SOME CONTRIBUTIONS TO THE THEORY OF ORDER STATISTICS

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1. Introduction and summary

This paper arose from the problem of proving the asymptotic normality of linear combinations of order statistics which was first posed by Jung [9]. In the course of this investigation, several facts of general interest in the study of moments of order statistics, which either had not been stated or had not been proved in their most satisfactory form, were established. These are collected in theorems 2.1 and 2.2 of section 2. Briefly we show in theorem 2.1 that any two order statistics are positively correlated, and in theorem 2.2 we give necessary and sufficient conditions for the existence of moments of quantiles and the convergence of the suitably normalized moments to those of the appropriate normal distribution.

Section 3 contains an "invariance principle" for order statistics more elementary than the one given by Hájek [7] but requiring fewer regularity conditions and adequate for our purposes in section 4. In an as yet unpublished paper, J. L. Hodges and the author give another application of this principle in deriving the asymptotic distribution of an estimate of location in the one sample problem.

Section 4 contains the principal results of the paper. We consider linear combinations of order statistics which do not involve the extreme statistics to a more significant extent than the sample mean does. For this class of statistics we establish asymptotic normality and convergence of normalized moments to those of the appropriate Gaussian distribution.

2. Some properties of moments of order statistics

Let $X_1, \cdots, X_n$ be a sample from a population with distribution F and density f which is continuous and strictly positive on $\{x|0 < F(x) < 1\}$. Then $F^{-1}(t)$ is well-defined and continuous for $0 < t < 1$, and for those values of t we may define $\psi(t) = f[F^{-1}(t)]$. We denote by $Z_{1,n} < \cdots < Z_{n,n}$ the order statistics of the sample.

The following two theorems will be proved in this section.

THEOREM 2.1. *Suppose that* $E(Z^2_{i,n}) + E(Z^2_{k,n}) < \infty$. *Then,* $\operatorname{cov}(Z_{i,n}, Z_{j,n}) \geq 0$.

THEOREM 2.2. *Suppose that* $\lim_{x\to\infty} |x|^\epsilon[1 - F(x) + F(-x)] = 0$ *for some* $\epsilon > 0$. *Then,*

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(a) *for any natural number* $k \geq 0, 0 < \alpha < 1$, *there exists* $N(k, \alpha, \epsilon)$ *such that* $E(Z_{r,n}^k)$ *exists for* $\alpha n \leq r \leq (1 - \alpha)n$ *and* $n \geq N(k, \alpha, \epsilon)$. *Conversely, if* $E|Z_{r,n}^k| < \infty$ *for some* k, n, *then for some* $\epsilon > 0$, $\lim_{x \to \infty} |x|^\epsilon(1 + F(-x) - F(x)) = 0$.

(b) *Then* $E[Z_{r,n} - F^{-1}(r/(n + 1))]^k = n^{-k/2}\sigma^k(p_n)\mu_k + o(n^{-k/2})$ *uniformly for* $\alpha n \leq r \leq (1 - \alpha)n$, n *sufficiently large, where* (i) $p_n = (r/n)$, (ii) $\sigma^2(p_n) = [(r/n)(1 - r/n)(\psi(r/n)]^{-2}$, *and* (iii) μ_k *is k-th central moment of the standard normal distribution.*

REMARK. Theorem 2.1, though useful and interesting, as we shall see in section 3, seems not to have appeared in the literature previously but was independently proved by Lehmann in a work, as yet unpublished, on positive dependance. Theorem 2.2(a) is trivial but seemed worth isolating. Theorem 2.2(b) has been proved in the literature, under assorted regularity conditions, by several authors, including Hotelling and Chu [3], Sen [11], [12], and Blom [2]. The last author obtains better estimates of the error than $o(n^{-k/2})$ under various conditions of differentiability and boundedness on F^{-1} and stipulations of the exact form of the tails of f. However, under the given minimal assumptions for $k = 1$, he shows that the error is $0(n^{-1/2})$ which is insufficient for our purposes.

To prove theorem 2.1 we require a lemma stated without proof in Tukey [13]. The elegant simplification of the author's original proof, which we present below, is due to Dr. S. S. Jogdeo.

LEMMA 2.1. *Let* X, Y *be random variables such that* $E(X^2) + E(Y^2) < \infty$ *and* $E(Y|X)$ *is a monotone increasing function of* X a.s.; *that is, there exists a monotone increasing function* $s(x)$, *such that* $s(X)$ *is a version of* $E(Y|X)$. *Then,* $\text{cov}(X, Y) \geq 0$.

PROOF. Let $s(x) = E(Y - E(Y)|x)$. Then since $s(x)$ is monotone increasing and $E(s(X)) = 0$, there exists a number c such that $s(x) \leq 0$ if $x \leq c$ and $s(x) \geq 0$ if $x \geq c$. But then, it is easily seen that

$$(2.1) \qquad \begin{aligned} \text{cov}(X, Y) &= E\{XE[Y - E(Y)|X]\} \\ &= E[(X - c)s(X)] \geq 0. \end{aligned} \qquad \text{Q.E.D.}$$

We now prove theorem 2.1. By lemma 2.1 it suffices to show that if $i < j$, $E(Z_{k,n}|Z_{i,n})$ is a continuous monotone increasing function of Z_{in}. It is well known that given $Z_{i,n}$, $Z_{j,n}$ is distributed as the $(j - i)$-th order statistic of a sample of $n - i$ from a population with density $f(x)/(1 - F(Z_{i,n}))$ for $x \geq Z_{i,n}$ and 0 otherwise. Then,

$$(2.2) \qquad E(Z_{k,n}|Z_{i,n}) = (j - i)\binom{n - i}{j - i}\int_0^1 F^{-1}[(1 - F(Z_{i,n}))t + F(Z_{i,n})]t^{j-i-1}(1 - t)^{n-j}\,dt,$$

by a standard representation of the expected value of an order statistic. (See Wilks [14], p. 236). Monotonicity of $E(Z_{j,n}|Z_{i,n})$ now follows readily since $(1 - s)t + s$ is monotone in s for $0 \leq t < 1$. Left and right continuity of $E(Z_{j,n}|Z_{i,n})$ also is a consequence of (2.2), the continuity of F and F^{-1}, and the dominated convergence theorem. Theorem 2.1 is proved.

We now proceed to the proof of theorem 2.2(a). The given condition is equivalent to

$$\lim_{s\to 0} s^{1/\epsilon}F^{-1}(s) = 0 = \lim_{s\to 1} (1 - s)^{1/\epsilon}F^{-1}(s). \tag{2.3}$$

Let j be the next largest natural number after $1/\epsilon$. Then $|F^{-1}(s)|^k \leq M^k[s(1 - s)]^{-kj}$. Upon again applying the standard fact that $F(Z_{r,n})$ has a beta $(r, n - r + 1)$ distribution we find that

$$\begin{aligned} E|Z_{r,n}|^k &= r\binom{n}{r}\int_0^1 |F^{-1}(s)|^k s^{r-1}(1 - s)^{n-r}\,ds \\ &\leq M^k\binom{n}{r}\int_0^1 s^{r-kj-1}(1 - s)^{n-r-kj}\,ds. \end{aligned} \tag{2.4}$$

Theorem 2.2(a), part 1, now follows upon taking $N(k, \alpha, \epsilon) = [kj/\alpha] + 1$ where $[x]$ is the greatest integer in x.

