SIGN-INVARIANT RANDOM ELEMENTS IN TOPOLOGICAL GROUPS

SIMEON M. BERMAN
COLUMBIA UNIVERSITY

1. Introduction

The concept of sign-invariant random variables was recently introduced by the writer in [1]: $X_1, \cdots, X_k$ are called sign-invariant if the 2^k joint distributions corresponding to the sets $(\epsilon_1 X_1, \cdots, \epsilon_k X_k)$, $\epsilon_1 \pm 1, \cdots, \epsilon_k = \pm 1$ are all the same. A family of random variables $\{X_t, t \in T\}$, where T is some index set, is called sign-invariant if every finite subfamily consists of sign-invariant random variables. An example is a family of independent random variables with symmetric distributions.

During the last several years, probabilists have been extending their interest from random variables and vectors in Euclidean space to random elements in abstract spaces, particularly topological groups. Grenander's monograph [3] contains a large bibliography of work up to 1963. In this paper we shall generalize some of the properties of sign-invariant random variables on the real line [1] to sign-invariant random elements in a commutative, locally compact topological group G having a countable base. This may be read independently of the previous paper. The definition of sign-invariant random elements is a direct extension of the definition given above for random variables: the X's are elements of the group, and $-X$ is the inverse of X under the group addition.

The Fourier transform was the main tool in the study of sign-invariant random variables on the line; this suggested the generalization to a commutative, locally compact group, which is the natural domain of Fourier analysis.

Section 2 contains the fundamental structure lemmas of sign-invariance. One of the main ones is that sign-invariant random elements are conditionally independent and symmetrically distributed, given a nontrivial sub-σ-field. In section 3 we present some group-theoretic results which characterize the convergence of a sequence of elements in a group G in terms of the convergence of their images under the mappings induced by the character group. The fundamental convergence theorem for series of sign-invariant random elements is given in section 4.

The main part of this paper, sections 5, 6, and 7, is about the stochastic process with sign-invariant increments: a stochastic process in G with a real interval parameter set such that increments of the process over nonoverlapping intervals are sign-invariant. An example is a process with symmetric independent incre-

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ments. We show that a process in G with sign-invariant increments has the same sample function characteristics as real-valued semimartingales and processes with independent increments. First we define separability for processes in G; then we show that almost all sample functions of a separable version of the process are bounded and have right- and left-hand limits at each point of an interval where such limits are definable; furthermore, there are at most a countable number of fixed points of discontinuity on an interval. These properties are found by studying the image of the sample functions under the mappings induced by the character group. These images are complex-valued processes closely related to martingales. This method of analysis was first used by Doob for processes on the real line ([2], p. 389).

We now formally state our assumptions. First, $(\Omega, \mathfrak{F}, P)$ is a probability space, where Ω is a set of points ω. Second, G is a commutative, locally compact topological group having a countable base. The latter assumption, which happens to make G a separable metric space ([7], p. 15), identifies the class of Baire sets in G with the class of Borel sets ([4], p. 220). A random element in G is a function $X = X(\omega)$, $\omega \in \Omega$, from Ω into G such that the inverse image of every Borel set in G is in $\mathfrak{F}$. The sum of two random elements is a random element because the Borel sets are the Baire sets ([4], p. 222). The argument ω of $X(\omega)$ will be understood but not explicitly recorded. Properties which are valid for all $\omega \in \Omega$, with the exception of a set of P-measure 0, will be said to hold a.e.(P).

Let Γ be the character group of G: the set of all representations γ of G into the set of complex numbers of modulus one, denoted by T. Also Γ is commutative, locally compact, and has a countable base ([7], p. 100). By the measure space $(\Gamma, \mathfrak{K}, \mu)$ we shall mean the group Γ with the σ-field $\mathfrak{K}$ of Borel sets and Haar measure μ. A property holding for all γ except for a set of μ-measure 0 will be said to hold a.e.(μ). We shall assume one more property for Γ, namely, that Γ is connected; this will be used from time to time. However, G cannot be compact, for then Γ is discrete, and cannot be connected.

2. Structure of sign-invariance

The following are generalizations to G of several real line propositions in [1].

LEMMA 2.1. *Let $X_1, \cdots, X_k$ be sign-invariant and let $f_j(x)$, $g_j(x)$ be bounded continuous, complex-valued functions on G such that $g_j(x) = g_j(-x)$, $j = 1, \cdots, k$, $x \in G$; then,*

$$E\left\{\prod_{j=1}^{k} f_j(X_j)g_j(X_j)\right\} = E\left\{\prod_{j=1}^{k} [\tfrac{1}{2}(f_j(X_j) + f_j(-X_j))]g_j(X_j)\right\}. \tag{2.1}$$

PROOF. Changing the sign of X_1 and using sign-invariance, we have

$$\begin{aligned} E\left\{f_1(X_1)g_1(X_1)\prod_{j=2}^{k} f_j(X_j)g_j(X_j)\right\} \\ = E\left\{f_1(-X_1)g_1(X_1)\prod_{j=2}^{k} f_j(X_j)g_j(X_j)\right\}; \end{aligned} \tag{2.2}$$

hence, the latter is also equal to the average of the quantities on either side of the equality sign, namely, to

$$E\left\{\tfrac{1}{2}[f_1(X_1) + f_1(-X_1)]g_1(X_1) \prod_{j=2}^{k} f_j(X_j)g_j(X_j)\right\}. \tag{2.3}$$

We obtain (2.1) by successively changing signs and averaging over $X_2, \cdots, X_k$.

