# AVERAGE MASSES OF THE DOUBLE GALAXIES

THORNTON PAGE
VAN VLECK OBSERVATORY, WESLEYAN UNIVERSITY

## 1. Abstract

It is possible to observe the line-of-sight projection of the orbital motion of one galaxy moving around another in a close pair, and also the separation projected on the tangent plane. The unknown orientation of the orbit is specified by two angles that can be considered independent random variables. Since there is a dynamical relation between mass, space separation, and orbital velocity (assuming circular orbits), and since the distribution of space separations has been determined from other data, it is possible to derive a statistical relation between the observable quantities and the mean mass, $\overline{M}$. Observations of apparent brightness can also be included, leading to a second statistical relation between observables and the mean ratio of mass to luminosity, $\overline{M/L}$.

New observational data are presented for 15 pairs of galaxies, and these are combined with data for 20 pairs previously reported [1] and 95 individual measurements in 44 close pairs reported by Humason and Mayall [8] to determine the average mass of one galaxy, $\overline{M} = (30 \pm 10) \times 10^{10}/h$ suns, and $\overline{M/L} \doteq 12h$ solar units, where $h$ is the ratio of the Hubble constant to the value assumed here, 100 km/sec/megaparsec, and the errors are root mean square.

When the data are considered in three groups: 14 pairs of spirals and Irr. types, 13 pairs of elliptical and SO types, and 14 mixed systems, it is found that the average mass of the ellipticals and SO types is $\overline{M}_E = (60 \pm 15) \times 10^{10}/h$, $\overline{M_E/L_E} = (94 \pm 38)h$, and of the spirals, $\overline{M}_S = (2. \pm 1.5) \times 10^{10}/h$, $\overline{M_S/L_S} \doteq 0.33h$, and that the data for mixed systems substantiate these figures. A formula is developed for the intrinsic variance of $M$ in terms of the residuals, but $\sigma_M^2$ proved indeterminate for the small samples ($n = 13, 14$).

Since the results for $\overline{M/L}$ are inconsistent with expectations based on other astrophysical data, several alternative hypotheses are investigated, and it is found (1) that an intergalactic medium is not likely to account for the discrepancy, and (2) that the assumption of radial motion (rather than circular

The work reported here was started at the McDonald Observatory of the University of Texas, continued at the Yerkes Observatory of the University of Chicago while the author was on leave from the Operations Research Office of the Johns Hopkins University, and was completed with support of a National Science Foundation Grant, G-11106.

orbits) equal to the velocity of escape decreases the mass estimates by only 50 per cent.

## 2. Introduction

The determination of average masses of galaxies from observations of motions of double galaxies has been treated in two earlier papers [1], [2]. For the statistical purposes of cosmology (average density of matter in space) and of galactic evolution (average masses of various morphological types) it offers the advantage of larger sample size than has yet been possible in measurement of individual masses from rotations [3]. Moreover, except for a few nearby systems it is observationally difficult to be sure that the whole mass of a galaxy is included in the rotation method.

Average masses of galaxies can also be determined from motions in clusters, as originally carried out by Sinclair Smith [4], but the validity of this approach has recently been brought into doubt by Ambartzumian [5] and the Burbidges [6], who postulate that the clusters have positive energy in order to bring their masses into accord with their luminosities.

In Holmberg's catalogue [7] there are 827 double and multiple systems, a list partially complete to about $14^{m}.3$ (and including some galaxies as faint as $15^{m}.7$), of which 695 are simple pairs with separations, $S$, ranging from less than $1'$ (minute of arc) to $10'$ and more. This distribution of separations, sharply peaked near $S = 0$, is in marked contrast to the expected distribution of 21,000 to 150,000 galaxies of the same brightness ($14^{m}.3$ to $15^{m}.7$) distributed at random over the sky; in fact, as Holmberg has shown, less than 13 per cent of the double galaxies should be optical pairs (one far behind the other). If the known clusters are avoided, as is the case here, the proportion of optical pairs will be a good deal smaller.

Holmberg has also shown [2] that the distribution of

$$S = \frac{r}{60D_p}\cos\phi = \frac{rh \times 10^{-4}}{60V}\cos\phi \tag{1}$$

is consistent with a distribution of space separations $r$,

$$p_r(r) = K\left[1 - \left(\frac{r}{r_m}\right)^3\right], \qquad 0.03r_m < r \leqq r_m, \tag{2}$$

where $K$ = a proportionality constant, $r$ = separation in astronomical units (a.u.), $r_m$ = maximum separation = $(47.5/h) \times 10^9$ a.u., $D_p$ = distance of the pair in parsecs, $\phi$ = inclination of $r$ to the tangent plane, $V$ = common radial velocity in km/sec, and

$$V = h \times 10^{-4}D_p \quad \text{(Hubble's law)}, \tag{3}$$

where

(4) $h \times 10^{-4} = H =$ Hubble's constant in km/sec/parsec.

Since there is some uncertainty as to the value of Hubble's constant, $h$ will be carried through this analysis to facilitate correction of the results. According to recent unpublished results $h$ is likely to be near 1.

It is unlikely that the double galaxies are in close proximity by chance; that is, that they are in hyperbolic orbits or chance collisions. Correcting Holmberg's estimate [7] for changes in the distance scale, the number of close approaches to distance $r_m$ or less per unit time and volume is

$$\frac{dm}{dt} = \sqrt{2}\,\pi r_m^2 n^2 \bar{v}. \tag{5}$$

Using $n_0 = 3.7h^3$ galaxies/$10^{20}$ psc$^3$ now, at $t = t_0$, $\bar{v} = 200$ km/sec $= 2.11 \times 10^{-4}$ psc/yr random velocity, $r_m = (47.5/h) \times 10^9$ a.u. $= (2.3/h) \times 10^5$ psc, $n(t) = n_0(t_0/t)^3$, $t_0 = (0.98/h) \times 10^{10}$ yr, the present age of the universe, and the average time for a galaxy to traverse a sphere of radius $r_m$,

$$\Delta t \fallingdotseq \frac{\pi r_m}{2\bar{v}} = \frac{1.44}{h} \times 10^9 \text{ yr}, \tag{6}$$

the total number of chance collisions in process is approximately

$$\begin{aligned} \frac{4\pi D_m^3}{3} \int_{t_0 - \Delta t}^{t_0} dm &= \frac{4}{3}\pi^2\sqrt{2}\, r_m^2 n_0^2 \bar{v} D_m^3 \int_{t_0-\Delta t}^{t_0} \left(\frac{t_0}{t}\right)^6 dt \\ &= \frac{4}{15}\pi^2\sqrt{2}\, r_m^2 n_0^2 \bar{v} D_m^3 t_0 \left[\left(\frac{1-\Delta t}{t_0}\right)^{-5} - 1\right] \\ &= 90 \text{ (independent of } h), \end{aligned} \tag{7}$$

where $D_m$ is the average distance of a $14\overset{\text{m}}{.}3$ galaxy, about $(5.1/h) \times 10^7$ psc, using Sandage's values [7] for field galaxies,

$$\log_{10} cz \equiv \log_{10} V = \log_{10} HD_m = 0.2m + 0.85 \pm 0.03. \tag{8}$$

Note that $D_m^{-3}$ also enters $n_0$, since

$$n_0 = \frac{N(m)}{\frac{4}{3}\pi D_m^3}, \tag{9}$$

where the number of field galaxies brighter than magnitude $m$ is

$$N(m) = 0.6m - 4.26, \tag{10}$$

according to Minkowski [9] and $n_0$ is calculated from equations (5), (8), (9), and (10). Also, from equations (1) and (8), the maximum separation of physical doubles of magnitude $m$ is given by $\log_{10} S_m = 4.05 - m/5$, corresponding to 15.5 at $m = 14.3$, somewhat larger than in Holmberg's catalogue [7].

This calculation simply confirms the fact that, in a uniform random distribution of galaxies in space, chance hyperbolic passages within $r_m$ of each other would only account for about ten per cent of the observed number of double galaxies. Since we avoid the major clusters where $n_0$ is larger than average, it can be assumed that the double galaxies studied here are moving in closed orbits. (See, however, section 8 below.)

### 3. The circular-orbit model

The quantities involved in a pair of galaxies observed from a large distance (observer at 0) are illustrated in figure 1, where $M_1$, $M_2$ = masses of the two galaxies in solar masses ($1.98 \times 10^{33}$ gm), $r$ = radius vector, in astronomical units (1 a.u. = $1.5 \times 10^{13}$ cm), $D_p$ = distance in parsecs, $S$ = angular separation in minutes of arc, $v$ = orbital velocity in a.u./yr (4.74 km/sec), $\phi$ = angle between $r$ and tangential plane (0 to $\pi/2$), $\psi$ = angle (not shown) between $v$ and plane of $OM_1M_2$ (0 to $2\pi$).

The projection of $r$ on the tangential plane is

$$a = r \cos \phi, \tag{11}$$

which is also related to observable quantities; using the fact that 1 a.u. subtends an angle of 1″ at distance 1 parsec,

$$a = 60\, SD_p = \frac{60\, SV}{h} \times 10^{-4}, \tag{12}$$

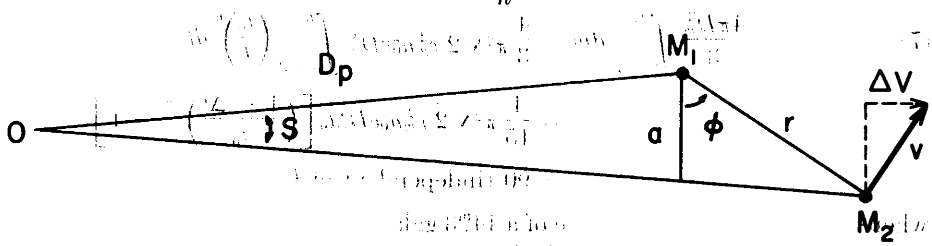


FIGURE 1

Circular-orbit model.