Conversely, if $E|Z_{n,r}|^\lambda < \infty$ for some $\lambda > 0$, then $\lim_{x\to\infty} x^\lambda P[|Z_{r,n}| \geq x] = 0$, which implies that

$$\lim_{x\to\infty} x^\lambda \int_x^\infty F^{r-1}(t)(1 - F(t))^{n-r}\,dF(t) = 0. \tag{2.5}$$

Choose t_0 such that $F(t_0) > 0$. Then

$$\begin{aligned} \int_x^\infty F^{r-1}(t)(1 - F(t))^{n-r}\,dF(t) &\geq F^{r-1}(t_0)\int_x^\infty (1 - F(t))^{n-r}\,dF(t) \\ &= F^{r-1}(t_0)\frac{(1 - F(x))^{n-r+1}}{(n - r + 1)}. \end{aligned} \tag{2.6}$$

We conclude that

$$\lim_{x\to\infty} x^{\lambda/(n-r+1)}(1 - F(x)) = 0, \tag{2.7}$$

and similarly,

$$\lim_{x\to\infty} x^{\lambda/(r+1)}F(-x) = 0. \tag{2.8}$$

Theorem 2.2(a) is proved.

The proof of 2.2(b) proceeds by a series of lemmas.

LEMMA 2.2. *Let $U_{1,n} < \cdots < U_{n,n}$ be the order statistics of a sample from the uniform distribution on* [0, 1]. *Let*

$$g_{n,k}(x) = k\binom{n}{k}x^{k-1}(1 - x)^{n-k}, \qquad 0 \leq x \leq 1 \tag{2.9}$$

denote the density of $U_{k,n}$. Let

$$g^*_{n,k}(x) = n^{-1/2}g_{n,k}(xn^{-1/2} + k(n + 1)^{-1}). \tag{2.10}$$

Then for every $\alpha > 0$ there exists $\tau(\alpha) > 0$, $M(\alpha) > 0$ such that $g^*_{n,k}(x) \leq M(\alpha)\exp - (\tau(\alpha)x^2)$ for $\alpha n \leq k \leq (1 - \alpha)n$.

PROOF. By Stirling's approximation,

$$k\binom{n}{k} \leq Cn^{n+1/2}k^{-(k-1/2)}(n - k)^{-[(n-k)+1/2]}, \tag{2.11}$$

where C is independent of n, k. Let $p_n = k/n$. Then,

$$g^*_{n,k}(x) \leq C p_n^{1/2}(1 - p_n)^{-1/2} p_n^{-k}(1 - p_n)^{n-k} \tag{2.12}$$

$$\left[n^{-1/2}x + \frac{np_n}{(n + 1)}\right]^{k-1} \left[1 - \left(n^{-1/2}x + \frac{np_n}{(n + 1)}\right)\right] n - k$$

for

$$- \frac{kn^{1/2}}{(n + 1)} \leq x \leq n^{1/2}\left(1 - \frac{k}{(n + 1)}\right). \tag{2.13}$$

Hence, after some simplification we obtain

$$g^*_{n,k}(x) \tag{2.14}$$
$$\leq C[p_n(1 - p_n)]^{-1/2}\left\{\left(1 + \frac{(x - \epsilon_n)}{p_n} n^{-1/2}\right)^{p_n - 1/n}\left(1 - \frac{(x - \epsilon_n)}{(1 - p_n)} n^{-1/2}\right)^{1 - p_n}\right\},$$

where $\epsilon_n = n^{1/2}p_n(n + 1)^{-1}$ and $-k(n + 1)^{-1} \leq n^{-1/2}x \leq (1 - k(n + 1)^{-1})$.

Consider the function

$$q(y, \epsilon) = M^{-\epsilon}(1 + ya^{-1})^{a-\epsilon}(1 - yb^{-1})^b \exp \lambda y^2/2, \tag{2.15}$$

where

$$0 \leq \lambda \leq \min [(a - \epsilon)/(a + b)^2, b/(a + b)^2] \leq 1, \tag{2.16}$$
$$b \geq 0, \qquad a \geq \epsilon \geq 0, \qquad M \geq a/(a - \epsilon).$$

Now,

$$\frac{\partial^2 \log q(y, \epsilon)}{\partial y^2} = \lambda - (a - \epsilon)(a + y)^{-2} - b(b + y)^{-2}, \tag{2.17}$$

and from the given restrictions on λ, it follows that for $-a \leq y \leq b$, $(\partial^2 q(y, \epsilon)/\partial y^2) \leq 0$. Moreover,

$$\frac{\partial \log q(y, \epsilon)}{\partial y} = \lambda y - (a - \epsilon)(a + y)^{-1} - b(b - y)^{-1}, \tag{2.18}$$

and from (2.18) we may see that,

$$\frac{\partial \log q(0, \epsilon)}{\partial y} \leq 0, \qquad \frac{\partial \log q(-\epsilon, \epsilon)}{\partial y} \geq 0, \tag{2.19}$$

since $\lambda \leq 1$.

Hence, $q(y, \epsilon)$ reaches its maximum, whatever be M, for $-\epsilon \leq y \leq 0$. We now show that for the given M, $q(y, \epsilon) \leq 1$, $-a \leq y \leq b$. Remark that $\log q(y, 0) \leq 0$ since $\log q(0, 0) = 0$, $\partial \log q(0, 0)/\partial y = 0$. But,

$$\frac{\partial \log q(y, \epsilon)}{\partial \epsilon} = -\log M - \log a^{-1}(a + y) \tag{2.20}$$

is ≤ 0 for M given, $-\epsilon \leq y \leq 0$, $\epsilon \geq 0$, and the inequality follows. We conclude that $(1 + ya^{-1})^{a-\epsilon}(1 - yb^{-1})^b \leq M^\epsilon \exp -\lambda y^2/2$ for $-a \leq y \leq b$, M and λ as given.

It follows from (2.18) and our preceding remarks that

$$g^*_{n,k}(x) \leq C[p_n(1 - p_n)]^{-1/2}M_n \exp - \lambda_n^2/2(x - \epsilon_n)^2 \tag{2.21}$$

for $-p_n \leq n^{-1/2}(x - \epsilon_n) \leq (1 - p_n)$ and $M_n = p_n(p_n - n^{-1})^{-1}$ and $\lambda_n = \min\{p_n - n^{-1}, (1 - p_n)\}$. But $-n^{1/2}p_n + \epsilon_n = -n^{1/2}k(n+1)^{-1}$, and λ_n and M_n can be uniformly bounded away from 0 and ∞ since $\alpha \leq p_n \leq (1 - \alpha)$. The lemma is therefore proved since $g^*_{n,k}(x)$ vanishes off the given range.

REMARK. It is well known that $n^{-1/2}g_{n,k}(n^{-1/2}x + k(n+1)^{-1})$, the density of $n^{1/2}[U_{k,n} - k(n+1)^{-1}]$, converges to a normal density uniformly on compacts if $\alpha \leq p_n \leq (1 - \alpha)$. More precisely,

$$\sup_{\alpha < p_n < (1-\alpha)} |g^*_{n,k}(x) - [\tau(p_n)]^{-1}\varphi(x/\tau(p_n))| \tag{2.22}$$

converges to 0 uniformly on bounded intervals if $\tau^2(p_n) = p_n(1 - p_n)$, and $\varphi(x)$ is the normal density (see, for example, Wilks [13], p. 270). It now follows from our lemma that $E(U_{k,n} - k(n+1)^{-1})^r = n^{-r/2}\tau^r(p_n)\mu^r + o(n^{-r/2})$ uniformly for $\alpha \leq p_n \leq (1 - \alpha)$ since we can, in particular, conclude that $n^{r/2}[U_{k,n} - k(n+1)^{-1}]^r$ is uniformly integrable for k in the given range. We now prove lemma 2.3.