Let (x, γ) denote the value in T of the character γ at the point x, $x \in G, \gamma \in \Gamma$; let $\bar{z}$ be the complex conjugate of z and $\mathcal{R}z$ the real part of z. From (2.1) and the relation $(-x, \gamma) = \overline{(x, \gamma)}$, we get

$$E\left\{\prod_{j=1}^{k} (X_j, \gamma_j)g_j(X_j)\right\} = E\left\{\prod_{j=1}^{k} \mathcal{R}(X_j, \gamma_j)g_j(X_j)\right\}, \qquad \gamma_1, \cdots, \gamma_k \in \Gamma. \tag{2.4}$$

When x is real, $|x|$ is the function which identifies x with $\max(-x, x)$. In [1], we made fundamental use of conditioning sign-invariant random variables with respect to their absolute values. Since the "positive" member of the pair $(x, -x)$ is not distinguished in an unordered group, we shall define the above conditioning for sign-invariant random elements in a more general way, as follows: let $\mathcal{E}$ be the class of all "even" complex-valued Borel functions on G, that is, g is in $\mathcal{E}$ if and only if $g(x) = g(-x)$, $x \in G$. We define the σ-field generated by $|X_1|, \cdots, |X_k|$ as the σ-field generated by the complex-valued random variables $g_1(X_1), \cdots, g_k(X_k)$, $g_1, \cdots, g_k \in \mathcal{E}$. We define conditional expectation given $|X_1|, \cdots, |X_k|$ as conditional expectation with respect to that σ-field. Then equation (2.1) is identical with the following:

$$E\left\{\prod_{j=1}^{k} f_j(X_j) \middle| |X_1|, \cdots, |X_k|\right\} = \prod_{j=1}^{k} \tfrac{1}{2}[f_j(X_j) + f_j(-X_j)]; \tag{2.5}$$

in fact, $(\frac{1}{2})[f(x) + f(-x)]$ is a function belonging to $\mathcal{E}$. From (2.4) we conclude:

LEMMA 2.2. *The elements $X_1, \cdots, X_k$ are sign-invariant if and only if they are conditionally independent, given $|X_1|, \cdots, |X_k|$, with the conditional joint characteristic function $\prod_{j=1}^{k} \mathcal{R}(X_j, \gamma_j)$.*

Putting γ_j in (2.4) equal to the identity of Γ, $j = h + 1, \cdots, k$, we obtain:

LEMMA 2.3. *The conditional characteristic function of $X_1, \cdots, X_h$, given $|X_1|, \cdots, |X_k|$, $1 \le h \le k$, is independent of $|X_{h+1}|, \cdots, |X_k|$.*

The conditional characteristic function of $X_1 + X_2$ and $X_3 + X_4$, given $|X_1 + X_2|$ and $|X_3 + X_4|$, is equal to the conditional characteristic function given $|X_1|, |X_2|, |X_3|, |X_4|, |X_1 + X_2|, |X_3 + X_4|$; this can be verified by the kind of calculation in the proof of lemma 2.1. More generally, we have:

LEMMA 2.4. *Let X_{ij}, $i = 1, \cdots, n_j$, $j = 1, \cdots, k$ be sign-invariant; then the conditional joint characteristic function of the k sums $\sum_{i=1}^{n_j} X_{ij}$, $j = 1, \cdots, k$ given their absolute values is the same as the conditional joint characteristic function given their absolute values as well as those of all their summands and partial summands.*

3. Convergence in topological groups

Here are several propositions relating convergence of elements in G with convergence of their images in T under the mappings induced by the characters of G.

Lemmas 3.1 and 3.4 appear on the real line in the following form in ([2], p. 335): if $\{x_n\}$ is a real sequence such that $\lim_{n\to\infty} e^{iux_n}$ exists for all u in a set of positive Lebesgue measure, then $\lim_{n\to\infty} x_n$ exists and is finite. We now generalize this result.

LEMMA 3.1. *Let $\{x_n\}$ be a sequence in G. If $\lim_{n\to\infty} (x_n, \gamma)$ exists for all γ, then $\lim_{n\to\infty} x_n$ exists in G.*

PROOF. Let $\hat{G} = G \cup \{\infty\}$ be the one-point compactification of G. The neighborhoods of $\{\infty\}$ in $\hat{G}$ are the complements of compact subsets of G because G is locally compact ([5], p. 150). Either all elements of $\{x_n\}$ are in a compact subset of G or some subsequence converges to $\{\infty\}$. We shall show that the latter is impossible.