The galaxies $M_1$ and $M_2$, separated by $r$ astronomical units, are seen from 0 at distance $D_p$ parsecs. The orbital velocity, $v$, is perpendicular to $r$, but not necessarily in the plane $OM_1M_2$.

using equation (4) with $V = (V_1 + V_2)/2$.

For the following analysis three assumptions are made:

(a) the relative orbit is circular,

(b) the galaxies are considered as point masses (no tidal effects),

(c) the density of intergalactic material, $\rho_i = 0$.

From (a) it follows that $v$ is perpendicular to $r$, and its component in the line of sight is

$$\Delta V_0 \equiv V_2 - V_1 = 4.74 v \cos \phi \cos \psi, \tag{13}$$

where 1 a.u./yr = 4.74 km/sec.

From Newtonian mechanics for point masses in circular orbits, with distances measured in a.u., time in years, and mass in solar masses,

$$v^2 = \frac{4\pi^2}{r}(M_1 + M_2) = \frac{4\pi^2}{r} NM. \tag{14}$$

Some of the systems considered below consist of widely separated, tight groups of galaxies; $N$ is the total number of such galaxies, and $M$ is the average mass.

Substituting equations (11), (12), and (13) in (14),

$$(\cos^3\phi \cos^2\psi)hM = \frac{6 \times 10^6 SV(\Delta V_0)^2}{(9.48\pi)^2 N} \doteq 675 \frac{SV}{N}[(\Delta V)^2 - \sigma_{\Delta V}^2]. \tag{15}$$

The observable quantities on the right are subject to observational errors, $\sigma_S$, $\sigma_V$, $\sigma_{\Delta V}$. In the mean, the observed $(\Delta V)^2$ is biased by the variance in $\Delta V$; hence each observed $(\Delta V)^2$ is reduced by $\sigma_{\Delta V}^2$ to be more nearly equal to the true value $(\Delta V_0)^2$. If there were no selection effects, and if $\phi$ and $\psi$ are independent of $M$, $S$, $V$, $\Delta V$, and $N$, equation (15) could be averaged over $n$ systems of galaxies, as in a previous study [1], to obtain an average $M$. For random orientations, the mean of $\cos^3\phi \cos^2\psi$ is $3\pi/32 = 0.2945$.

In actual fact, the pairs and groups of galaxies to which this analysis will be applied are selected with respect to $S$, and the effect of such selection depends on the distribution of $r$, equation (2). As Holmberg has shown [2], this distribution of $r$ leads to a regression of $(\Delta V_0)^2$ on $SV$ from which $\overline{M}$ can be determined.

The joint probability distribution,

$$p_{M,r,\phi,\psi} = p_M p_r p_\phi p_\psi \tag{16}$$

on the reasonable assumption that the orientation angles $\phi$ and $\psi$ are independent of each other and of $r$ and $M$, and on the more doubtful assumption that $M$ is independent of $r$. The probability densities are

$$p_\phi = \cos\phi, \qquad p_\psi = \frac{1}{2\pi}, \tag{17}$$

where $0 \leqq \phi \leqq \pi/2$, and $0 \leqq \psi \leqq 2\pi$. Note that $\Delta V$ is always reckoned positive.

The variable, $r$, is now transformed to $a$ by equation (11), with Jacobian $1/\cos\phi$, and the conditional distribution

$$p_{\phi,\psi|a,M} = \frac{1}{2\pi} \frac{p_r\left(\dfrac{a}{\cos\phi}\right)}{\displaystyle\int_0^{\pi/2} p_r\left(\frac{a}{\cos\phi}\right) d\phi}, \tag{18}$$

from which is obtained the expected value of

$$(\Delta V_0)^2 = (9.48\pi)^2 \frac{NM}{a} \cos^3\phi \cos^2\psi \tag{19}$$

given $a$ and $M$:

$$\begin{aligned} E\{(\Delta V_0)^2 | a, M\} &= (9.48\pi)^2 \frac{NM}{a} \int_0^{2\pi} \cos^2\psi \, d\psi \frac{I_1(a)}{2\pi I_0(a)} \\ &= A_i(a)M, \end{aligned} \tag{20}$$

where $\int_0^{2\pi} \cos^2 \psi \, d\psi = \pi$ and

$$I_k(a) = \int_0^\alpha p_r\left(\frac{a}{\cos \phi}\right) \cos^{3k} \phi \, d\phi. \tag{21}$$

Likewise,

$$\begin{aligned} E\{(\Delta V_0)^4 | a, M\} &= (9.48\pi)^4 \left(\frac{NM}{a}\right)^2 \int_0^{2\pi} \cos^4 \psi \, d\psi \frac{I_2(a)}{2\pi I_0(a)}, \\ &= B_i(a) M^2, \end{aligned} \tag{22}$$

an expression that is useful in computing variances. Note that $\int_0^{2\pi} \cos^4 \psi \, d\psi = 3\pi/4$.

Using Holmberg's distribution $p_r(r)$, equation (2), the integrals must be limited to a range $0 \leqq \phi \leqq \alpha = \cos^{-1} a/r_m$ since $p_r$ is zero for $r \geqq r_m$. Then

$$\frac{I_0(a)}{K} = \alpha - \frac{1}{2} \sin \alpha \cos \alpha - \frac{1}{2} \cos^3 \alpha \log \frac{1 + \sin \alpha}{\cos \alpha}, \tag{23}$$

$$\frac{I_1(a)}{K} = \frac{1}{3} \cos^2 \alpha \sin \alpha + \frac{2}{3} \sin \alpha - \alpha \cos^3 \alpha, \tag{24}$$

$$\begin{aligned} \frac{I_2(a)}{K} = \frac{1}{6} \cos^5 \alpha \sin \alpha &+ \frac{5}{24} \cos^3 \alpha \sin \alpha + \frac{15}{48} \cos \alpha \sin \alpha \\ &+ \frac{15}{48} \alpha - \cos^3 \alpha \left[\frac{1}{3} \cos^2 \alpha \sin \alpha + \frac{2}{3} \sin \alpha\right], \end{aligned} \tag{25}$$

where

$$\cos \alpha = \frac{a}{r_m}. \tag{26}$$

The measured quantity $\Delta V$ is assumed to be a normal random variable with mean $\Delta V_0$ and variance $\sigma_V^2 = \sigma_\Delta^2 / W_\Delta$, where $W_\Delta$ is a weighting factor, and $\sigma_\Delta$ is the standard deviation of unit weight. Hence

$$E\{(\Delta V)^2 | a\} = E\{(\Delta V_0)^2 | a\} + \frac{\sigma_\Delta^2}{W_\Delta} \tag{27}$$

or

$$\begin{aligned} E\{(\Delta V)^2 | a\} - \frac{\sigma_\Delta^2}{W_\Delta} &= \underset{(M)}{E} \{E[(\Delta V_0)^2 | a, M]\} \\ &= E\{M\} (9.48\pi)^2 \frac{N}{2} \frac{I_1(a)}{a I_0(a)} \end{aligned} \tag{28}$$

from equation (20).

A plot of $I_1(a)/a I_0(a)$ confirms the fact, noted by Holmberg, that it can be approximated by

$$\frac{I_1(a)}{aI_0(a)} \doteqdot \frac{0.4}{a} + \frac{0.6}{r_m} \tag{29}$$

over the interval $0.03 \leqq a/r_m \leqq 1$, and the accuracy of fit is shown by the following values (see also figure 6).

| | | | | | |
|---|---|---|---|---|---|
| $\cos\alpha \equiv a/r_m =$ | 0.05 | 0.10 | 0.25 | 0.50 | 0.75 |
| $I_1(a)/I_0(a) =$ | 0.4456 | 0.4686 | 0.5462 | 0.6939 | 0.8474 |
| $0.4 + 0.6a/r_m =$ | 0.43 | 0.46 | 0.55 | 0.70 | 0.85 |

Equation (28) is a regression of the form

$$Y_i = \overline{M}A_i(a) \tag{30}$$

with the observable quantities

$$Y_i = (\Delta V)^2 - \frac{\sigma_\Delta^2}{W_\Delta}, \tag{31}$$

$$\begin{aligned} A_i(a) &= (9.48\pi)^2 \frac{N_i}{2} \frac{I_1(a)}{aI_0(a)} \\ &\doteqdot (9.48\pi)^2 \frac{N_i}{2}\left(\frac{0.4}{a} + \frac{0.6}{r_m}\right) \\ &= 5.92 \times 10^{-8} h \frac{N_i}{2}\left(\frac{10^4}{SV} + 0.19\right), \end{aligned} \tag{32}$$

substituting equation (12) and $r_m$ from equation (2). The weights for the observation equations (30) must be inversely proportional to the variance in $Y$ which is, to a first approximation,

$$\begin{aligned} \sigma_{Y|a}^2 &= 4(\Delta V_0)^2 \frac{\sigma_\Delta^2}{W_\Delta} \\ &= 4A_i(a)\overline{M} \frac{\sigma_\Delta^2}{W_\Delta}. \end{aligned} \tag{33}$$

It is to be noted that the observational errors in $a$, that is, in $S$ and $V$, are negligible compared to those in $\Delta V$. The angular separation $S$ is measured to $\pm 0'.1$ (minute of arc), and $S$ ranges from $0'.7$ to $40'$; hence $\sigma_S/S$ is of the order 0.1 or less, and $\alpha_V/V$ is of the same order. However, it will be shown below that $\sigma_\Delta = 90$ km/sec, $W_\Delta$ ranges from 0.1 to 20, and $\Delta V$ from 1 to 600 km/sec. Hence $\sigma_Y^2/Y^2$ is much larger than $\sigma_S^2/S^2 + \sigma_V^2/V^2$ and the latter can be neglected.