LEMMA 2.3. *Let F satisfy the general conditions of this section, and in addition, suppose that $f(x)$ is $\geq \lambda > 0$ for all x such that $0 < F(x) < 1$. Then, the conclusion of theorem 2.2(b) holds.*

PROOF. We remark first that the given conditions imply that $\{x \mid 0 < F(x) < 1\}$ is an open interval, and hence X_1 is bounded and $E(Z^r_{k,n})$ exists for every k, r. Let $U_{k,n} = F(Z_{k,n})$. Then $U_{1,n} < \cdots < U_{n,n}$ are the order statistics of a sample from the uniform distribution on [0, 1]. Then, by the mean value theorem,

$$n^{1/2}[Z_{k,n} - F^{-1}(k(n+1)^{-1})] = [\psi(U^*_{k,n})]^{-1}V_{k,n}, \tag{2.23}$$

where $U^*_{k,n}$ lies between $U_{k,n}$ and $k(n+1)^{-1}$ and $V_{k,n} = n^{1/2}(U_{k,n} - k(n+1)^{-1})$. Then,

$$\begin{aligned} &|E[n^{r/2}[Z_{k,n} - F^{-1}(k(n+1)^{-1})]^r - \sigma^r(p_n)\mu_r| \\ &\leq \sup\,[|V_{k,n}| \leq A]|[\psi(U^*_{k,n})]^{-r} - [\psi(p_n)]^{-r}|E|V_{k,n}|^r \\ &+ [\psi(p_n)]^{-r}|\int_{[|V_{k,n}| \leq A]} V^r_{k,n}\,dP - [\tau(p_n)]^{-1} \\ &\int_{[|x| \leq A]} x^r\varphi\left(\frac{x}{\tau(p_n)}\right) dx| \\ &+ \frac{[\psi(p_n)]^{-r}}{\tau(p_n)} \int_{[|x| \leq A]} x^r\varphi\left(\frac{x}{\tau(p_n)}\right) dx \\ &+ \lambda^{-r}\int_{[|V_{k,n}| \geq A]} |V_{k,n}|^r\,dP. \end{aligned} \tag{2.24}$$

Since $|V_{k,n}| \leq A \Leftrightarrow |U_{k,n} - k(n+1)^{-1}| \leq An^{-1/2}$ implies $|U^*_{k,n} - p_n| \leq An^{-1/2} + p_n n^{-1}$ and since ψ is uniformly continuous on the interval $[\beta, 1 - \beta]$ strictly contained in [0, 1], we may conclude, using lemma 2.2, that, as $n \to \infty$, the first two terms on the right-hand side of the inequality (2.30) go to 0 uniformly for $\alpha \leq p_n \leq (1 - \alpha)$. Again by lemma 2.2 the last term goes to 0 as

$A \to \infty$, uniformly in n, and the third term is evidently $o(1)$ as $A \to \infty$ uniformly for $\alpha \le p_n \le (1 - \alpha)$. The lemma follows.

We now prove theorem 2.2(b). Let $0 < \alpha - \delta$, and let $c = F^{-1}(\alpha - \delta)$, $d = F^{-1}(1 - (\alpha - \delta))$. Define

(i) $f_{c,d}(x) = f(x)$ for $c \le x \le d$,

(ii) $\qquad = f(c)$ for $c - (\alpha - \delta)[f(c)]^{-1} \le x \le c$,

(iii) $\qquad = f(d)$ for $d \le x \le d + (\alpha - \delta)[f(d)]^{-1}$.

Define $F_{c,d}$ to be the distribution with density $f_{c,d}$, $\psi_{c,d}$ to be the corresponding $f_{c,d}(F_{c,d}{}^{-1})$. Given our original sample, $X_1, \cdots, X_n$ generates a sample $\hat{X}_1, \cdots, \hat{X}_n$ from $f_{c,d}$ by defining $\hat{X}_i = X_i$ if $c \le X_i \le d$, $\hat{X}_i = T_1^i$ if $X_i < c$, $\hat{X}_i = T_2^i$ if $X_i > d$, where $\{T_1^i\}$, $\{T_2^i\}$, $1 \le i \le n$ are distributed independently of each other and the X_i's according to the uniform distribution on $(d - (\alpha - \delta)[f(c)]^{-1}, c)$ and $(d, d + (\alpha - \delta)[f(d)]^{-1})$ respectively. Let $\hat{Z}_{1,n} < \cdots < \hat{Z}_{n,n}$ denote the order statistics of $\{\hat{X}_i\}$, $1 \le i \le n$. Then, by lemma 2.3, $E(\hat{Z}_{k,n} - F^{-1}(k(n+1)^{-1}))^r = n^{-r/2}\sigma(p_n) + o(n^{-r/2})$ uniformly for $\alpha \le p_n \le (1 - \alpha)$, since for n sufficiently large

$$F_{c,d}^{-1}(k(n+1)^{-1}) = F^{-1}(k(n+1)^{-1}) \tag{2.25}$$

and $\psi_{c,d} = \psi$ if $\alpha \le p_n \le (1 - \alpha)$. Hence, to prove the theorem, it suffices to show that $n^{r/2}E|Z_{k,n} - \hat{Z}_{k,n}|^r \to 0$ uniformly for $\alpha \le p_n \le (1 - \alpha)$.

Suppose that $c < 0$, $d > 0$. The cases where c, d have the same sign may be dealt with similarly. Then

$$|Z_{k,n} - \hat{Z}_{k,n}| = |Z_{k,n} - \hat{Z}_{k,n}|(I[Z_{k,n} < c] + I[Z_{k,n} > d]) \tag{2.26}$$

where $I(A)$ is the indicator function of the event A. We may conclude that

$$\begin{aligned} n^{r/2}E|Z_{k,n} - \hat{Z}_{k,n}|^r \le\ & n^{r/2}E[(|Z_{k,n}| + |c|)^r I[Z_{k,n} < c]] \\ & + E[(|Z_{k,n}| + d + (\alpha + \delta)[f(d)]^{-1})^r I[Z_{k,n} > d]]\}. \end{aligned} \tag{2.27}$$

It therefore suffices to show

$$E(|n^{1/2}Z_{k,n}|^r I[Z_{k,n} < c]) \text{ and } E(|n^{1/2}Z_{k,n}|^r I[Z_{k,n} > d]) \to 0 \tag{2.28}$$

since it then follows that $|c|^r E(I[Z_{k,n} < c]) \to 0$. The other term behaves similarly.

By assumption there exists a natural number j such that $|F^{-1}(y)| \le M[y(1 - y)]^{-j}$. Now,

$$\begin{aligned} E|n^{1/2}Z_{k,n}|^r I[Z_{k,n} < c] &= E(|n^{1/2}F^{-1}(U_{k,n})|^r I[U_{k,n} < \alpha - \delta]) \\ &\le M^r n^{r/2} E(U_{k,n}{}^{-rj}(1 - U_{k,n})^{-rj} I[U_{k,n} < \alpha - \delta]) \\ &= M^r n^{r/2} \int_{[x < \alpha - \delta]} k \binom{n}{k} x^{k-rj-1}(1 - x)^{n-k-rj}\, dx. \end{aligned} \tag{2.29}$$

Without loss of generality, take r to be a natural number and choose n sufficiently large so that $(\alpha - \delta/2) \le (k - rj)/(n - 2rj + 1)$ for all $k \ge \alpha n$. Then,

$$(2.30)\qquad n^{r/2}\int_{[x<\alpha-\delta]} k\binom{n}{k} x^{k-rj-1}(1-x)^{n-k-rj}\,dx$$
$$= n^{r/2}\frac{n(n-1)\cdots(n-2rj+1)}{(k-rj)\cdots(k-1)(n-k-rj+1)\cdots(n-k+1)}$$
$$\int_{[x<\alpha-\delta]}(k-rj)\binom{n-2rj}{k-rj}x^{k-rj-1}(1-x)^{(n-2rj)-(k-rj)}\,dx.$$

Now, $x < \alpha - \delta$ and $(k - rj) \geq (\alpha - \delta/2)(n - 2rj + 1)$ imply

$$(2.31)\qquad (n-2rj)^{1/2}(x-(k-rj)(n-2rj+1)^{-1}) < -(n-2rj)^{1/2}\delta/2.$$

Hence, the expression on the right of (2.30) is not larger than

$$(2.32)\qquad n^{r/2+2rj}\int_{[x<-(n-2rj)^{1/2}\delta/2]} g^{*}_{(n-2rj)(k-rj)}(x)\,dx$$
$$\leq Me^{-K(n-2rj)}(n-2rj)^{-1/2}n^{(r/2)+2rj},$$

where K, M depend only on α, by lemma 2.2 and the well-known approximation to the tail of the normal distribution (Feller [5], p. 166). The theorem is proved.