Let $\{x'_n\}$ be a subsequence converging to $\{\infty\}$, and let f be a real-valued continuous function on Γ with compact support. We have

$$\lim_{n\to\infty} \int_\Gamma (x'_n, \gamma) f(\gamma)\, d\mu(\gamma) = 0 \tag{3.1}$$

by the Riemann-Lebesgue theorem ([7], p. 116). Here the interchange of order of limit and integration is permitted by dominated convergence; hence,

$$\int_\Gamma \lim_{n\to\infty} (x'_n, \gamma) f(\gamma)\, d\mu(\gamma) = 0. \tag{3.2}$$

The validity of this equation for every f implies that $\lim_{n\to\infty} (x'_n, \gamma) = 0$, a.e.$(\mu)$; however, we have $|(x'_n, \gamma)| \equiv 1$, a contradiction. This negates the possibility that $x'_n \to \infty$.

In accordance with the first alternative, the elements of $\{x_n\}$ belong to some compact subset of G and so must have at least one point of accumulation. We shall show that there is at most one such point: this will finish the proof. If y_1 and y_2 are accumulation points, then, as the hypothesis implies, they must satisfy the equation $(y_1, \gamma) = (y_2, \gamma)$ identically in $\gamma \in \Gamma$. This implies $y_1 = y_2$ ([7], p. 99).

LEMMA 3.2. *Let $f(t)$ be a function on a real number set S to G, and τ a point of accumulation of S from below (above). If $\lim (f(t), \gamma)$, $t \uparrow \tau$ $(t \downarrow \tau)$, $t \in S$, exists for all $\gamma \in \Gamma$, then the corresponding limit of $f(t)$ exists in G.*

PROOF. It is enough to prove that $f(t)$ approaches a unique limit along every sequence in S converging to τ from below (above), because G has a countable base. The discussion in ([2], p. 409) demonstrates that it is even enough to show the existence of a limit along each monotone sequence in S converging to τ. The existence of such a limit is a consequence of lemma 3.1 and the hypothesis of our lemma.

We use the connectedness of Γ to strengthen our results.

LEMMA 3.3. *Let $2H$ be the image of a set $H \subset \Gamma$ under the mapping*

$$\gamma \to \gamma + \gamma = 2\gamma. \tag{3.3}$$

If $\mu(H) > 0$, then any subgroup of Γ containing $2H$ is identical with Γ.

PROOF. Any subgroup containing H is identical with Γ; in fact, $H - H$ is a neighborhood of the identity in Γ, and since Γ is connected, it is generated by such a neighborhood ([7], pp. 13, 50). This implies that a subgroup containing $2H$ also contains the subgroup 2Γ. To complete the proof, we show that $\Gamma = 2\Gamma$. Let G' be the annihilator of 2Γ, that is, the subgroup of G defined by the relation $G' = \{x\colon (x, 2\gamma) = 1, \gamma \in \Gamma\}$. If $x_0 \in G'$, then we have the following chain of implications: $(2x_0, \gamma) = (x_0, 2\gamma) = 1$ for all $\gamma \in \Gamma$; $2x_0 = 0$; x_0 is an element of finite order; $x_0 = 0$ ([7], p. 110); G' contains only 0; $\Gamma = 2\Gamma$.

LEMMA 3.4. *The hypotheses of lemmas 3.1 and 3.2 are fulfilled if the limits of $(x_n, 2\gamma)$ and $(f(t), 2\gamma)$ are assumed to exist for all γ in some set H, $H \in \mathcal{H}$, $\mu(H) > 0$.*

PROOF. The γ-set for which such a limit exists is a subgroup of Γ. Lemma 3.3 applies to such a subgroup.

4. Convergence of sums of sign-invariant random elements

As is well known, a series of independent random variables converges in distribution if and only if it converges with probability 1. In [1], we proved this relation for a series of sign-invariant random variables. The proof depended in part on some facts from the theory of series of independent random variables. Now we prove a general theorem for series of sign-invariant random elements in G, one which does not depend on the case of independence. The method used is an extension of a martingale analysis first used by Doob on the real line ([2], p. 335).

In the following, all random elements are defined on the common probability space Ω.

LEMMA 4.1. *The complex-valued function (X, γ), where X is a random element in G and γ is in Γ, is measurable with respect to the product σ-field $\mathcal{F} \times \mathcal{H}$ in $\Omega \times \Gamma$.*

PROOF. The mapping from $\Omega \times \Gamma$ into $G \times \Gamma$ defined by $\omega, \gamma \to X, \gamma$ is $\mathcal{F} \times \mathcal{H}$ measurable; and the mapping of $G \times \Gamma$ into T defined by $x, \gamma \to (x, \gamma)$ is continuous; therefore, the composite mapping is $\mathcal{F} \times \mathcal{H}$ measurable.