Because the dynamics of the multiple systems are less precisely represented by equation (14) than the pure pairs, the weights $w_i^2 = \sigma_\Delta^2/\sigma_Y^2$ were modified by the factor $1/N_i$,

$$w_i = \frac{W_\Delta}{N_i A_i(a)} \tag{34}$$

and the least squares solution of equation (30) is

$$\hat{M} = \frac{\sum^n w_i A_i Y_i}{\sum^n w_i A_i^2} = \frac{\sum^n \frac{W_\Delta Y_i}{N_i}}{\sum^n \frac{W_\Delta A_i}{N_i}} \tag{35}$$

$$= \frac{3.38 \times 10^7}{h} \frac{\sum^n W_\Delta \left[ (\Delta V)^2 - \frac{\sigma_\Delta^2}{W_\Delta} \right] \frac{1}{N_i}}{\sum^n W_\Delta \left[ \frac{10^4}{SV} + 0.19 \right]}.$$

It is to be noted that, since 0.19 is generally small compared to $10^4/SV$ in equation (35), this solution is nearly the same as the average of equation (15) used in a previous study [1], with $\overline{\cos^3 \phi \cos^2 \psi} = 0.20$ and weights $w_i = W_\Delta/SV$. Hence the results of this more refined analysis are not expected to differ significantly from the earlier results, except for the modified value of $h$, now believed to be 0.75 to 1.0 (corresponding to the constant 75 to 100 km/sec/megaparsec in the Hubble law). However, it is now possible to determine the root mean square error in $\hat{M}$ from the least square residuals,

$$\sigma_{\hat{M}}^2 = \frac{\sum^n w_i Y_i^2 - \hat{M} \sum^n w_i A_i Y_i}{(n-1) \sum w_i A_i^2} \tag{36}$$

$$= \frac{\sum^n \frac{W_\Delta Y_i^2}{N A_i}}{\frac{n-1}{N} \sum W_\Delta A_i} - \frac{(\hat{M})^2}{n-1}.$$

The regression is plotted and the values of $\hat{M}$ and $\sigma_{\hat{M}}$ are computed in section 7 below.

The uncertainty in $\hat{M}$ represented by $\sigma_{\hat{M}}^2$ can be ascribed to three causes: (a) the observational errors, represented almost entirely by $\sigma_Y^2$, equation (33); (b) the random distribution of $\phi$ and $\psi$, averaged out in the integrals of equation (20); and (c) the inherent variability of $M$, represented here by $\sigma_M^2$. With the kind help of Professor J. Neyman at the Fourth Berkeley Symposium, a formula was derived for $\sigma_M^2$, starting with equation (22) in which the expected value of $M^2$ is replaced by $(\overline{M})^2 + \sigma_M^2$. Using the expected value of the *measured* $(\Delta V)^4$,

$$E\{(\Delta V)^4 | a, M\} = E\{(\Delta V_0)^4\} + 6E\{(\Delta V_0)^2\} \frac{\sigma_\Delta^2}{W_\Delta} + \frac{3\sigma_\Delta^4}{W_\Delta^2} \tag{37}$$

and of the *measured* $(\Delta V)^2$ from equation (27),

$$\sigma_{Y_a}^2 = B_i \sigma_M^2 + (B_i - A_i^2)(\overline{M})^2 + 4A_i \frac{\overline{M}\sigma_\Delta^2}{W_\Delta} + \frac{2\sigma_\Delta^4}{W_\Delta^2}. \tag{38}$$

The expected value of the sum of least squares can be written in the form

$$E\{S_0^2\} = \sum^n w_i\sigma_Y^2 - (\sum^n A_i^2 w_i)\sigma_{\hat{M}}^2 \tag{39}$$

$$= \sum^n w_i\sigma_Y^2 - \frac{\sigma_Y^2 \sum^n A_i^2 w_i^2}{\sum^n A_i^2 w_i}$$

$$= \sigma_Y^2 \left( \sum^n w_i - \frac{\sum^n A_i^2 w_i^2}{\sum^n A_i^2 w_i} \right).$$

Substituting equation (38) in (39), replacing $(\overline{M})^2$ with $E\{(\hat{M})^2\} - \sigma_{\hat{M}}^2$ and solving for $\sigma_M^2$ results in a complex expression involving terms containing $S_0^2$, $(\hat{M})^2$, $\hat{M}\sigma_\Delta^2$, and $\sigma_\Delta^4$, together with various sums of combinations of $w_i$, $A_i$, and $B_i$. The application of this formula will be discussed in section 7; it is displayed in the appendix.

## 4. Mean ratio of mass to luminosity

Another observable quantity is the apparent magnitude of each galaxy, defined as $m' = -2.5 \log_{10} l'$, where $l'$ is the apparent brightness. Knowing the distance, $D_p$, from equation (3), the intrinsic brightness or luminosity $L$, also in solar units, can be determined from the inverse square law, and since $M$ is correlated with $L$ for stars, it is to be expected that $M/L$ will show less variability than $M$.

Correcting for absorption of light by interstellar dust within our own galaxy, the apparent magnitude that would be observed at distance $D_p$ from a galaxy of luminosity $L$ becomes

$$m = m' - 0.25 \csc b = -2.5 \log_{10} l, \tag{40}$$

where $b$ = galactic latitude and

$$\frac{l}{l_s} = L\left(\frac{d_s}{D_p}\right)^2, \tag{41}$$

where $l_s$ is the brightness of the sun at distance $d_s$. If the sun were at a distance of 10 psc it would have a magnitude of 5.26 (its "absolute magnitude"); hence $-2.5 \log_{10} l_s = 5.26$ for $d_s = 10$, and

$$L = \left(\frac{D_p}{10}\right)^2 10^{0.4(5.26-m)} \tag{42}$$

$$= \left(\frac{V}{h}\right)^2 10^{0.4(20.26-m)}.$$

Introducing the factor $(1/\sum_N L)(V/h)^2 \sum^N 10^{0.4(20.26-m)} = 1$ in equation (30), where $\sum^N L$ is the sum of luminosities of $N$ galaxies in one system, so that $NM/\sum^N L = M/L$,

$$(43) \qquad Y_i = \frac{N_i M}{\sum_j^{N_i} L_j} \frac{V_i^2 A_i}{h^2 N_i} \sum_j^{N_i} 10^{0.4(20.26 - m_j)}$$

$$\doteqdot \frac{2.96}{h} (\overline{M/L}) \left( 10^{-4} \frac{V_i}{S_i} + 0.19 \times 10^{-8} V_i^2 \right) \sum_j^{N_i} 10^{0.4(20.26 - m_j)},$$

where Holmberg's approximation of $A_i$, equation (32), has been substituted. Equation (43) is a regression of the form

$$(44) \qquad Y_i = (\overline{M/L}) C_i(V, S)$$

and the least squares solution, with weights $w_i'$ is

$$(45) \qquad \widehat{M/L} = \frac{\sum^n w_i' C_i Y_i}{\sum^n w_i' C_i^2}.$$

The observational errors in $m$ range from $0^m.1$ to $0^m.5$. Ignoring the small term $0.19 \times 10^{-8} V^2$, the variance in $C$ due to errors in the measured quantities $V$, $S$, and $m$ is

$$(46) \qquad \sigma_C^2 = C^2 \left[ \frac{\sigma_V^2}{V^2} + \frac{\sigma_S^2}{S^2} + (0.92)^2 \sigma_m^2 \right] \cdot$$

Since $(0.92)^2 \sigma_m^2$ is of the same order as $\sigma_V^2 / V^2$, the variance in $C$ is negligible compared with $\sigma_Y^2$.

However, because the luminosities vary widely, the weights used do not contain the factor $1/N_i$,

$$(47) \qquad w_i' = \frac{W_\Delta}{C_i(V, S)}$$

so that

$$(48) \qquad \widehat{M/L} = \frac{\sum^n W_\Delta Y_i}{\sum^n W_\Delta C_i}$$

and, as before,

$$(49) \qquad \sigma^2_{\widehat{M/L}} = \frac{\sum^n W_\Delta \frac{Y_i^2}{C_i}}{(n-1) \sum^n W_\Delta C_i} - \frac{(\widehat{M/L})^2}{n-1}$$

The factor $1/h$ in $C_i(V, S)$ means that, as expected, the estimates of $M/L$ and $\sigma_{M/L}$ are proportional to $h$. Both quantities are computed in section 7.