REMARK. The hypothesis that f be continuous and positive throughout on the carrier of F may obviously, if one is interested in the moments of a single percentile $Z_{[\alpha n]n}$, be weakened to f continuous in some neighborhood of $F^{-1}(\alpha)$. Our results thus contain the results of Hotelling and Chu [3] and Sen [11], [12]. Upon putting supplementary conditions on the local behavior of f, we may similarly obtain better estimates of the error term thus refining the results of Blom.

3. An invariance principle for the quantile function

We keep the general assumptions of section 2. Let us define a process on [0, 1] by,

$$(3.1)\qquad Z_n(t) = n(Z^{*}_{k,n} - Z^{*}_{(k-1)n})t + Z^{*}_{k,n}(1-k) + kZ^{*}_{(k-1)n}$$

on $[(k-1)/n, k/n)$; $1 \leq k \leq n$, where $Z^{*}_{k,n} = Z_{k,n} - F^{-1}(k/(n+1))$ and $Z^{*}_{0,n} = 0$, $Z_n(1) = Z^{*}_{n,n}$.

Then, for every n, $Z_n(t)$ is a process with continuous sample functions. For each $0 < \alpha < \beta < 1$, there is a natural correspondence between $\{Z_n(t), \alpha \leq t \leq \beta\}$ and a probability P_n which belongs to the set $\mathcal{P}(C[\alpha, \beta])$ of all probability measures on the set of all continuous functions on $[\alpha, \beta]$ endowed with the uniform norm and the appropriate Borel field.

Let Q_n, Q be members of $\mathcal{P}(C[\alpha, \beta])$. Let $Q_n(t)$, $Q(t)$ be processes with continuous sample functions on $[\alpha, \beta]$, inducing the measures Q_n, Q on $C[\alpha, \beta]$. Then Q_n converges to Q in the sense of Prohorov if and only if for every h continuous and bounded on $C[\alpha, \beta]$, $\mathcal{L}[h(Q_n(t))] \to \mathcal{L}[h(Q(t))]$. Hâjek [8] has shown that a necessary and sufficient condition for Prohorov convergence to a $Q \in \mathcal{P}[C[\alpha, \beta]]$ is,

$$\lim_{\delta\to 0}\limsup_{n\to\infty} P[\sup_{s,t\in[\alpha,\beta],|t-s|<\delta}|Q_n(s) - Q_n(t)|\geq \epsilon] = 0 \tag{3.2}$$

for every $\epsilon > 0$, and that,

$$\mathcal{L}[Q_n(s_1), \cdots, Q_n(s_k)] \to \mathcal{L}[Q(s_1), \cdots, Q(s_k)] \tag{3.3}$$

for all $s_1, \cdots, s_k \in [\alpha, \beta]$.

For $Z_n(t)$ condition (3.2) is readily seen to be equivalent to

$$\lim_{\delta\to 0}\limsup_{n\to\infty} P[\sup_{k,m\in[\alpha n,\beta n],|k-m|<\delta n}|Z^*_{k,n} - Z_{m,n}| \geq \epsilon] = 0. \tag{3.4}$$

After obtaining the main theorem of this section we were informed of an unpublished monograph of Hájek [7] in which a more general theorem than ours is stated under a regularity condition. Since in our simpler situation the regularity condition is unnecessary and our proof quite short, we felt it worthwhile to include theorem 3.1. Our original proof has been further simplified by a lemma ascribed to Rubin [7].

THEOREM 3.1. *Let $Z_n(t)$ be as above, $0 < \alpha < \beta < 1$. Then there exists a centered Gaussian process $Z(t)$ on $[\alpha, \beta]$ with continuous sample functions and covariance $s(1 - t)/\psi(s)\psi(t)$, $s \leq t$, such that $n^{1/2}Z_n(t)$ converges to $Z(t)$ in the sense of Prohorov on $[\alpha, \beta]$.*

PROOF. Let $U_n(t) = n(U^*_{k,n} - U^*_{(k-1)n})t + U^*_{k,n}(1 - k) + kU^*_{(k-1),n}$ on $[(k - 1)/n, k/n)$, $1 \leq k \leq n$, where $= U^*_0 = 0$, $U^*_{k,n}F(Z_{k,n}) - k/(n + 1)$, $U_n(1) = Z^*_{n,n}$. Let $V_n(t) = (U_{k,n} - U_{(k-1),n})^{-1}(U^*_{(k-1),n} - U^*_{k,n})t + (U_{k,n} - U_{(k-1),n})^{-1}\{U_{(k-1),n}U^*_{k,n} - U_{k,n}U^*_{(k-1),n}\}$ on $[U_{(k-1),n}, U_{k,n})$, $1 \leq k \leq n + 1$, where $U_{(n+1),n} = 1$, $U^*_{(n+1),n} = 0$.

Now, $V_n(t)$ is essentially a version of the empirical cumulative, and Donsker [4] has shown that $n^{1/2}V_n(t)$ converges on $[0, 1]$, in the sense of Prohorov, to a Gaussian process $V(t)$ centered at 0 with continuous sample functions and covariance $s(1 - t)$ for $s \leq t$, a process known as the Brownian bridge.

From this follows (Rubin) lemma 3.1.

LEMMA 3.1. *The process $n^{1/2}U_n(t)$ converges in the sense of Prohorov on $[0, 1]$ to the Brownian bridge.*

PROOF. Clearly, (3.2) is satisfied in this case. To prove (3.1) remark that,

$$\begin{aligned}(3.5)\quad & P[\sup_{|s-t|<\delta} n^{1/2}|U_n(s) - U_n(t)| \geq \epsilon] \\ & \leq P[\sup_{|k-m|<2\delta n} n^{1/2}|U^*_{k,n} - U^*_{m,n}| \geq \epsilon] \qquad \text{for } n \geq \delta^{-1} \\ & \leq P[\sup_{|k-m|<2\delta n} n^{1/2}|U^*_{k,n} - U^*_{m,n}| \geq \epsilon, \max_{1\leq j\leq n}|U_{k,n} - k/(n + 1)| < \delta] \\ & \quad + P[\max_{1\leq j\leq n} |U_{k,n} - k/(n + 1)| \geq \delta].\end{aligned}$$

Now $|k/n - m/n| < 2\delta$, $|U_{k,n} - k/(n + 1)| \leq \delta$, $|U_{m,n} - m/(n + 1)| \leq \delta$ implies $|U_{k,n} - U_{m,n}| < 5\delta$ for $n \geq \delta^{-1}$. We conclude that

$$\begin{aligned}(3.6)\quad & P[\sup_{|k-m|<2\delta n} n^{1/2}|U^*_{k,n} - U^*_{m,n}| \geq \epsilon, \max_{1\leq j\leq n} |U_{k,n} - k/(n + 1)| \leq \delta] \\ & \qquad \leq P[\sup_{|s-t|<5\delta} n^{1/2}|V_n(s) - V_n(t)| \geq \epsilon] \to 0.\end{aligned}$$

Therefore, to prove the lemma it suffices to show that

$$P[\max_{1 \leq k \leq n} |U_{k,n} - k/(n+1)| \geq \epsilon] \to 0. \tag{3.7}$$

But,

$$\begin{aligned} P[\max_{1 \leq k \leq n} |U_{k,n} - k/(n+1)| \geq \epsilon] \\ \leq \sum_{n\alpha \leq k \leq n(1-\alpha)} P[U_{k,n} - k/(n+1)| \geq \epsilon] \\ + P[\max_{1 < k < \alpha n} |U_{k,n} - k/(n+1)| > \epsilon] \\ + P[\max_{(1-\alpha)n \leq k \leq n} |U_{k,n} - k/(n+1)| \geq \epsilon]. \end{aligned} \tag{3.8}$$

Choose α such that $\alpha < \epsilon/2$. Then

$$P[\max_{1 \leq k \leq \alpha n} |U_{k,n} - k/(n+1)| \geq \epsilon] \leq P[U_{[\alpha n]n} \geq \epsilon/2] \to 0. \tag{3.9}$$

Similarly, $P[\max_{(1-\alpha)n \leq k \leq n} |U_{k,n} - k/(n+1)| \geq \epsilon] \to 0$ and by lemma 2.2,

$$\sum_{n\alpha \leq k \leq n(1-\alpha)} P[|U_{k,n} - k/(n+1)| \geq \epsilon] \leq 2nM(\alpha) \exp - Kn \to 0. \tag{3.10}$$

In fact, by the Borel-Cantelli lemma, $\max |U_{k,n} - k/(n+1)|$ converges almost surely to 0. Lemma 3.1 follows.