THEOREM 4.1. *Let $\{X_n, n = 1, 2, \cdots,\}$ be a sequence of sign-invariant random elements in G. The following three conditions are equivalent:*

(i) *$\sum_{n=1}^{\infty} X_n$ converges in distribution;*
(ii) *for every ϵ, $0 < \epsilon < 1$, the γ-set for which $\prod_{n=1}^{\infty} |\Re(X_n, \gamma)| > \epsilon$ has positive μ-measure, a.e.(P);*
(iii) *$\sum_{n=1}^{\infty} X_n$ converges a.e.(P).*

PROOF. It is well known that (iii) implies (i); we shall prove that (i) implies (ii), and that (ii) implies (iii).

By the continuity theorem for characteristic functions [3], (i) implies that the characteristic function of $X_1 + \cdots + X_n$, namely, $E\{\prod_{k=1}^{n} \mathcal{R}(X_k, \gamma)\}$, converges to a characteristic function $\varphi(\gamma)$, $\gamma \in \Gamma$. Applying the bounded convergence theorem to $\prod_{k=1}^{n} \mathcal{R}(X_k, \gamma)$, we get

$$E\left\{\prod_{n=1}^{\infty} |\mathcal{R}(X_n, \gamma)|\right\} \geq \lim_{n\to\infty} E\left\{\prod_{k=1}^{n} \mathcal{R}(X_k, \gamma)\right\} = \varphi(\gamma), \qquad \gamma \in \Gamma. \tag{4.1}$$

Let A be a set in $\mathfrak{F}$ with indicator function I_A, and let θ be the identity of Γ; then, by (4.1), we have

$$\begin{aligned} \lim_{\gamma\to\theta} &\left| P(A) - E\left\{I_A \prod_{n=1}^{\infty} |\mathcal{R}(X_n, \gamma)|\right\}\right| \\ &= \lim_{\gamma\to\theta} \left| E\left\{I_A\left(1 - \prod_{n=1}^{\infty} |\mathcal{R}(X_n, \gamma)|\right)\right\}\right| \leq \lim_{\gamma\to\theta} [1 - \varphi(\gamma)] = 0. \end{aligned} \tag{4.2}$$

Now let A denote the ω-set for which $\prod_{n=1}^{\infty} |\mathcal{R}(X_n, \gamma)| \leq \epsilon$, a.e.$(\mu)$. The set A belongs to $\mathfrak{F}$ by lemma 4.1 and Fubini's theorem. The latter also implies that $\prod_{n=1}^{\infty} |\mathcal{R}(X_n, \gamma)| \leq \epsilon$, for almost all $\omega \in A$, a.e.(μ); therefore,

$$E\left\{I_A \prod_{n=1}^{\infty} |\mathcal{R}(X_n, \gamma)|\right\} \leq \epsilon P(A), \qquad \text{a.e.}(\mu). \tag{4.3}$$

Comparing this with (4.2), we see that $P(A) \leq \epsilon P(A)$, so that $P(A) = 0$; thus (i) implies (ii).

We now prove (iii). Let $I(\epsilon, \gamma)$ be the indicator function of the $\mathfrak{F} \times \mathfrak{K}$-measurable set where $\prod_{n=1}^{\infty} |\mathcal{R}(X_n, \gamma)| > \epsilon$. The sequence

$$\left(\sum_{k=1}^{n} X_k, \gamma\right) I(\epsilon, \gamma) \Big/ \left\{\prod_{k=1}^{n} \mathcal{R}(X_k, \gamma)\right\}, \qquad n = 1, 2, \cdots, \tag{4.4}$$

is bounded by $1/\epsilon$, so that all its expectations exist for each γ. Let $\mathfrak{F}_n$ denote the σ-field in Ω generated $X_1, \cdots, X_n, |X_{n+1}|, |X_{n+2}|, \cdots$, $n = 1, 2, \cdots$; we have $\mathfrak{F}_n \subset \mathfrak{F}_{n+1} \subset \mathfrak{F}$ for every n. For each γ, the sequence in (4.4) is a martingale with respect to the sequence of σ-fields $\{\mathfrak{F}_n\}$. In fact, application of the conditional expectation operator $E(\cdot|\mathfrak{F}_n)$ to the $n + 1^{\text{st}}$ element in (4.4) leaves $I(\epsilon, \gamma)$ and the denominator unchanged, and transforms $(\sum_{k=1}^{n+1} X_k, \gamma)$ into $\mathcal{R}(X_{n+1}, \gamma)(\sum_{k=1}^{n} X_k, \gamma)$ by lemma 2.3. The martingale convergence theorem applies to the sequence in (4.4) because it is bounded: for each γ the limit exists a.e.(P). From this, lemma 4.1, and Fubini's theorem, we infer the existence of the limit a.e.(μ), a.e.(P). The sequence obtained from (4.4) by squaring each element, namely,