## 5. The observational data

In a previous study [1] measurements of $V$, $\Delta V$, and $S$ were reported for 20 pairs of galaxies, 15 of them in multiple systems. Since then, Humason, Mayall,

## TABLE I

### New Measures of Differential Velocities

$\Delta_y$ is the separation of the two spectra on the film.
$V'$ is the measured velocity before correction for observer motion.
$\Delta V$ is reckoned positive when the first nebula listed in the pair has the larger velocity of recession.
$W_V$ is the weight of the observation of $V$.
$W_\Delta$ is the weight of the observation of $\Delta V$.
For NGC 2820 one film included was reported previously in table I [1], p. 66.

| NGC | Holm-berg | No. of Films | Estimated Line Intensities | | | | | | $\Delta_y$ (mm.) | $V'$ (km/sec) | $W_V$ | $\Delta V$ (km/sec) | $W_\Delta$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | *NII* | $H_\alpha$ | 3727 | *H* | *K* | Other | | | | | |
| 2535 | 94a | 3 | +5 | +8 | 0 | | | | 0.75 | 4073 | 4.0 | 1 | 3.4 |
| 2536 | b | | +5 | +8 | 0 | | | | | | | | |
| 2719 | 105a | 2 | +3 | +20 | +8 | | | $N_1H_\beta$ | 0.18 | 3181 | 3.4 | −137 | 2.7 |
| Anon | b | | +1/2 | +10 | +5 | | | $N_1H_\beta$ | | | | | |
| 2820 | 124a | 2 | +3 | +10 | +3 | | | $N_1H_\beta$ | 1.72 | 1673 | 2.3 | 2 | 2.2 |
| 2814 | c | | +1/2 | +3 | +1/2 | | | | | | | | |
| 2820 | 124a | 2 | +2 | +8 | +2 | | | $N_1H_\beta$ | 0.98 | 1602 | 3.0 | 211 | 2.6 |
| Anon | d | | +1/2 | +10 | +8 | | | $N_1H_\beta$ | | | | | |
| 3455 | 221a | 2 | +2 | +5 | | | | | 1.52 | 1140 | 2.0 | −54 | 2.2 |
| 3454 | b | | 0 | +1 | | | | | | | | | |
| Anon | 231a | 1 | +5 | +8 | | | | | 0.27 | 6155 | 1.4 | 318 | 1.3 |
| Anon | b | | +3 | +5 | | | | | | | | | |
| 3993 | 308a | 1 | +1/2 | +1 | | | | | 1.27 | 4800 | 1.2 | 62 | 1.2 |
| 3997 | b | | +5 | +10 | | | | | | | | | |
| 3995 | 309a | 2 | +5 | +10 | +10 | | | $N_1H_\beta H_\gamma$ | 0.75 | 3242 | 2.1 | 218 | 2.5 |
| 3994 | b | | +8 | +10 | +5 | | | $N_1H_\beta H_\gamma$ | | | | | |
| 3995 | 309a | 1 | +5 | +8 | +3 | | | $N_1$ | 1.63 | 3360 | 1.5 | 91 | 1.3 |
| 3991 | c | | +1/2 | +10 | +3 | | | $N_1$ | | | | | |
| 5278 | — | 1 | +5 | +5 | | | | | 0.27 | 7545 | 1.4 | −43 | 1.2 |
| 5279 | | | +1 | +3 | | | | | | | | | |
| Anon | 541a | | +1 | +5 | | | | | 0.54 | 4680 | 1.2 | −464 | 1.2 |
| Anon | b | | +1 | +5 | | | | | | | | | |
| 5480 | 588a | 1 | +5 | +10 | 0 | | | | 1.32 | 1870 | 1.2 | −305 | 0.5 |
| 5481 | b | | 0 | 0 | 0 | −5 | −3 | | | | | | |
| 5506 | 604a | 1 | | | +8 | −1 | −2 | $N_1N_2H_\beta H_\gamma$ | 1.57 | 2040 | 1.4 | 311 | 1.2 |
| 5507 | b | | | | | −3 | −2 | G | | | | | |
| 5775 | 685a | 1 | +3 | +5 | | | | | 1.80 | 1545 | 1.1 | 40 | 1.2 |
| 5774 | b | | +1/2 | +3 | | | | | | | | | |
| 6068 | 727a | 2 | +5 | +8 | | | | | 0.84 | 3964 | 2.4 | −71 | 2.5 |
| Anon | b | | +5 | +8 | | | | | | | | | |

and Sandage [8] have reported 806 individual velocities of galaxies, and Page has measured 15 more pairs, as indicated in table I. It is to be noted that $\Delta V$ is measured *directly* in Page's 35 pairs by obtaining spectrograms (with the B-Spectrograph on the 82-inch telescope of the McDonald Observatory) showing spectra of both galaxies in a pair, side by side. In order for this to be possible, the separation $S$ must be less than 4′.8; that is, the 35 pairs were selected for

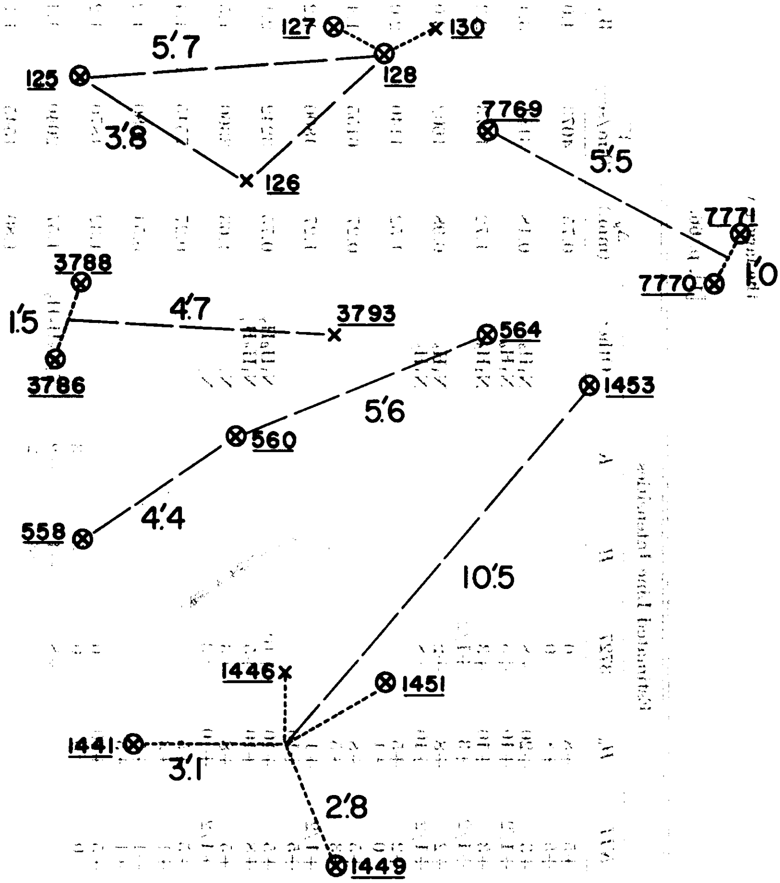


FIGURE 2

Relative projected positions of galaxies in multiple systems. Each circled $x$ represents the approximate position of a galaxy with a measured radial velocity. Uncircled crosses represent galaxies for which the velocity has not been measured. Each multiple system is connected by dashed lines (the systems are separated by large distances in the sky). As explained in the text, the systems NGC 125, 126, 127, 128, 130, and NGC 558, 560, 564 were dropped. NGC 1453 is considered a satellite of the group NGC 1441, 1446, 1449, 1451, with total system mass $5M$. The pair NGC 7770, 7771 is treated as an isolated system of total mass $2M$, and NGC 7769 is treated as a satellite of NGC 7770, 7771 with total system mass $3M$.

$S < 4'.8$ and magnitude brighter than about $14^m.3$, although they are a very small sample of all doubles that satisfy these criteria. Most of them were drawn from Holmberg's catalogue [7], in which a further criterion is applied: $S \leqq 2(A_1 + A_2)$, where $A_1$ and $A_2$ are the largest dimensions of the two member galaxies.

Pairs were initially located in the Humason and Mayall lists [8] simply by identifying all pairs from Holmberg's catalogue and adding further pairs with small $S$. In this manner 97 multiple systems were found with two or more measured radial velocities (including those measured by Page), 61 of them listed in Holmberg's catalogue. However, many of these systems were small groups of the "trapezium type" as defined by Ambartzumian [5], to which the dynamical formula, equation (14), does not apply.

The position of each galaxy in the 97 systems was plotted in the manner of figure 2, together with all NGC objects within $3S$. Then, all obvious trapezium-type systems were eliminated from the list. Pairs of groups were retained if (a) the individual separations from the center of each group were less than $S/3$, where $S$ refers to the separation between the centers of groups, and if (b) radial velocities were available for 2/3 of the members of each group. This procedure left 66 systems: the 42 pure pairs listed in table III, where the nearest neighboring NGC galaxy is much more than $3S$ distant, and the 24 pairs of groups or pairs within groups satisfying criteria (a) and (b) above, listed in table IV. The systems eliminated are listed in table II.

This rough screening does not guarantee that the *space* separations satisfy criterion (a), and since there may be doubts about the applicability of equation (14) to multiple groups, they have been kept separate in the analysis of section 7 below.

## 6. Observational errors

In order to apply equations (35), (36), (47), and (48) it is necessary to estimate the mean square errors in the observed values of $\Delta V$, $V$, and $S$. It is clear that these errors, or the associated weights, vary widely, due to differences in method of measurement, in spectrographic dispersion, in photographic emulsion, and in the inherent character of the lines in spectra of galaxies.

The spectrograms obtained by Page [1] were all made with the same spectrograph, at the same dispersion, and on the same type of photographic emulsion (Eastman red-sensitive 103aF film). Moreover, they all show *two* spectra—the slit of the spectrograph was oriented to bisect the two members of a pair—and $\Delta V$ could be measured directly, in most cases, by repeated settings (of a cross hair carried in a microscope on an accurate measuring engine) on the same line in first one and then the other spectrum. Suitable precautions were taken to line up the cross hair with the image of the slit, and to correct for curvature of this image. Ideally, this method of measurement saves a factor of two in $\sigma^2_V$.