We require the following generalization to processes of a well-known theorem of Slutsky.

LEMMA 3.2. *Let $\{Q_n\}$ be a sequence of processes with continuous sample functions on $[\alpha, \beta]$ which converge in the sense of Prohorov to Q on $[\alpha, \beta]$. Let $\{V_n\}$ be a sequence of processes with continuous sample functions such that*

$$P[\sup_{\alpha \leq t \leq \beta} |V_n(s) - b(s)| \geq \epsilon] \to 0 \tag{3.11}$$

for every $\epsilon > 0$ and a fixed continuous function $b(s)$. Then,

(a) *the sequence $Q_n(s)V_n(s)$ converges in the sense of Prohorov to $b(s)Q(s)$;*

(b) *the sequence $Q_n(s) + V_n(s)$ converges to $b(s) + Q(s)$.*

PROOF. We prove (a); the proof of (b) is similar. It is clear that

$$\mathcal{L}[V_n(s_1)Q_n(s_1), \cdots, V_n(s_k)Q_n(s_k)] \to \mathcal{L}[b(s_1)Q(s_1), \cdots, b(s_k)Q(s_k)] \tag{3.12}$$

for all $\alpha < s_i < \beta$, k finite, by the convergence of Q_n and V_n and the ordinary Slutsky theorem. It remains to show that V_nQ_n satisfies (3.2). Let $M_{1,n} = \sup_{\alpha \leq t \leq \beta} |Q_n(t)|$, $M_{2,n} = \sup_{\alpha \leq t \leq \beta} |V_n(t)|$. Then by an elementary inequality

$$\begin{aligned} P[\sup_{s,t \in [\alpha,\beta], |t-s| < \delta} |Q_n(s)V_n(s) - Q_n(t)V_n(t)| \geq \epsilon] \\ \leq P[\sup_{s,t \in [\alpha,\beta], |t-s| < \delta} M_{1,n}|V_n(t) - V_n(s)| \geq \epsilon/2] \\ + P[\sup_{s,t \in [\alpha,\beta], |t-s| < \delta} M_{2,n}|Q_n(s) - Q_n(t)| \geq \epsilon/2]. \end{aligned} \tag{3.13}$$

But by the convergence of Q_n and V_n, there exists M_a such that $P[M_{1,n} \leq M_a] \geq 1 - a$ and $P[M_{2,n} \leq M_a] \geq 1 - a$ for all n. We conclude that

$$\begin{aligned} \lim_{\delta \to 0} \limsup_n P[\sup_{s,t \in [\alpha,\beta], |t-s| \leq \delta} M_{1,n}|V_n(t) - V_n(s)| \geq \epsilon/2] \\ \leq \lim_{\delta} \limsup_n P[\sup_{s,t \in [\alpha,\beta], |t-s| < \delta} |V_n(t) - V_n(s)| > \epsilon/2M_a] + a. \end{aligned} \tag{3.14}$$

The lemma follows by applying a similar argument to the second term of (3.13).

Now, define $\tilde{Z}_n(t) = F^{-1}(U_n(t) + nt/(n+1)) - F^{-1}(nt/(n+1))$, $\alpha \leq t \leq \beta$.

By the mean value theorem,

$$n^{1/2}\tilde{Z}_n(t) = (\psi[Y_n(t)])^{-1}n^{1/2}U_n(t) \tag{3.15}$$

where $Y_n(t)$ lies between $nt/(n+1)$ and $U_n(t) + nt/(n+1)$. The process $[\psi(Y_n(t))]^{-1}$ necessarily possesses continuous sample functions on $[\alpha, \beta]$. From the convergence of $n^{1/2}U_n(t)$ it follows that

$$P[\sup_{0\le t\le 1} |U_n(t)| \ge \epsilon] \to 0 \tag{3.16}$$

for every $\epsilon > 0$, and therefore, since $[\psi(x)]^{-1}$ is uniformly continuous for $0 < a < x < b < 1$, that

$$P[\sup_{\alpha\le t\le\beta} |[\psi(Y_n(t))]^{-1} - [\psi(t)]^{-1}| \ge \epsilon] \to 0 \tag{3.17}$$

for every $\epsilon > 0$. Hence by lemma 3.2 we obtain that $n^{1/2}\tilde{Z}_n(t)$ converges on $[\alpha, \beta]$ to $\hat{Z}(t)[\psi(t)]^{-1}$, a centered Gaussian process with the covariance structure given in the statement of theorem 3.1. Now, to show that (3.2) holds for $n^{1/2}Z_n(t)$, it suffices to check that (3.4) is satisfied. But $Z_n(k/n) = \tilde{Z}_n(k/n) + Z^*_{k,n}$. By the necessity of (3.2),

$$\begin{aligned} 0 &= \lim_{\delta\to 0} \limsup_n P[\sup_{s,t\in[\alpha,\beta],|t-s|<\delta} n^{1/2}|\tilde{Z}_n(s) - \tilde{Z}_n(t)| \ge \epsilon] \\ &\ge \lim_{\delta\to 0} \limsup_n P[\sup_{\alpha n<k,m<\beta n,|k-m|<\delta n} n^{1/2}|Z^*_{k,n} - Z^*_{m,n}| \ge \epsilon], \end{aligned} \tag{3.18}$$

and the theorem follows.

4. Convergence of linear systematic statistics

Let $\{a_{k,n}\}$ $1 \le k \le n$, $n \ge 1$ be a double sequence of constants. Form the statistic $T_n = \sum_{k=1}^n a_{k,n} Z_{k,n}$. Such quantities are known as systematic statistics and are of use in estimation and testing (cf. Jung [9] and Blom [2]). The convergence of moments of T_n and the asymptotic normality of T_n have been investigated by several writers, including Jung [9], Blom [2], Hâjek [7], and more recently, Gastwirth, Chernoff, and Johns (private communication), [6], and Govindarajulu [15] under various regularity conditions.

Our conditions are somewhat simpler, though by no means inclusive. We are essentially able to deal with all systematic statistics which involve the extremal statistics to the same extent as the sample mean or less.

Let us define $M_n(t) = \sum_{k<nt} a_{k,n}$. Then $M_n(t)$ is of bounded variation and

$$T_n - \int_0^1 F^{-1}(nt/(n+1))\, dM_n(t) = \int_0^1 Z_n(t)\, dM_n(t). \tag{4.1}$$

We then have a modification and generalization of a theorem of Hâjek [7].