$$\left(\sum_{k=1}^{n} X_k, 2\gamma\right) I(\epsilon, \gamma) \Big/ \left\{\prod_{k=1}^{n} \mathcal{R}^2(X_k, \gamma)\right\}, \tag{4.5}$$

also has a limit a.e.(μ), a.e.(P). By definition of $I(\epsilon, \gamma)$, the limit of the denominator exists and is greater than ϵ^2 for almost every pair ω, γ for which $I(\epsilon, \gamma) = 1$; therefore, the limit of $(\sum_{k=1}^{n} X_k, 2\gamma)$ exists for all such pairs. The latter exists on a set of positive μ-measure, a.e.(P); in fact, the set of γ's for which $I(\epsilon, \gamma) = 1$

has positive μ-measure, a.e.(P), by (ii). From this and from lemmas 3.1 and 3.4 (iii) follows.

5. Stochastic processes with sign-invariant increments

Let $X(t)$, $a \leq t \leq b$, be a stochastic process on $(\Omega, \mathfrak{F}, P)$ to G. The process is said to have sign-invariant increments if for every finite set of disjoint intervals $(a_1, b_1), \cdots, (a_k, b_k)$, in $[a, b]$ the random elements $X(b_1) - X(a_1), \cdots, X(b_k) - X(a_k)$ are sign-invariant. An important example of such a process is one with independent symmetric increments.

Let $\{t_n, n = 1, 2, \cdots\}$ be a sequence of distinct points in $[a, b]$; assume $t_1 = a$, $t_2 = b$. For $n \geq 2$, let $t_0^{(n)}, t_1^{(n)}, \cdots, t_n^{(n)}$ be the first $n + 1$ elements of $\{t_n\}$ arranged so that $a = t_0^{(n)} < t_1^{(n)} < \cdots < t_n^{(n)} = b$. Define the following:

$$
\begin{aligned}
X_{nk} &= X(t_k^{(n)}) - X(t_{k-1}^{(n)}), && k = 1, \cdots, n, \\
\mathcal{B}_n &= \sigma\text{-field in } \Omega \text{ generated by } && |X_{nk}|, k = 1, \cdots, n, \\
\mathcal{B}^{(n)} &= \sigma\text{-field in } \Omega \text{ generated by } && \mathcal{B}_n, \mathcal{B}_{n+1}, \cdots, \\
\mathcal{B} &= \bigcap_{n=1}^{\infty} \mathcal{B}^{(n)}.
\end{aligned}
\tag{5.1}
$$

These σ-fields are all contained in $\mathfrak{F}$.

LEMMA 5.1. *For every $\gamma \in \Gamma$, $n \geq 2$, we have*

$$
\begin{aligned}
E\{(X(b) - X(a), \gamma) | \mathcal{B}_n\} &= E\{\mathcal{R}(X(b) - X(a), \gamma) | \mathcal{B}_n\} \\
&= E\{\mathcal{R}(X(b) - X(a), \gamma) | \mathcal{B}^{(n)}\} = \prod_{k=1}^{n} \mathcal{R}(X_{nk}, \gamma), \qquad \text{a.e.}(P).
\end{aligned}
\tag{5.2}
$$

PROOF. Write $X(b) - X(a)$ as $X_{n1} + \cdots + X_{nn}$; then the first equality in (5.2) follows from lemma 2.4 (for the case $k = 1$). Lemma 2.2 implies the equality of the first and last members of (5.2).

We now link the third member of (5.2) to the others. For any integer $m > 0$, the conditional characteristic function of $X(b) - X(a)$ given $\mathcal{B}_n$ is the same as the conditional characteristic function given $\mathcal{B}_n, \cdots, \mathcal{B}_{n+m}$, by lemma 2.4. The latter converges a.e.(P) to the conditional characteristic function, given $\mathcal{B}^{(n)}$ for $m \to \infty$, by the martingale theorem for conditional expectations ([2], p. 331).

LEMMA 5.2. *For $\gamma \in \Gamma$, we have*

$$
E\{\mathcal{R}(X(b) - X(a), \gamma) | \mathcal{B}\} = \lim_{n \to \infty} \prod_{k=1}^{n} \mathcal{R}(X_{nk}, \gamma), \qquad \text{a.e.}(P). \tag{5.3}
$$

PROOF. Let $n \to \infty$ in (5.2) and apply the previously mentioned martingale theorem.

LEMMA 5.3. *Lemmas 5.1 and 5.2 remain valid if a and b are replaced by any two elements t' and t'', $t' < t''$, of the sequence $\{t_n\}$, and the product of $\mathcal{R}(X_{nk}, \gamma)$ is restricted to X_{nk}-increments over subintervals of $[t', t'']$.*

PROOF. By Lemma 2.3, the conditional characteristic function of a subset of $\{X_{nk}, k = 1, \cdots, n\}$, given all of the absolute values $|X_{nk}|$, depends only on those absolute values corresponding to members of the subset. This makes it possible to prove our lemma in the same way as lemmas 5.1 and 5.2.