The identifiable lines in spectra of galaxies at dispersion 300 to 400 Å/mm are few; most of Page's measures refer to $H_\alpha$ 6563, and $N$ II 6584 emission lines,

## TABLE II

### Close Multiple Galaxies not Suitable for Analysis

Parentheses set off groups; galaxies with measured velocities are in boldface type.

| NGC or Holmberg Number | Approximate Separations Maximum | Approximate Separations Minimum | Reason for Rejection |
|---|---|---|---|
| (**80**, 81), **83** | 5′ | 3′ | No $V$, NGC 81 |
| **125**, 126, (**127, 128**, 130) | 6 | 1 | Trapezium |
| **495**, 496, 498, **499**, 501 | 7 | 4 | Trapezium |
| 558, **560, 564** | 6 | 4 | Trapezium |
| (**584**, 586), **596** | 40 | 8 | No $V$, NGC 586 |
| (733, **736**, 738, 739, 740), (**750, 751**) | 30 | 2 | Only 1 $V$ in first group |
| 1396, **1399, 1404**, 1408 | 11 | 8 | Trapezium |
| **1400**, 1402, **1407** | 12 | 10 | Trapezium |
| **2911**, 2912, **2914** | 5 | 2 | No $V$, NGC 2912 |
| **3613, 3619**, 3625 | 28 | 15 | Trapezium |
| **3681, 3684, 3686**, 3691 | 13 | 12 | Trapezium |
| 6027 **a**, b, c, **d**, e | 0.5 | 0.2 | Trapezium |
| (6959, 6961, **6962, 6964**, 6967), **6963** | 10 | 2 | Trapezium |
| 7006, 3 companions, **a, b, c** | 16 | 7 | Trapezium |
| **7240, 7242a**, 7242b | 4 | 0.5 | No $V$, NGC 7242b |
| 7383–7390, **7385, 7386** | 5 | 2 | Trapezium |
| (**7611, 7617, 7619, 7626**), (7615, 7621, **7623**) | 11 | 3 | No $V$, NGC 7615, 7621 |
| Ho 6 **a, b**, c, **d, e, f, g** | 3 | 1 | Trapezium |
| Ho 123 **a, b**, c | 2 | 0.5 | No $V$ for $c$ |
| Ho 124 **a**, b, **c, d** | 4 | 2 | Trapezium |
| Ho 130 **a, b**, c, d | 2 | 1 | Trapezium |
| Ho 172 **a, b, c** | 3 | 3 | Trapezium |
| Ho 173 **a, b**, NGC 3165 | 8 | 5 | Trapezium |
| Ho 212 **a, b, c** | 10 | 8 | Trapezium |
| Ho 308 **a, b**, c, d | 3 | 3 | Trapezium |
| Ho 368 **a, b, c**, d, e, f | 18 | 4 | Trapezium |
| Ho 413 **a**, b, **c**, d | 12 | 5 | Trapezium |
| Ho 694 **a, b**, NGC 5839, 5845, **5850** | 10 | 8 | Trapezium |
| Ho 719 **a, b**, c | 16 | 14 | Trapezium |
| Ho 792 a, **b, c, d** | 3 | 2 | Trapezium |
| Ho 795 **a, b**, c, d, e, f, g, h, i, j | 12 | 3 | Trapezium |
| (Pairs with small projected separations, $a = 60SV/h \times 10^{-4}$) | | | |
| Ho 17 **a, b** | | 24.0 | $ah$ = 3200 psc |
| Ho 17 **a, c** | | 36.0 | 4800 psc |
| Ho 105 **a, b** | | 0.4 | 3700 psc |
| Ho 215 **a, b** | | 1.3 | 6100 psc |
| Ho 240 **a, c** | | 2.9 | 6100 psc |
| Ho 270 **a, b** | | 1.3 | 2800 psc |
| Ho 409 **a, b** | | 4.4 | 4400 psc |
| Ho 486 **a, b** | | 0.7 | 1700 psc |
| Ho 694 **a, b** | | 0.7 | 4200 psc |
| Ho 710 **a, b** | | 0.5 | 4000 psc |
| Ho 714 **a, b** | | 0.8 | 5050 psc |
| NGC **750, 751** | | 0.4 | 6100 psc |
| NGC **4038, 4039** | | 1.2 | 5000 psc |
| NGC **5544, 5545** | | 0.6 | 5400 psc |
| (Systems with $\Delta V$ poorly determined) | | | |
| Ho 272 **a, b** | | 1.5 | $W_\Delta$ = 0.5 |
| Ho 369 **ab–c** | | 18.0 | 0.28 |
| Ho 397 **a, b** | | 7.5 | 0.29 |
| Ho 411 **a, b** | | 3.7 | 0.10 |
| Ho 422 **a, b** | | 4.2 | 0.07 |
| NGC **1316, 1317** | | 7.3 | 0.43 |
| NGC **1600, 1601** | | 1.5 | 0.34 |
| NGC **5857, 5859** | | 2.0 | 0.15 |
| NGC **5898, 5903** | | 7.3 | 0.11 |
| NGC **6927–Anon** | | 2.1 | 0.11 |
| NGC **6962, 6954** | | 2.3 | 0.43 |

TABLE III

DATA FOR 42 PURE PAIRS ($N = 2$)

Magnitudes and types are from Holmberg [11] or Mayall and Sandage [8] except for those in parentheses, which are from Holmberg [7]. Letter designations (F = faint, etc.) are from the "New general catalogue" [12].
†Too small projected separation for final analysis.

| NGC | Holm-berg | Mag. | Type | $S$ | $V$ (km/sec) | $\Delta V$ (km/sec) | $W_\Delta$ | Obs. |
|---|---|---|---|---|---|---|---|---|
| 2535 | 94a | 13.2 | (Sbc) | 1.′75 | 3983 | 1 | 3.44 | P |
| 2536 | b | (14.3) | (EO) | | | | | |
| 2672 | 99a | 13.2 | E1 | 0.6 | 3885 | 431 | 0.47 | H |
| 2673 | b | 14.4 | EO | | | | | |
| 2719 | 105a | (14.0) | (Sab) | 0.4† | 3143 | 137 | 2.66 | P |
| Anon | b | (14.5) | (ESO) | | | | | |
| 3190 | 175a | 12.0 | Sa | 6.0 | 1250 | 52 | 1.10 | H |
| 3193 | b | 11.9 | E2 | | | | | |
| 3227 | 187a | 11.3 | Sb | 2.3 | 1110 | 217 | 10.75 | H |
| 3226 | b | 12.6 | E1 | | | | | |
| 3395 | 215a | (12.1) | Sc | 1.3† | 1599 | 7 | 5.23 | P |
| 3396 | b | (13.1) | Irr | | 1660 | 108 | 0.92 | M |
| 3455 | 221a | (12.6) | (E) | 3.8 | 1034 | 54 | 2.17 | P |
| 3454 | b | (13.4) | (ESO) | | | | | |
| Anon | 231a | (13.8) | (E) | 0.8 | 6086 | 318 | 1.27 | P |
| Anon | b | (14.1) | (Sbc) | | | | | |
| 3769 | 270a | (12.3) | SBc | 1.3† | 750 | 28 | 2.24 | P |
| Anon | b | (14.1) | Sa | | | | | |
| 3998 | 310a | (11.8) | (SO) | 3.0 | 990 | 339 | 2.12 | M |
| 3990 | b | (13.3) | (SO) | | | | | |
| 4382 | 397a | 10.1 | SO | 7.5 | 720 | 1 | 0.29 | H |
| 4394 | b | 11.8 | SBb | | | | | |
| 4438 | 409a | 10.9 | Sap | 4.4† | 346 | 901 | 0.43 | H |
| 4435 | b | 11.9 | SBO | | | | | |
| 4461 | 411a | 12.0 | SO | 3.7 | 1061 | 1504 | 0.10 | H |
| 4458 | b | (12.5) | EO | | | (Optical?) | | |
| 4490 | 414a | 10.1 | Sc | 3.5 | 695 | 155 | 1.17 | P |
| 4485 | b | 12.2 | Sc | | | | | |
| 4550 | 422a | 12.6 | E7 | 4.2 | 594 | 628 | 0.07 | H |
| 4551 | b | (12.8) | E4 | | | | | |
| 4568 | 427a | 11.7 | Sc | 1.3 | 2000 | 18 | 3.28 | P |
| 4567 | b | 12.0 | Sc | | 2270 | 129 | 0.80 | M |
| 4649 | 448a | 9.9 | E2 | 2.8 | 901 | 112 | 0.68 | P |
| 4647 | b | 12.1 | Sc | | 1277 | 204 | 0.74 | M |
| 4762 | 478a | 11.0 | Sa | 10.9 | 1101 | 593 | 0.83 | H |
| 4754 | b | 11.6 | SBO | | | | | |
| 4782 | 485a | (12.8) | EO | 0.7 | 4194 | 628 | 1.00 | P |
| 4783 | b | (13.2) | EO | | | | | |
| 4809 | 486a | (13.1) | Irr | 0.7† | 824 | 57 | 1.17 | P |
| 4810 | b | (13.2) | Irr | | | | | |
| 5194 | 526a | 8.9 | Sc | 4.4 | 574 | 90 | 1.00 | P |
| 5195 | b | 10.5 | Irr | | 598 | 104 | 2.76 | H |

