = \lambda_n\phi_n$, then it is enough if $\sum_n |\lambda_n| a_n < \infty$, to have sample function continuity for the corresponding process.

Another problem is to obtain representations of ξ in terms of white noise integrals, and to attempt to develop a theory of "canonical" representations.

Let $\mathcal{B}$ be the σ-field of Borel subsets of G/K. Given a nonnegative measure μ on $\mathcal{B}$, which assigns finite mass to any compact subset, *the white noise associated with* μ is a map W which associates with each μ-finite subset $B \in \mathcal{B}$ a random variable $W(B)$ on Ω, such that the following conditions hold: (i) the distribution of $W(B)$ is Gaussian; (ii) $E(W(B)) = 0$; (iii) $E(W(B)^2) = \mu(B)$; (iv) if A and B are compact subsets in $\mathcal{B}$, and if $A \cap B = \varnothing$, then $W(A)$ and $W(B)$ are independent; and (v) if $A_1, A_2, \cdots$ is a mutually disjoint countable collection of Borel sets, and $\bigcup_{i=1}^{\infty} A_i$ has finite μ-measure, then $W(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} W(A_i)$, the sum being, say, in L_2.

Given such a white noise, one can, by using standard techniques, define stochastic integrals $\int_A h(u)\, dW(u)$ for $A \in \mathcal{B}$ and $h \in L_2(\mu)$. Such integrals are called white noise integrals. The problem is to see if the process ξ can be represented as the integral of a kernel with respect to white noise associated with one or more measures μ, and to decide if a particular one of these representations is "canonical" in terms of being the only one with "natural" properties. For an elegant account of this idea formulated by Lévy, the reader is referred to Hida [10].

Some progress can be made on this problem. Denote by H the smallest closed subspace of $L_2(\Omega, \mathcal{S}, P)$ containing $\{\xi(a), a \in G/K\}$. For $B \in \mathcal{B}$, let $\mathcal{S}_B$ be the smallest sub-σ-field of $\mathcal{S}$ generated by $\{\xi(a), a \in B\}$, and let H_B be the smallest closed subspace of H containing $\{\xi(a), a \in B\}$. Let π_B be the projection on H_B. Because ξ is Gaussian, $\pi_B(\xi(a)) = E(\xi(a)|\mathcal{S}_B)$. One can now study the von Neumann algebra generated by the projections $\{\pi_B, B \in \mathcal{B}\}$, by using the well-known decomposition theory of these algebras. This theory, coupled with the fact that there is a natural unitary equivalence between the space H and the reproducing kernel, Hilbert space H_f, whose reproducing kernel is $f(a, b)$, ought to enable us to get a generalized canonical representation for ξ, much in the same way as in Hida's paper [10]. We are able to do this in some favorable cases, but not in general.

The other class of problems which arises is the discussion of Markov properties of the process ξ as envisaged by Lévy [13] and McKean [16]. To be specific, suppose we are in case IV described above. If $\mathcal{C} = \{C_\alpha\}$ is a family of smooth surfaces in G/K, each of which disconnects G/K into an interior I_α (containing the point eK), with compact closure $I_\alpha \cup C_\alpha$ and an exterior E_α, then ξ is said to have the Markov property relative to $\mathcal{C}$ if for each α, the σ-fields $\mathcal{S}_{I_\alpha}$ and $\mathcal{S}_{E_\alpha}$ are independent conditional on the σ-field $\mathcal{S}_{C_\alpha}$. Lévy defined this notion having in mind the family of concentric spheres in R^d, and McKean pointed out the general formulation as well as the connection between the Markov property and certain Dirichlet problems, for the special kernels (1.1).

We have been able to make some progress on these questions also. To keep things simple, let us confine attention to those cases where G/K is two-point homogeneous, (see [9]) and a process ξ whose sample functions are continuous. In that case, a point $a \in G/K$ corresponds to a pair $(|a|, \theta)$ where $|a|$ is the distance of a from o, and θ is a point of the unit sphere in G/K. The distance $|a|$

ranges over a finite or infinite interval according to whether or not G/K is compact. We can now imitate McKean's procedure in [16] and obtain an expansion

$$\xi(a) = \sum_{n,\ell} \xi_n^\ell(|a|)\chi_n^\ell(\theta), \qquad a \sim (|a|, \theta) \tag{4.1}$$

for the process ξ. This expansion converges in $L^2(\Omega)$. Here, $\xi_n^\ell(\cdot)$ are certain Gaussian processes, and χ_n^ℓ are the spherical harmonics for the natural action of K on the unit sphere in G/K. One can express the covariance of ξ_n^ℓ in terms of the covariance of ξ. It is then also possible to relate the Markov property (relative to spheres) for ξ to the Markov properties of the sequence ξ_n^ℓ of one-parameter processes. The details of these results and the illustrations of their applicability must await the fuller exposition of this paper mentioned above.

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