LEMMA 5.4. *For* ϵ, $0 < \epsilon < 1$, *the* γ-*set for which* $|E\{\mathcal{R}(X(b) - X(a), \gamma)|\mathcal{B}\}| > \epsilon$ *has positive* μ-*measure, a.e.* (P).

PROOF. For $A \in \mathfrak{F}$, we have, by bounded convergence,

$$P(A) = \lim_{\gamma \to \theta} E\{I_A(X(b) - X(a), \gamma)\}. \tag{5.4}$$

Here we are permitted to interchange the order of limit and expectation because Γ has a countable base, and, consequently, the convergence $\gamma \to \theta$ may be defined as sequential convergence.

The proof is now similar to that of the first part of theorem 4.1. Let A be the ω-set for which $|E\{\mathcal{R}(X(b) - X(a), \gamma)|\mathcal{B}\}| \leq \epsilon$, a.e.$(\mu)$. By Fubini's theorem, this inequality holds a.e.(P) on A, a.e.(μ); hence, since $A \in \mathcal{B}$, we have

$$\begin{aligned} |E\{I_A(X(b) - X(a), \gamma)\}| \\ = |E\{I_A E[\mathcal{R}(X(b) - X(a), \gamma)|\mathcal{B}]\}| \leq \epsilon P(A), \qquad \text{a.e.}(\mu). \end{aligned} \tag{5.5}$$

This and (5.4) show that $P(A) \leq \epsilon P(A)$; hence, $P(A) = 0$.

LEMMA 5.5. *Let* t *be a point of* (a, b) *such that there is a subsequence* $\{s_n\}$ *of* $\{t_n\}$ *converging to* t *from below (above). Then* $\lim_{n\to\infty} X(s_n)$ *exists a.e.*(P).

PROOF. Without loss of generality, we may assume that $\{s_n\}$ is strictly increasing (decreasing). We shall show that the series of sign-invariant random elements $\sum_{n=1}^{\infty} [X(s_{n+1}) - X(s_n)]$ converges, a.e.(P). By theorem 4.1, it is sufficient to show that for ϵ, $0 < \epsilon < 1$, the infinite product $\prod_{n=1}^{\infty} |\mathcal{R}(X(s_{n+1}) - X(s_n), \gamma)|$ is greater than ϵ on a γ-set of positive μ-measure, a.e.(P). This condition is satisfied in our case. Define a sequence of partitions by

$$t_0^{(n)} = s_1, \qquad t_1^{(n)} = s_2, \cdots, t_{n-1}^{(n)} = s_n, \qquad t_n^{(n)} = t, \qquad n = 2, 3, \cdots, \tag{5.6}$$

and apply lemmas 5.2 and 5.4.

We now define a class $\mathcal{C}$ of complex-valued functions $g(t)$, $a \leq t \leq b$: g is said to belong to $\mathcal{C}$ if it approaches a limit along any monotone subsequence of $\{t_n\}$, that is, if $\{s_n\}$ is a monotone subsequence of $\{t_n\}$, then $\lim_{n\to\infty} g(s_n)$ exists. The following proposition is implicit in the proof of the martingale sample function theorem ([2], p. 361).

LEMMA 5.6. *A (complex-valued) stochastic process on* $[a, b]$, *whose restriction to the parameter set* $\{t_n\}$ *is a martingale, has the property that almost all sample functions belonging to the class* $\mathcal{C}$.

Next we define a class $\mathfrak{D}$ of functions $f(t)$, $a \leq t \leq b$, taking values in G. The function f is said to belong to $\mathfrak{D}$ if it approaches a limit along any monotone subsequence of $\{t_n\}$. The next proposition relates membership of f in $\mathfrak{D}$ to the membership in $\mathcal{C}$ of its images in T under the mappings induced by the group of characters.

LEMMA 5.7. *If there is a set* $H \in \mathcal{H}$, $\mu(H) > 0$, *such that* $(f(\cdot), 2\gamma)$ *belongs to* $\mathcal{C}$ *for all* $\gamma \in H$, *then* f *belongs to* $\mathfrak{D}$.

PROOF. Apply lemmas 3.1 and 3.4.

Here is the fundamental sample function theorem for processes in G with sign-invariant increments.

THEOREM 5.1. *Almost all sample functions of $X(t)$ belong to $\mathfrak{D}$.*

PROOF. We shall show that almost all sample functions satisfy the hypothesis of lemma 5.7.

The proof is similar to that of theorem 4.1 in the part where we show that (ii) implies (iii). Fix ϵ, $0 < \epsilon < 1$, and let $I(\epsilon, \gamma)$ be the indicator function of the set in $\Omega \times \Gamma$, where $|E\{\mathfrak{R}(X(b) - X(a), \gamma)|\mathfrak{B}\}|$ is greater than ϵ; $I(\epsilon, \gamma)$, is $\mathfrak{F} \times \mathfrak{K}$ measurable, by lemmas 4.1, 5.1, and 5.2.