TABLE III (Continued)

| NGC | Holm-berg | Mag. | Type | S | V (km/sec) | ΔV (km/sec) | $W_\Delta$ | Obs. |
|---|---|---|---|---|---|---|---|---|
| 5257 | 532a | (12.9) | (Sb) | 14 | 6645 | 175 | 1.03 | P |
| 5258 | b | (13.3) | (Sb) | | | | | |
| Anon | 541a | (13.6) | ? | 1.5 | 4750 | 464 | 1.21 | P |
| Anon | b | (14.0) | (ESO) | | | | | |
| 5427 | 573a | 12.0 | Sbc | 2.6 | 2211 | 96 | 2.19 | P |
| 5426 | b | 12.7 | Sbc | | | | | |
| 5480 | 588a | (12.1) | (E) | 3.1 | 2010 | 305 | 0.54 | P |
| 5481 | b | (13.2) | (E) | | | | | |
| 5506 | 604a | (13.3) | (SO) | 3.9 | 1987 | 311 | 1.19 | P |
| 5507 | b | (13.9) | (E) | | | | | |
| 5576 | 632a | 12.0 | E4 | 2.8 | 1601 | 185 | 0.54 | H |
| 5574 | b | 13.4 | SBO | | | | | |
| 5775 | 685a | 12.2 | Sb | 4.5 | 1555 | 40 | 1.15 | P |
| 5774 | b | 12.7 | Sc | | | | | |
| 5930 | 710a | (13.6) | (E2) | 0.5† | 2782 | 175 | 2.61 | P |
| 5929 | b | (14.1) | (EO) | | | | | |
| 5954 | 714a | (13.1) | (Sc) | 0.8† | 2166 | 30 | 3.76 | P |
| 5953 | b | (13.2) | (E) | | | | | |
| 6068 | 727a | (13.3) | (Sbc) | 2.0 | 4186 | 71 | 2.45 | P |
| Anon | b | (14.1) | (ESO) | | | | | |
| 7714 | 810a | (13.4) | Spec | 2.0 | 2982 | 38 | 1.10 | M |
| 7715 | b | (14.3) | SBc | | | | | |
| 1888 | — | (pB) | Sb | 0.9 | 2935 | 0 | 1.19 | M |
| 1889 | | 14.4 | EO | | | | | |
| 2693 | — | 13.3 | E2 | 0.9 | 5082 | 167 | 0.83 | H |
| 2694 | | 15.5 | EO | | | | | |
| 3799 | — | (cF) | SBa | 1.4 | 3386 | 17 | 2.95 | P |
| 3800 | | (F) | Sb | | | | | |
| 4038 | — | 10.8 | {Sc | 1.2† | 1443 | 44 | 4.89 | P |
| 4039 | | | {Sc | | 1460 | 13 | 0.83 | H |
| 4105 | — | 12.0 | E2 | 1.3 | 1805 | 283 | 1.34 | H |
| 4106 | | 12.4 | SBO | | | | | |
| 5278 | — | (pF) | (Sb) | 1.3 | 7687 | 43 | 1.24 | P |
| 5279 | | (F) | (Sb) | | | | | |
| 5544 | — | (F) | (E) | 0.6† | 3095 | 10 | 0.05 | P |
| 5545 | | | (E) | | | | | |
| 5857 | — | 13.9 | Sb | 2.0 | 4719 | 48 | 0.15 | H |
| 5859 | | 13.2 | Sb | | | | | |
| 5898 | — | 12.6 | EO | 7.3 | 2385 | 308 | 0.11 | H |
| 5903 | | 12.7 | E2 | | | | | |
| 6658 | — | 14.1 | SO | 9.5 | 4557 | 100 | 1.34 | H |
| 6661 | | 13.2 | SO | | | | | |

with a gain in accuracy (due to the Doppler factor $\Delta\lambda/\lambda$, and to the increasing dispersion in the red) over $O$ II 3727 emission and the $H$ and $K$ absorption lines in the blue. However, seven of Page's 35 spectra showed only absorption lines, which cannot be measured as accurately as emission lines. In addition, some galaxies have diffuse lines, or tilted lines. Therefore, weights $w_{ijk}$ were assigned to each measurement as described previously [1] and the weighted mean was recorded for each spectrogram, together with the summed "lines weight," which

TABLE IV

PAIR DATA FOR 24 MULTIPLE SYSTEMS

Magnitudes and types are from [11] or Mayall and Sandage [8] except for those in parentheses, which are from Holmberg [7].
Velocity for object NGC 224 is taken to be $10^4 \times$ distance in parsecs.
The objects NGC 224, 3607, 5846, and 750 have too small projected separation for final analysis.

| NGC | Holmberg | Mag. | Type | $N$ | $S$ | $V$ (km/sec) | $\Delta V$ (km/sec) | $W_\Delta$ | Obs. |
|---|---|---|---|---|---|---|---|---|---|
| 224 | 17a | 4.33 | Sb | 3 | 24.0 | 46 | 52 | 20.70 | H |
| 221 | b | 9.06 | E2 | | | | | | |
| 224 | 17a | 4.33 | Sb | 3 | 36.0 | 46 | 27 | 18.20 | H |
| 205 | c | 8.89 | SBO | | | | | | |
| Anon | 143a | (13.7) | E | 2 | 1.6 | 5902 | 67 | 2.69 | P |
| | b | (13.9) | E | | | | | | |
| 3607 | 240a | 11.0 | SO | 2 | 2.9 | 729 | 258 | 1.15 | H |
| 3605 | c | 14.0 | E4 | | | | | | |
| 3605, 7 | 240ac | — | SO, E4 | 3 | 7.3 | 923 | 388 | 1.84 | H |
| 3608 | b | 12.1 | E1 | | | | | | |
| 3627 | 246a | 9.5 | Sb | 2 | 20.7 | 610 | 45 | 1.34 | H |
| 3623 | b | 9.9 | Sa | | | | | | |
| 3623, 7 | 246ab | — | Sa, Sb | 3 | 34.4 | 669 | 118 | 1.40 | H |
| 3628 | c | 10.23 | Sb | | | | | | |
| 3788 | 272a | 12.6 | S | 2 | 1.5 | 2540 | 415 | 0.54 | P |
| 3786 | b | 13.2 | S | | | | | | |
| 3995 | 309a | (13.3) | Sb | 2 | 2.0 | 3236 | 218 | 2.53 | P |
| 3994 | b | (13.7) | E | | | | | | |
| 3995 | 309a | (13.3) | Sb | 3 | 4.1 | 3354 | 91 | 1.26 | P |
| 3991 | c | (14.1) | S | | | | | | |
| 4278 | 369a | 11.2 | E1 | 2 | 3.7 | 918 | 450 | 1.08 | P |
| 4283 | b | 13.3 | EO | | | 839 | 447 | 1.15 | H |
| 4278, 83 | 369ab | — | EO, E1 | 3 | 18.0 | 799 | 81 | 0.28 | H |
| 4274 | c | 10.8 | Sa | | | | | | |
| 5846 | 694a | 11.2 | EO | 2 | 0.7 | 2037 | 510 | 1.64 | H |
| Anon | b | 14.1 | E2 | | | 2060 | 547 | 1.13 | M |
| 7771 | 820a | 13.1 | SBb | 2 | 1.0 | 4523 | 62 | 0.74 | M |
| 7770 | b | 14.5 | Sb | | | | | | |
| 7770, 71 | 820ab | — | SBb, Sb | 3 | 5.5 | 4544 | 42 | 0.96 | M |
| 7769 | c | 12.5 | Sc | | | | | | |
| 750 | — | 13.7 | EO | 2 | 0.4 | 5293 | 4 | 1.29 | H |
| 751 | | 14.1 | EO | | | | | | |
| 1316 | — | 10.0 | Irr | 2 | 7.3 | 1820 | 185 | 0.43 | H |
| 1317 | | 12.1 | Sa | | | | | | |
| | | 13.9 | | | | | | | |
| 1441, 49, | — | 14.6 | Sa, So | 5 | 10.5 | 3960 | 203 | 1.43 | H |
| 51 | | 14.5 | E3 | | | | | | |
| 1453 | | 12.9 | E1 | | | | | | |
| 1600 | — | 12.2 | E5 | 2 | 1.5 | 4811 | 167 | 0.34 | H |
| 1601 | | 15.1 | So | | | | | | |
| 2562 | — | 14.0 | Sa | 2 | 4.7 | 4758 | 188 | 1.34 | H |
| 2563 | | 13.7 | SO | | | | | | |
| 6927 | — | 15.6 | So | 2 | 2.1 | 4588 | 142 | 0.11 | H |
| Anon | | | E7 | | | | | | |
| 6927, 28, | — | 15.6 | | | | | | | |
| Anon | | 13.8 | SO, E7 | 4 | 5.2 | 4571 | 304 | 0.92 | H |
| | | — | | | | | | | |
| 6930 | | 14.0 | Sa, Sb | | | | | | |
| 6962 | — | 12.8 | Sb | 2 | 2.3 | 4212 | 351 | 0.43 | H |
| 6964 | | 14.2 | E4 | | | | | | |
| 7576 | — | 13.8 | Sa | 2 | 10.5 | 3625 | 281 | 1.00 | H |
| 7585 | | 12.7 | SOp | | | | | | |

varied from 1/2 to 16. Deviations from the mean, $\delta_{ijk}$, determine a "measurement error" or standard deviation $\sigma_m$, where

$$\sigma_m^2 = \sum_i \sum_j \sum_k \frac{w_{ijk}\,\delta_{ijk}^2}{m-n} \tag{50}$$

$= (49 \text{ km/sec})^2$ for $\Delta V$, and $(76 \text{ km/sec})^2$ for $V$, and the subscripts refer to the $k$th line on the $j$th spectrogram of the $i$th pair of galaxies, in a total of $m$ lines measured on $n$ spectrograms.