THEOREM 4.1. *Under the general conditions of section 2, suppose that there exists $\alpha > 0$ such that $a_{k,n} = 0$ for $k \le \alpha n$, $k \ge (1-\alpha)n$ for all $n \ge N$. Suppose that there exists $M(t)$ of bounded variation in $[\alpha, 1-\alpha]$ such that $M_n(t) \to M(t)$ on a dense set of t, $\alpha \le t \le (1-\alpha)$ and that $\overline{V}(M_n) \le M' < \infty$ for all n where $\overline{V}(M_n)$ denotes the total variation of M_n. Then,* (4.2) $$\mathcal{L}[n^{1/2}(T_n - \int_0^1 F^{-1}(t)\, dM_n(t))] \to N(0, \sigma^2(M, F))$$

where N denotes the normal distribution and

(4.3) $$\sigma^2(M, F) = 2 \int_0^1 \int_0^t s(1 - t)[\psi(s)\psi(t)]^{-1}\, dM(s)\, dM(t).$$

PROOF. We remark that since M is constant off $(\alpha, 1 - \alpha)$ and the integrand is bounded in that interval, by our assumptions $\sigma^2(M, F) < \infty$. To prove the theorem it suffices to show that

(1) $$\mathcal{L}\left(\int_0^1 n^{1/2} Z_n(t)\, dM_n(t)\right) \to \mathcal{L}\left(\int_0^1 Z(t)\, dM(t)\right),$$

and that

(2) $$\int_0^1 |F^{-1}(t) - F^{-1}(nt/(n + 1))| dM_n(t) = o(n^{-1/2}),$$

since by (4.1) it readily follows that $\int_0^1 Z(t)\, dM(t)$ has the desired distribution. By theorem 3.1 and a theorem of Prohorov ([10], p. 166), relation (1) holds if

(4.4) $$\int_0^1 f(t)\, dM_n(t) \to \int_0^1 f(t)\, dM(t)$$

uniformly for equicontinuous, uniformly bounded (compact) sets of continuous functions f on $[\alpha, (1 - \alpha)]$. But this readily follows from our assumptions upon using the method of proof of Helly's theorem. Relation (2) follows trivially since

(4.5) $$\left|F^{-1}\left(\frac{nt}{(n + 1)}\right) - F^{-1}(t)\right| \le \frac{M''t}{(n + 1)}$$

for $t \in [\alpha, (1 - \alpha)]$ by the mean value theorem and continuity of $\psi(t)$. Theorem 4.1 is proved. The following corollaries are immediate.

COROLLARY 4.1. *If* $\overline{V}(M_n - M) = o(n^{-1/2})$, *then theorem* 4.1 *holds with* $\int_0^1 F^{-1}(t)\, dM_n(t)$ *replaced by* $\int_0^1 F^{-1}(t)\, dM(t)$.

COROLLARY 4.2. *If*

(4.6) $$M_n(t) = n^{-1} \sum_{kn^{-1} < t} h(kn^{-1}),$$

that is, $a_{k,n} = n^{-1}h(kn^{-1})$, $h = 0$ *on* $[\alpha, (1 - \alpha)]^c$, *and h is continuously differentiable or, more generally, obeys a Lipschitz condition of order* $> \frac{1}{2}$ *on* $[\alpha, (1 - \alpha)]$, *then theorem* 4.1 *holds with* $\int_0^1 F^{-1}(t)\, dM_n(t)$ *replaced by* $\int_0^1 F^{-1}(t)h(t)\, dt$ *and* $M(t) = \int_0^t h(s)\, ds$.

PROOF. The condition is clearly sufficient to guarantee

(4.7) $$\int_0^1 F^{-1}(t)\, dM_n(t) = n^{-1} \sum_{k=1}^n h(kn^{-1})$$

to equal $\int_0^1 F^{-1}(t)h(t)\, dt + o(n^{-1/2})$.

REMARK. (1) In particular, corollary 4.1 applies if

(4.8) $$a_{k,n} = \int_{(k-1)/n}^{k/n} h(t)\, dt + o(n^{-3/2})$$

uniformly for $\alpha n \leq k \leq (1-\alpha)n$ for some function $h(t)$ in $L_1([\alpha, (1-\alpha)])$. This provides an alternative system of weights for the estimates considered by Jung.

(2) Corollary 4.2 establishes the asymptotic normality of the trimmed and Winsorized means of Tukey (see Bickel [1]).

THEOREM 4.2. *Under the conditions of theorem* 4.1 *if* $|x|^{\epsilon}[1 - F(x) + F(-x)]$ *tends to* 0 *as* $x \to \infty$ *for some* $\epsilon > 0$, $E(T_n^k)$ *exists eventually for every natural number* k *and*

$$n^{k/2}E(T_n - \int_0^1 F^{-1}(t)\, dM_n(t))^k \to \sigma^2(M, F)\mu_k, \qquad \text{as} \quad n \to \infty. \tag{4.9}$$

PROOF. By the linearity property of the expectation and (4.1),

$$\begin{aligned} E\left(T_n - \int_0^1 F^{-1}(t)\, dM_n(t)\right)^k \\ = \int_\alpha^{(1-\alpha)} \int_\alpha^{(1-\alpha)} E\left(\prod_{i=1}^k [Z_n(s_i) - F^{-1}(s_i)]\right) \prod_{i=1}^k dM_n(s_i). \end{aligned} \tag{4.10}$$

An easy extension of theorem 2.2 implies that

$$n^{k/2}E\left(\prod_{i=1}^k [Z_n(s_i) - F^{-1}(s_i)]\right) \to E\left[\prod_{i=1}^k Z(s_i)\right] \tag{4.11}$$

uniformly for $\alpha \leq s_i \leq (1-\alpha)$. We conclude that

$$E\left(T_n - \int_0^1 F^{-1}(t)\, dM_n(t)\right)^k \to E\left[\int_\alpha^{(1-\alpha)} Z(t)\, dM(t)\right]^k, \tag{4.12}$$

and the theorem is proved.

REMARK. (1) This establishes convergence of the variance for the trimmed and Winsorized means as stated in Bickel [1].

(2) Under the conditions of corollaries 4.1 or 4.2, $\int_0^1 F^{-1}(t)\, dM_n(t)$ may be replaced by $\int_0^1 F^{-1}(t)\, dM(t)$. We can now prove the following theorem.

THEOREM 4.3. *Suppose* $E(X_1^2) < \infty$. *Let* $M_n(t)$ *defined as before tend to* $M(t)$ *on a dense set in* $[0, 1]$, $\overline{V}(M_n) < \infty$ *on* $[0, 1]$. *Assume, furthermore, that for some* $\alpha > 0$, $|a_{k,n}| \leq M''n^{-1}$ *for all* $k \leq \alpha n$, $k \geq (1-\alpha)n$. *Then,*

$$\mathcal{L}[n^{1/2}(T_n - E(T_n))] \to N(0, \sigma^2(M, F)). \tag{4.13}$$

PROOF. We require first the following lemma.

LEMMA 4.1. *Let* X_n *be a sequence of random variables. Let* $Y_{m,n}$ *be another double sequence of random variables such that,*

(i) $\mathcal{L}(Y_{m,n}) \to \mathcal{L}(Y_m)$ *for each* m *as* $n \to \infty$,

(ii) $\mathcal{L}(Y_m) \to \mathcal{L}(Y)$ *as* $m \to \infty$,

(iii) $\limsup_m \limsup_n P[|X_n - Y_{m,n}| \geq \delta] = 0$,

for every $\delta > 0$. *Then,* $\mathcal{L}(X_n) \to \mathcal{L}(Y)$.

PROOF. First note that

$$(4.14)\quad \begin{aligned}&|P[Y_{m,n} < x] - P[X_n < x]|\\ &\le P[Y_{m,n} < x, X_n \ge x] + P[X_n < x, Y_{m,n} \ge x]\\ &\le P[x - \delta < Y_{m,n} < x, X_n \ge x] + P[X_n < x, x \le Y_{m,n} < x + \delta]\\ &\quad + 2P[|X_n - Y_{m,n}| \ge \delta]\\ &\le P[x - \delta < Y_{m,n} < x + \delta] + 2P[|X_n - Y_{m,n}| \ge \delta].\end{aligned}$$

Let x, $x - \delta$, $x + \delta$ be points of continuity of $\mathfrak{L}(Y_m)$ for all m and of $\mathfrak{L}(Y)$ as well. Then,

$$(4.15)\quad \begin{aligned}&\limsup_n |P[Y_{m,n} < x] - P[X_n < x]|\\ &\qquad \le P[x - \delta < Y_m < x + \delta] + 2 \limsup_n P[|X_n - Y_m| \ge \delta].\end{aligned}$$

Now, take the lim sup as $m \to \infty$, reducing the second term to 0, the first to $P[x - \delta < Y < x + \delta]$, and finally the limit as $\delta \to 0$. It follows that

$$(4.16)\quad \lim_n P[X_n < x] = \lim_m \lim_n P[Y_{m,n} < x] = P[Y < x].$$

The lemma is proved.