For each element t' of the sequence $\{t_n\}$, we define the σ-field $\mathfrak{F}_{t'} \subset \mathfrak{F}$ as the σ-field generated by $X(s)$, $a \leq s \leq t'$. For each $\gamma \in \Gamma$, the complex-valued stochastic process

$$\frac{(X(t) - X(a), \gamma)I(\epsilon, \gamma)}{E\{\mathfrak{R}(X(t) - X(a), \gamma)|\mathfrak{B}\}}, \qquad a \leq t \leq b, \tag{5.7}$$

restricted to the parameter set $\{t_n\}$, is a bounded (by $1/\epsilon$) martingale with respect to the family of σ-fields $\{\mathfrak{B}, \mathfrak{F}_t, t \in \{t_n\}\}$; in fact, the proof of this assertion is the same as the proof that (4.4) is a martingale. The sample functions of this process are almost all elements of the class $\mathfrak{C}$, by lemma 5.6, for each $\gamma \in \Gamma$. The process obtained from (5.7) by squaring the variables, that is, the process

$$\frac{(X(t) - X(a), 2\gamma)I(\epsilon, \gamma)}{E^2\{\mathfrak{R}(X(t) - X(a), \gamma)|\mathfrak{B}\}}, \qquad a \leq t \leq b, \tag{5.8}$$

also has almost all sample functions in the class $\mathfrak{C}$; in fact, the mapping $z \to z^2$ is continuous.

The stochastic process $E^2\{\mathfrak{R}(X(t) - X(a), \gamma)|\mathfrak{B}\}$, is nonincreasing a.e.$(P)$ on the parameter set $\{t_n\}$; in fact, if t', t'' are elements of $\{t_n\}$ with $t' < t''$, then

$$\begin{aligned} &E^2\{\mathfrak{R}(X(t'') - X(a), \gamma)|\mathfrak{B}\} \\ &\quad = E^2\{\mathfrak{R}(X(t'') - X(t'), \gamma)|\mathfrak{B}\} \cdot E^2\{\mathfrak{R}(X(t') - X(a), \gamma)|\mathfrak{B}\} \\ &\quad \leq E^2\{\mathfrak{R}(X(t') - X(a), \gamma)|\mathfrak{B}\}, \qquad \text{a.e.}(P), \end{aligned} \tag{5.9}$$

by lemmas 5.2 and 5.3. It follows that almost all sample functions of the denominator in (5.8) are elements of the class $\mathfrak{C}$; consequently, the numerator sample functions are almost all in $\mathfrak{C}$, for each $\gamma \in \Gamma$. By Fubini's theorem, we infer that the t-functions $(X(t), 2\gamma)I(\epsilon, \gamma)$ belong to $\mathfrak{C}$, a.e.(μ), a.e.(P). This implies that $(X(t), 2\gamma)$ belongs to $\mathfrak{C}$, a.e.(μ), on the set where $I(\epsilon, \gamma) = 1$, a.e.(P). The set of γ's on which $I(\epsilon, \gamma) = 1$ has positive μ-measure, a.e.(P), by lemma 5.4; therefore, almost all sample functions satisfy the hypothesis of lemma 5.7. The proof is complete.

6. Properties of separable processes

A stochastic process $X(t)$, $a \leq t \leq b$, on $(\Omega, \mathfrak{F}, P)$ to G is said to be separable relative to the class of closed sets, or simply, separable, if there exists a sequence

$\{t_n\}$ in $[a, b]$ and a set $\Lambda \in \mathfrak{F}$ of probability 0 such that for any open interval $I \subset [a, b]$ and any closed set $F \subset G$, the ω-sets $\{X(t') \epsilon F, t' \epsilon \{t_n\} \cap I\}$ and $\{X(t) \epsilon F, t \in I\}$ differ by a subset of Λ. We may assume that $\{t_n\}$ is dense in $[a, b]$ because a separability sequence can always be enlarged without affecting its role. In this section $X(t)$ will be assumed separable with a dense separability sequence in $[a, b]$.

LEMMA 6.1. *The sample functions of $X(t)$ are bounded on $[a, b]$, a.e.(P).*

PROOF. By theorem 5.1, $X(t)$ has limits along every monotone subsequence of $\{t_n\}$, a.e.(P); therefore, $X(t)$ is bounded over the sequence $\{t_n\}$, a.e.(P). Separability implies boundedness over $[a, b]$, a.e.(P).

We now relate the separability of $X(t)$ to the separability of the complex-valued stochastic process $(X(t), \gamma)$ for each γ.