Where two or more spectrograms are obtained of the same pair of galaxies, the deviations are found to be larger than expected from the weights $\sum w_{ijk}$ and $\sigma_m$ as determined by equation (49); that is, there is an additional variance between spectrograms, $\sigma_p^2$, as noted earlier by Mayall and Aller [3]. The combination of measurements from different spectrograms requires the weights

(51)
$$w_{ij} = \frac{\sigma_m^2 + \sigma_p^2}{\dfrac{\sigma_m^2}{\sum w_{ij}} + \sigma_p^2},$$

which must be found by trial and error. The deviations of plate means from the final mean $\sigma_{ij}$ determine $\sigma_p$,

(52)
$$\sigma_p^2 + \sigma_m^2 = \frac{\sum_i \sum_j w_{ij}^2 \delta_{ij}}{n - N}$$

for $n$ spectrograms of $N$ pairs of galaxies. The results are shown in table V.

TABLE V

INTERNAL ERRORS IN PAGE'S MEASURES

| | $\Delta V$ | $V$ |
|---|---|---|
| No. of lines measured, $m$ | 335 | 152 |
| No. of spectrograms, $n$ | 57 | 51 |
| No. of pairs of galaxies, $N$ | 17 | 17 |
| Measurement error, $\sigma_m$ | 47 km/sec | 76 km/sec |
| Plate error, $\sigma_p$ | 75 km/sec | 71 km/sec |
| Standard deviation, $\sigma$ | 90 km/sec | 104 km/sec |

Relative to the standard deviation $\sigma$ the weight of a determination of $\Delta V$ or $V$ for the $i$th pair of galaxies is

(53)
$$W_i = \sum_j \frac{\sigma^2}{\dfrac{\sigma_m^2}{\sum_k w_{ijk}} + \sigma_p^2}.$$

The quantity determined by equation (53) applied to Page's measurements of $\Delta V$ is designated $W_\Delta$; this and other weights used in reference to $\Delta V$ are all based on the standard deviation $\sigma_\Delta = 90$ km/sec. The next problem is to obtain these weights for Humason's and Mayall's data.

Humason [8] gives "estimated errors" for his measures on individual galaxies that vary from $\pm 10$ to $\pm 300$ km/sec and mentions that his probable measurement error is $\pm 11$ km/sec and his probable plate error, $\pm 24$ km/sec. His individual measurements were not available, but 114 velocities measured by both Humason and Mayall can be used to check the accuracy of Humason's estimated errors.

Through the kindness of Dr. Mayall, all of his individual measurements in [8] were made available for this study and I am indebted to Mr. A. Kruszewski for the following analysis. From internal differences, the best estimate of $\sigma_m$ in Mayall's measures (that is, the r.m.s. error of a single line measurement of unit weight) based on 1377 individual line measurements, was found, as expected, to vary with slit width and emulsion, between ±83 km/sec with 4-second-of-arc slit width on Eastman IIa0 emulsion to ±154 km/sec on Eastman 103a0 emulsion with 8″ slit width. Using the former $\sigma_m^2 = 6946$ as standard, relative weights $w_s$ for all the combinations of emulsion and slit widths used by Mayall are given in table VI.

TABLE VI

RELATIVE WEIGHTING FACTORS, $w_s$, FOR MAYALL'S VELOCITY MEASURES

| | Slit Width | | | | | |
|---|---|---|---|---|---|---|
| Emulsion | 4″ | 5″ | 6″ | 7″ | 8″ | 10″ |
| IIa0 | 1.00 | 0.65 | 0.46 | — | — | — |
| I E S | 0.56 | — | 0.47 | 0.43 | 0.39 | 0.32 |
| I 1200 | — | — | 0.43 | 0.41 | 0.39 | — |
| 103a0 | — | 0.39 | 0.38 | — | 0.35 | — |
| Agfa | — | — | 0.52 | — | 0.47 | — |
| Ilford | — | — | 0.43 | — | — | — |
| Ia0 | — | — | 0.33 | — | — | — |

Using these weights for means of Mayall's measures on each plate, Kruszewski determined $\sigma_p^2 = 2154$, $\sigma_p = \pm 46$ km/sec, from 134 spectra of 59 different objects, a result that showed no significant dependence on the emulsion used. The root mean square error of Mayall's unit weight $\sigma_M$ (one spectrum, IIa0 emulsion, 4″ slit and "lines weight" 1.0) is then given by

$$\sigma_M^2 = \sigma_m^2 + \sigma_p^2 = 9100(\text{km/sec})^2 \tag{54}$$

and the weight of a velocity determination is

$$W_M = \sum_j \frac{\sigma_m^2 + \sigma_p^2}{\sigma_m^2/w_s w_j + \sigma_p^2}, \tag{55}$$

where $w_s$ is the weighting factor for slit width and emulsion given in table VI, $w_j$ is the summed lines weight in Mayall's table V [8] for the $j$th spectrum, and the sum is taken over all the measured spectra of one object.

In the case of Humason's measures, it was assumed that his "estimated errors" $e$ in table I [8] are relatively correct. The differences, Mayall minus Humason in table VII [8] were analyzed, weighting Mayall's measures by $W_M$ and Humason's by $(100/e)^2 = W_H$. These differences show that $\sigma_H^2/\sigma_M^2 = 1.33$ ±40(r.m.s.), where $\sigma_H$ is the r.m.s. error of an observation for which $e = \pm 100$

km/sec. Equation (54) then gives $\sigma_H = 110$; that is, Humason's "estimated errors" are close to his actual r.m.s. errors as determined by the overlap between his and Mayall's measures.

Finally, as a check on the error determinations, the variance of 17 differences in measured velocities, Mayall-Page and Humason-Page (listed in table VII)

TABLE VII

COMPARISON WITH OTHER MEASUREMENTS

The calculated weight for $M$ = Mayall's observations is $\sigma_M = 95$ km/sec; for $H$ = Humason's observations, his "estimated error" is listed.
The objects NGC 3395, 3396, 4038, and 4039 were dropped from final analysis due to small separation.
The data for NGC 4485 are cited in table 2 of [1].

| NGC | Observer | Weight or Error | No. of Plates | Red-shift | Page Red-shift | Page Weight | Page Plates | P − O Difference |
|---|---|---|---|---|---|---|---|---|
| 2535 | $H$ | 75 | 1 | 4243 | 4073 | 3.4 | 3 | −170 |
| 3395 | $M$ | 1.9 | 1 | 1751 | 1632 | 6.7 | 4 | −119 |
| 3396 | $M$ | 2.3 | 1 | 1643 | 1639 | 6.7 | 4 | −4 |
| 3995 | $M$ | 2.5 | 1 | 3347 | 3369 | 4.2 | 3 | 22 |
| 4038 | $H$ | 75 | 1 | 1673 | 1669 | 4.9 | 4 | −4 |
| 4039 | $H$ | 50 | 2 | 1660 | 1625 | 4.9 | 4 | −35 |
| 4278 | $H$ | 40 | 2 | 624 | 700 | 0.6 | 1 | 76 |
| 4283 | $H$ | 65 | 1 | 1071 | 1150 | 0.2 | 1 | 79 |
| 4485 | $H$ | — | — | 625 | 568 | 1.0 | 1 | −57 |
| 4490 | $H$ | 50 | 2 | 625 | 723 | 1.0 | 1 | 98 |
| 4567 | $M$ | 1.9 | 1 | 2284 | 2087 | 1.8 | 3 | −197 |
| 4568 | $M$ | 1.8 | 1 | 2413 | 2069 | 1.8 | 3 | −344 |
| 4647 | $M$ | 2.0 | 1 | 1448 | 1340 | 0.4 | 1 | −108 |
| 4649 | $M$ | 1.4 | 1 | 1244 | 855 | 0.6 | 1 | −389 |
| 4649 | $H$ | 50 | 1 | 1389 | 855 | 0.6 | 1 | −534 |
| 5194 | $H$ | 35 | 2 | 438 | 466 | 1.5 | 1 | 28 |
| 5195 | $H$ | 35 | 3 | 542 | 550 | 1.5 | 1 | 8 |

were computed. Each squared difference $(V_M - V_P)^2$, was weighted by the factor $W_M W_P/(W_M + W_P\,\sigma_M^2/\sigma_P^2)$, where $W_P$ is the weight for $V$ from table V, and the factor $\sigma_P^2/\sigma_M^2 = 10800/9100 = 1.19$ adjusts the weights to the scale of $\sigma_P$. Similarly, each $(V_H - V_P)^2$ was weighted by $8900/(e^2 + 8900/W_P)$, where $e$ is Humason's estimated error, and $8900 = 10800/(1.1)^2$ adjusts the weights to the scale of $\sigma_P$. The variance of these differences, $\sum W(V - V_P)^2/(n-1)$, is close to $2\sigma_P$, confirming the consistency of these estimated errors.

The weights (relative to $\sigma_\Delta = 90$ km/sec) of each individual velocity determination by Mayall and Humason are combined to give the weight of each difference $\Delta V = V_2 - V_1$,

$$W_\Delta = \frac{W_1 W_2}{W_1 + W_2} \tag{56}$$

and in the cases of two determinations these weights were applied to obtain a weighted mean. The data for 42 pure pairs are presented in table III and for 24 multiple systems in table IV.

Only 52 of these 66 systems can be used in equations (35), (36), (47), and (48), however. In the other 14 systems the quantity $SV$ is so small that $a = 60SV/h \times 10^{-4}$ is less than the lower limit of $r$ in equation (2); that is, $(10^4/SV) + 0.19$ is greater than 4.75, and the projected separation $a$ is less than $0.028\, r_m = 1.3 \times 10^9$ a.u. = 6400 psc, which is smaller than the average diameter of a galaxy. Projected separations as small as this cannot include random $\phi$, $\psi$; the two galaxies must be one behind the other, on the average, or else one *inside* the other. The 14 systems eliminated from consideration because $a$ is too small ($SV < 2200$) are listed in the second part of table II.