By theorem 4.2, if $U_n(t)$ is the distribution function of the measure which assigns mass $1/n$ to i/n, $1 \le i \le n$, then

$$(4.17)\quad \operatorname{var} n^{1/2} \int_\beta^{(1-\beta)} Z_n(t)\, dU_n(t) \to 2 \int_\beta^{(1-\beta)} \int_\beta^t s(1 - t)[\psi(s)\psi(t)]^{-1}\, ds\, dt.$$

Let $\lambda_1 = F^{-1}(\beta)$, $\lambda_2 = F^{-1}(1 - \beta)$. Then,

$$(4.18)\quad 2 \int_\beta^{(1-\beta)} \int_\beta^t s(1 - t)[\psi(s)\psi(t)]^{-1}\, ds\, dt = 2 \int_{\lambda_1}^{\lambda_2} \int_{\lambda_1}^y F(x)(1 - F(y))\, dx\, dy$$

by a change of variable. The latter integral may readily be evaluated. Thus,

$$(4.19)\quad \begin{aligned}2 \int_{\lambda_1}^{\lambda_2} \int_{\lambda_1}^y F(x)\, dx\, dy &= 2\lambda_2 \int_{\lambda_1}^{\lambda_2} F(x)\, dx - 2 \int_{\lambda_1}^{\lambda_2} y d\left(\int_{\lambda_1}^y F(x)\, dx\right)\\ &= \int_{\lambda_1}^{\lambda_2} y^2\, dF(y) + \beta\lambda_1^2 - (1 - \beta)\lambda_2^2 + 2\lambda_2 \int_{\lambda_1}^{\lambda_2} F(x)\, dx,\end{aligned}$$

and

$$(4.20)\quad 2 \int_{\lambda_1}^{\lambda_2} F(y) \left(\int_{\lambda_1}^y F(x)\, dx\right) dy = \int_{\lambda_1}^{\lambda_2} d\left(\int_{\lambda_1}^y F(x)\, dx\right)^2 = \left(\int_{\lambda_1}^{\lambda_2} F(x)\, dx\right)^2.$$

We obtain, therefore, after some simplification,

$$(4.21)\quad \begin{aligned}&2 \int_\beta^{(1-\beta)} \int_\beta^t s(1 - t)[\psi(s)\psi(t)]^{-1}\, ds\, dt\\ &\qquad = \int_{\lambda_1}^{\lambda_2} t^2\, dF(t) + \beta(\lambda_1^2 + \lambda_2^2) - \left(\int_{\lambda_1}^{\lambda_2} t\, dF(t) + \beta(\lambda_1 + \lambda_2)\right)^2.\end{aligned}$$

Since $E(X_1^2) < \infty$, it readily follows that as $\beta \to 0$,

$$(4.22)\quad \begin{aligned}&2 \int_\beta^{(1-\beta)} \int_\beta^y s(1 - t)[\psi(s)\psi(t)]^{-1}\, ds\, dt \to \operatorname{var} X_1\\ &\qquad = 2 \int_0^1 \int_0^t s(1 - t)[\psi(s)\psi(t)]^{-1}\, ds\, dt.\end{aligned}$$

Let $\alpha \geq \beta_m \rightarrow 0$ and define

$$Y_{m,n} = \int_{\beta_m}^{(1-\beta_m)} [Z_n(t) - E(Z_n(t)] \, dM_n(t). \tag{4.23}$$

By theorem 4.2, $\mathcal{L}(Y_{m,n}) \rightarrow \mathcal{L}(Y_m)$ as $n \rightarrow \infty$, where Y_m is

$$N\left[0, \int_{\beta_m}^{(1-\beta_m)} s(1-t)[\psi(s)\psi(t)]^{-1} \, dM(s) \, dM(t)\right]. \tag{4.24}$$

Of course, $\mathcal{L}(Y_m) \rightarrow N(0, \sigma^2(M, F))$ as $m \rightarrow \infty$, where

$$\sigma^2(M, F) \leq (M'') \operatorname{var} X_1 + 2 \int_\alpha^{(1-\alpha)} \int_\alpha^t s(1-t)[\psi(s)\psi(t)]^{-1} \, dM(s) \, dM(t), \tag{4.25}$$

which is finite. Now

$$P[n^{1/2}|Y_{m,n} - (T_n - E(T_n))| \geq \epsilon] \leq \epsilon^{-2} n \operatorname{var} (Y_{m,n} - T_n) \tag{4.26}$$

by Tchebichev's inequality. But,

$$\operatorname{var} (Y_m - T_n) = | \sum_{k,\ell \in [\beta_m n, (1-\beta_m)n]^c} a_{k,n} a_{\ell,n} \operatorname{cov} (Z_{k,n}, Z_{\ell,n}) \leq n^{-2}[M'']^2 \sum_{k,\ell \in [\beta_m n, (1-\beta_m)n]^c} \operatorname{cov} (Z_{k,n}, Z_{\ell,n}) \tag{4.27}$$

by theorem 2.1. Now it follows that

$$n \operatorname{var} (Y_{m,n} - T_n) \leq M'' n \operatorname{var} \left(\overline{X} - \int_{\beta_m}^{(1-\beta_m)} Z_n(t) \, dU_n(t)\right) \tag{4.28}$$

where $\overline{X}$ is the sample mean. But again, by theorem 2.1,

$$n \operatorname{var} \left(\overline{X} - \int_{\beta_m}^{(1-\beta_m)} Z_n(t) \, dU_n(t)\right) \leq n \operatorname{var} \overline{X} - n \operatorname{var} \int_{\beta_m}^{(1-\beta_m)} Z_n(t) \, dU_n(t). \tag{4.29}$$

We conclude that,

$$\limsup_n P[n^{1/2}|Y_{m,n} - (T_n - E(T_n))| \geq \epsilon] \leq \operatorname{var} X_1 - \int_{\beta_m}^{(1-\beta_m)} \int_{\beta_m}^t s(1-t)[\psi(s)\psi(t)]^{-1} \, ds \, dt. \tag{4.30}$$

By our previous remarks we see that the requirements of lemma 4.1 are satisfied and the theorem is proved.

REMARK. Theorem 4.3 implies the asymptotic normality of the estimates considered by Jung.

The following corollary is immediate.

COROLLARY 4.2. *Under the conditions of theorem* 4.3, $n \operatorname{var} T_n \rightarrow \sigma^2(M, F)$.

In the general case we can only establish the following corollary.