LEMMA 6.2. *Let $X(t)$ be separable, with separability sequence $\{t_n\}$ and null set $\Lambda \in \mathfrak{F}$; then, for each γ, the stochastic process $(X(t), \gamma)$ is separable with respect to the closed sets of T with the same separability sequence $\{t_n\}$ and null set $\Lambda \in \mathfrak{F}$.*

PROOF. Let C be a closed set in T, and C_γ its inverse image in G under mapping $x \to (x, \gamma)$. Then C_γ is closed in G because the mapping is continuous. The ω-sets $\{X(t') \in C_\gamma, t' \epsilon \{t_n\} \cap I\}$ and $\{X(t) \in C_\gamma, t \in I\}$ differ by a subset of Λ because $X(t)$ is separable in G. This implies that the ω-sets $\{(X(t'), \gamma) \in C, t' \in \{t_n\} \cap I\}$ and $\{(X(t), \gamma) \in C, t \in I\}$ differ by a subset of Λ.

We now relate the local properties of $X(t)$ to those of the *separable* complex-valued processes $(X(t), \gamma)$, $\gamma \in \Gamma$. The known properties of the latter are used to obtain those of the former.

LEMMA 6.3. *Let τ be a point of $[a, b]$. If for every subsequence $\{s_n\}$ of $\{t_n\}$ converging from below (above) to τ the limit of $X(s_n)$ exists a.e.(P), then the general one-sided limit of $X(s)$ for $s \uparrow \tau$ ($s \downarrow \tau$) exists a.e.(P).*

PROOF. If $\lim_{n\to\infty} X(s_n)$ exists a.e.(P), then $\lim_{n\to\infty} (X(s_n), \gamma)$ exists a.e.(P) for each $\gamma \in \Gamma$. This implies the existence a.e.(P) of the general one-sided limit of $(X(s), \gamma)$ for $s \uparrow \tau$ ($s \downarrow \tau$), for each γ; in fact, this follows from known properties of separable complex-valued processes ([2], p. 55). (In that reference, the separability is with respect to closed *intervals;* here separability is with respect to closed *sets.* The latter implies the former.) The existence of the limit a.e.(P) for each γ implies its existence for all γ in any countable subset of Γ; hence, the limit exists for all γ in a dense subset of Γ because Γ has a countable base, and so has a dense denumerable subset.

The existence of the limit on a dense γ-set Γ' implies its existence for all $\gamma \in \Gamma$. We shall show, in fact, that the sample functions of the process $(X(t), \gamma)$, $\gamma \in \Gamma$, are uniformly (in t) approximable by sample functions of a process $(X(t), \gamma')$, $\gamma' \in \Gamma'$. In the topology of Γ, the convergence $\gamma' \to \gamma$ is characterized by $(x, \gamma - \gamma') \to 1$, uniformly in x on compact subsets of G ([7], p. 100). Almost all sample functions of $X(t)$ lie in a compact subset of G, by lemma 6.1; hence, for a fixed sample function, if γ' is sufficiently close to γ, the absolute value of the difference

$$|(X(t), \gamma) - (X(t), \gamma')| = |(X(t), \gamma - \gamma') - 1| \tag{6.1}$$

is close to 0 uniformly in t.

An application of lemma 3.2 now completes the proof.

THEOREM 6.1. *There are at most a countable number of fixed points of discontinuity of $X(t)$, a.e.(P).*

PROOF. Lemmas 5.5 and 6.3 imply that the process is continuous a.e.(P) at each point of $[a, b]$. The statement of the lemma now follows from a general theorem that a process which is stochastically continuous at each point on an interval has at most a countable number of fixed points of discontinuity. This was stated and proved by Doob ([2], p. 356) for real-valued processes, and generalized by Pakshirajan [6] to processes in a metric space. As we stated in section 1, our group G is a metric space.

THEOREM 6.2. *Almost all sample functions of $X(t)$ have right- and left-hand limits at all points of (a, b).*

PROOF. According to theorem 5.1, almost all sample functions are elements of $\mathfrak{D}$; thus, for each γ, almost all sample functions of the complex-valued process $(X(t), \gamma)$ are elements of $\mathfrak{C}$. The latter process is separable, by lemma 6.2; therefore, for each γ, almost all the sample functions have right- and left-hand limits at all points of (a, b) (cf. [2], p. 361). By the argument in the proof of lemma 6.3, almost all these sample functions have right- and left-hand limits at all points of (a, b) for *all* $\gamma \in \Gamma$. This property of $(X(t), \gamma)$, $\gamma \in \Gamma$, induces the same on the sample functions of $X(t)$, by lemma 3.2.

7. Supplementary note

After completion of the paper, it occurred to me that the above results are valid in a modified form when Γ is not connected. In this case, the properties given above for Γ are, in fact, valid for the subgroup Γ' of Γ generated by a neighborhood of the identity. Let G' be the subgroup of G which is mapped onto one under all characters in Γ'; then, Γ' is the dual of the factor group G/G'; therefore, the convergence theorems, proved above for G when Γ is connected, are valid for G/G' when Γ is not connected.

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