COROLLARY 4.3. *Under the conditions of theorem* 4.3, *if* $n^{1/2}\overline{V}(M_n - M) \rightarrow 0$ *on* $[0, 1]$, *then,*

$$E(T_n) \rightarrow \int_0^1 F^{-1}(t) \, dM(t). \tag{4.31}$$

PROOF. Since clearly

$$E\left(\int_{\beta_m}^{(1-\beta_m)} Z_n(t)\, dM_n(t)\right) \to 0, \tag{4.32}$$

it suffices to show that $\limsup_m \limsup_n |E(T_n - \hat{Y}_{m,n})| \to 0$ where

$$\hat{Y}_{m,n} = \int_{\beta_m}^{(1-\beta_m)} Z_n(t)\, dM_n(t) + \int_{\beta_m}^{(1-\beta_m)} F^{-1}(t)\, dM_n(t). \tag{4.33}$$

But,

$$|E(T_n - Y_{m,n})| \le n^{-1}M'' \sum_{k \in [\beta_m n, (1-\beta_m)n]^c} |E(Z_{k,n})|. \tag{4.34}$$

Define $R_n(t)$ to be the measure assigning mass $1/n$ to k/n if $E(Z_{k,n}) \ge 0$, $-(1/n)$ otherwise, $1 \le k \le n$. Then,

$$n^{-1} \sum_{\beta_m n \le k \le (1-\beta_m)n} |E(Z_{k,n})| = E \int_{\beta_m}^{(1-\beta_m)} Z_n(t)\, dR_n(t) + \int_{\beta_m}^{(1-\beta_m)} |F^{-1}(t)|\, dR_n(t) \to \int_{\beta_m}^{(1-\beta_m)} |F^{-1}(t)|\, dt. \tag{4.35}$$

Now, $n^{-1} \sum_{k=1}^{n} |E(Z_{k,n})| \le n^{-1} \sum_{k=1}^{n} E|Z_{k,n}| = E|X_1|$. The corollary follows from (4.34) and (4.35).

This result is, of course, unsatisfactory since it is precisely as an asymptotically normal estimate of $\int_0^1 F^{-1}(t)\, dM(t)$ that T_n is usually employed. Slightly less general but more satisfactory is corollary 4.4.

COROLLARY 4.4. *Under the conditions of theorem* 4.3, *if there exists* A *such that* $f(x)$ *is monotone for* $|x| \ge A$, *and* $n^{1/2}\overline{V}(M_n - M) \to 0$ *on* $[0, 1]$, *then*

$$n^{1/2}\left[E\left(T_n - \int_0^1 F^{-1}(t)\, dM(t)\right)\right] \to 0, \tag{4.36}$$

and hence $n^{1/2}(T_n - \int_0^1 F^{-1}(t)\, dM(t))$ *has asymptotically an* $N(0, \sigma^2(M, F))$ *distribution.*

PROOF. Let $0 < \beta_m + \delta < \min(F(-A), 1 - F(A), \alpha)$. Denote $F^{-1}(\beta_m + \delta)$ by λ_m. Let

$$\begin{aligned} f_m(x) &= f(x), \quad x < \lambda_m \\ &= f(\lambda_m), \quad \lambda_m \le x < \lambda_m + (1 - \beta_m)[f(\lambda_m)]^{-1}. \end{aligned} \tag{4.37}$$

Define,

$$\begin{aligned} X_i(m) &= X_i, \quad X_i < \lambda_m, \\ &= T_i, \quad X_i > \lambda_m, \end{aligned} \tag{4.38}$$

where $\{T_i\}$ $1 \le 1 \le n$ is a sequence of random variables uniform on $(\lambda_m, \lambda_m + (1 - \beta_m)[f(\lambda_m)]^{-1})$ and independent of each other and of the X_i. Let $Z_{1,n}(m) < \cdots < Z_{n,n}(m)$ denote the order statistics of the $X_i(m)$. Then, clearly,

$$\sum_{k \le \beta_m n} a_{k,n}(Z_{k,n} - Z_{k,n}(m)) < nM''(|Z_{([\beta_m n]+1)n}| + K)I[Z_{([\beta_m n]+1)n} > \lambda_m] \tag{4.39}$$

where $K = \max(|\lambda_m|, |\lambda_m + (1 - \beta_m)[f(\lambda_m)]^{-1}|)$. We can show by arguments similar to those employed in the proof of theorem 2.2 that

$$(4.40)\qquad n^{1/2}(E[\sum_{k\leq\beta_m n}(Z_{k,n}-Z_{k,n}(m)]) \to 0$$

for every fixed m. Since

$$(4.41)\qquad n^{1/2}E\left(\int_{\beta_m}^{(1-\beta_m)} Z_n(t)\,dM_n(t)\right) = n^{1/2}\left(E(T_n) - \int_{\beta_m}^{(1-\beta_m)} F^{-1}(t)\,dM(t) + o(n^{-1/2})\right) \to 0,$$

by the remarks following theorem 4.2, we conclude from our previous remark that we need only show

$$(4.42)\qquad \limsup_m \limsup_n n^{1/2}\left|E\left(\sum_{k\leq\beta_m n} a_{k,n}Z_{k,n}(m) - \int_0^{\beta_m} F^{-1}(t)\,dM_n(t)\right)\right| = 0$$

and a similar proposition for the upper tail.

It clearly suffices to establish that

$$(4.43)\qquad n^{-1/2}\sum_{k\leq\beta_m n}|E(Z_{k,n}(m) - F^{-1}(k/(n+1))| \to 0.$$

Let F_m be the distribution of $X_1^{(m)}$. Then, F_m is either convex or concave depending on whether f_m is monotone increasing or decreasing for $x \leq A$. Hence, F_m^{-1} is concave or convex, and by Jensen's inequality,

$$(4.44)\qquad E(Z_{k,n}(m) - F_m^{-1}(k/(n+1)) = E[F_m^{-1}(U_{k,n})] - F_m^{-1}(E(U_{k,n}))$$

has the same sign for all k. But $F_m^{-1}(k/(n+1)) = F^{-1}(k/(n+1))$ for $k \leq \beta_m n$, and we conclude that,

$$(4.45)\qquad n^{-1/2}\sum_{k\leq\beta_m n}|E(Z_{k,n}(m)) - F^{-1}(k/(n+1))| = n^{-1/2}|E(\sum_{k\leq\beta_{m,n}} Z_{k,n}(m) - F^{-1}(k/(n+1))|.$$

Now $\limsup_m \limsup_n n^{-1/2}E[\sum_{k\leq\beta_m n}(Z_{k,n}(m) - F^{-1}(k/(n+1))] = 0$ readily follows from theorem 4.2, and the identity

$$(4.46)\qquad E[n^{-1}\sum_{k=1}^{n} Z_{k,n}(m)] = \int_0^1 F_m^{-1}(t)\,dt, \qquad \text{Q.E.D.}$$

Remark. The condition $E(X_1) < \infty$ clearly suffices for corollaries 4.3 and 4.4.

Jung [9] and Blom [2] have shown convergence of moments under various conditions. The condition of Jung in the case of convergence of the mean may be weakened to $a_{k,n} = n^{-1}\,a(k/n)$ when a has at least two derivatives bounded on $[0, 1]$. Corollary 4.2 then holds with the error being $0(n^{-1})$ rather than just $o(n^{-1/2})$ as shown in corollary 4.4.

This completes our present study of linear systematic statistics. Clearly, there are still many open questions. The restriction on $M(t)$ leaves statistics which involve the extremal order statistics in a more significant fashion than the mean undealt with. On the other hand, the restriction $E(X_1^2) < \infty$ seems too restrictive for systematic statistics involving the extremes to a lesser extent than the mean. Hájek [7], using his more general invariance principle, states a theorem which covers some situations we cannot deal with. Unfortunately his regularity conditions do not cover the mean itself.

The invariance principle of section 3, simple though it is, has other interesting applications. In a forthcoming paper J. L. Hodges and the author have applied it to determine the behavior of

$$\operatorname*{med}_{k \leq n} \tfrac{1}{2}[Z_{k,(2n)} + Z_{(2n-k+1),(2n)}], \tag{4.47}$$

an asymptotically nonnormal robust estimate of location, which is much easier to compute than the Hodges-Lehmann estimate $\text{med}_{i \leq j} (X_i + X_j)/2$.

◇ ◇ ◇ ◇ ◇

Note added in proof. Results similar to theorem 2.2 (a) and (b) have appeared in W. Van Zwet, Convex Transformations of Random Variables, Thesis, Amsterdam, 1964. (In particular, 2.2(a) was noted and a stronger form of 2.2(b) proved under the assumption that f is continuously differentiable.)

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