## 7. Results for the circular-orbit model

The least squares solutions for $h\overline{M}$ and $\overline{M/hL}$, from equations (35) and (47), are given for all 52 systems in the first line of table VIII. The root mean square

TABLE VIII

AVERAGE MASS DETERMINATION, DOUBLE GALAXIES

| Set | $n$ | $\Sigma N$ | $\frac{h\overline{M}}{10^{10}}$ | $\frac{h\sigma_M}{10^{10}}$ | $\left(\frac{\overline{M}}{hL}\right)$ | $\frac{1}{h}\sigma_{M/L}$ |
|---|---|---|---|---|---|---|
| All systems | 52 | 116 | 30.8 | 10.5 | 12.2 | 12.2 |
| Pure pairs only | 33 | 66 | 26.3 | 14.2 | 8.7 | 14.4 |
| $W_\Delta > 0.5$ only | 41 | 90 | 28.3 | 8.9 | 11.2 | 8.3 |
| Spirals and Irr. only | 17 | 36 | 4.22 | 3.9 | 0.67 | 1.7 |
| Pure pairs only | 10 | 20 | 2.42 | 2.3 | 0.32 | 0.32 |
| $W_\Delta > 0.5$ only | 14 | 30 | 2.13 | 1.5 | 0.33 | 0.35 |
| Ellipticals and SO only | 18 | 37 | 66.0 | 27. | 101. | 72. |
| Pure pairs only | 13 | 27 | 63.5 | 38. | 97. | 98. |
| $W_\Delta > 0.5$ only | 13 | 27 | 59.3 | 15. | 94. | 38. |
| Mixed systems | 17 | 40 | 31.8 | 18. | 48.5 | 25. |
| Pure pairs only | 10 | 20 | 27.6 | 23. | 42.5 | 35. |
| $W_\Delta > 0.5$ only | 14 | 33 | 31.3 | 19. | 48.5 | 28. |
| Ellipticals, SO, and mixed ($W_\Delta > 0.5$ only) | 27 | 42 | 63.1 | 23. | | |

errors of these determinations, $\sigma_M$, from equation (36) and $\sigma_{\widehat{M/L}}$, from equation (48), are of the same order as the quantity determined, which is not surprising in view of the large observational errors in $\Delta V$ and the fact that, at each projected separation, $a = 60SV/h \times 10^{-4}$, the random orientation described by $\phi$ and $\psi$

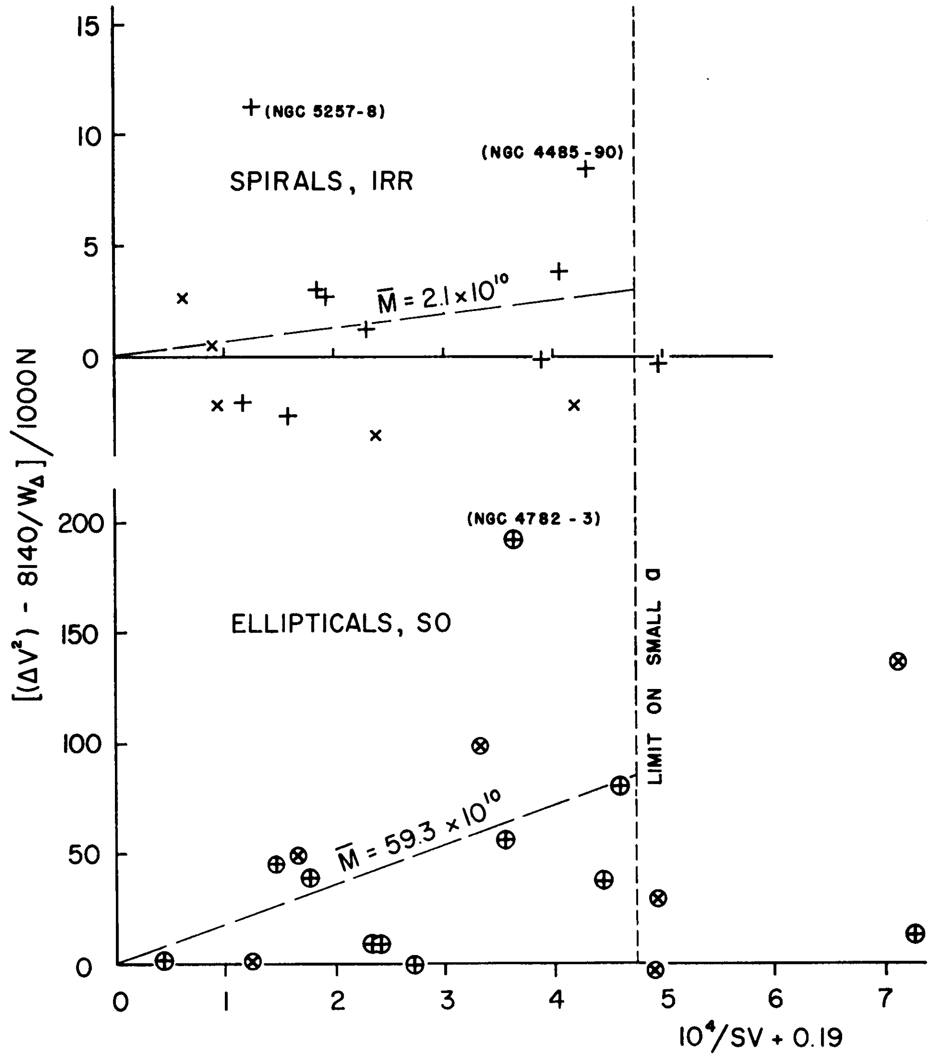


FIGURE 3

The regression $Y_i = \overline{M}A_i$ for spirals and Irr, and for ellipticals and SO. The crosses and circled crosses refer to pure pairs only; the $x$'s and circled $x$'s to multiple systems. The dashed lines represent the least squares solutions for all systems of weight $W_\Delta > 0.5$; points of lower weight are not shown. Values of $M$ are given in solar units. Points to the right of $(10^4/SV) + 0.19 = 4.75$, corresponding to the lower limit of the projected separation, $a$, are omitted from the least squares solution. Note the difference in scales of the ordinate.

will produce a spread in $\Delta V$ from $\Delta V = 0$ to $\Delta V = v$, its theoretical maximum. The evidence for dispersion in $M$ and $M/L$ will be discussed later.

The second line of table VIII gives the least squares solutions for 33 pure pairs only and the third line for 41 systems (including 13 multiple systems), excluding values for 11 systems of low weight, $W_\Delta$. The fact that these three solutions do not differ significantly is evidence that the multiple systems included can be treated as pairs; that is, equations (14) and (28) apply to a fair approximation. The exclusion of low-weight data reduces the uncertainty in $M$ and $M/L$ in a satisfactory way, showing that the estimates of $W_\Delta$ are reasonably consistent.

In the previous study [1] evidence was adduced for a bimodal distribution of $M$, with the "heavy weights" roughly 30 times as massive as "light weights." M. Schwarzschild [10] later noted that the systems containing elliptical and SO galaxies are the heavyweights; spiral and irregular galaxies, the lightweights. Dividing the present data, therefore, into three groups, solutions were made for $hM$ and $M/hL$ for (a) 17 systems containing spirals and Irr. types only, (b) 18 systems containing elliptical and SO types only, and (c) 17 mixed systems.

The present results, given in table VIII, clearly confirm the previous conclusion; using the 14 observations of weight, $W_\Delta > 0.5$, $h\overline{M}_S = (2.1 \pm 1.5) \times 10^{10}$ and (13 observations) $h\overline{M}_E = (59.3 \pm 15.) \times 10^{10}$, about 30 times larger. The regressions are plotted separately in figure 3. The fact that both $\sigma_{\hat{M}}/\overline{M}$ and $\sigma_{\widehat{M/L}}/(M/L)$ remain about the same justifies, to some extent, this division of the observational material. Further justification is found when the 27 ellipticals, SO, and mixed systems are treated as if the only mass were that of elliptical and SO types; that is, by replacing $N\overline{M}$ by $N_E\overline{M}_E$ (neglecting $\overline{M}_S$) in equation (38) and solving for $h\overline{M}_E$. The result given in the last line of table VIII corroborates $h\overline{M}_E = 60 \times 10^{10}$, whereas the mean $h\overline{M}$ (and $\overline{M/hL}$) for the mixed systems was significantly smaller, between the means $h\overline{M}_S$ and $h\overline{M}_E$ ($\overline{M_S/hL_S}$ and $\overline{M_E/hL_E}$). If $\overline{M}_S$ were included as $\overline{M}_E/30$ in this treatment of mixed pairs, the agreement would be even closer.

The mean ratios, $\overline{M_S/hL_S}$ and $\overline{M_E/hL_E}$, show an even greater difference, the former being considerably *lower* than expected for spirals with nuclei of population II stars, or even for irregulars with population I stars only, and the latter being *higher* than expected for systems of stars of population II without an admixture of nonluminous matter. An adjustment of $h$ will of course change both determinations in the *same* direction; however, these results may indicate that $h$ must be increased to 2 ($H = 200$ km/sec/megaparsec), and that the elliptical galaxies contain a large portion of nonluminous matter (5 or 10 times the mass of luminous stars).

The regressions of $Z_i$ on $C_i$ are plotted in figure 4 for spirals and ellipticals separately. Although, in each case, one or two points seem to determine the solution, the weighting factor $W_\Delta/C_i$ in fact reduces the effect, and solutions omitting these points are changed only to $\overline{M/hL} = 0.6 \pm 0.6$ for spirals, $93 \pm 45$

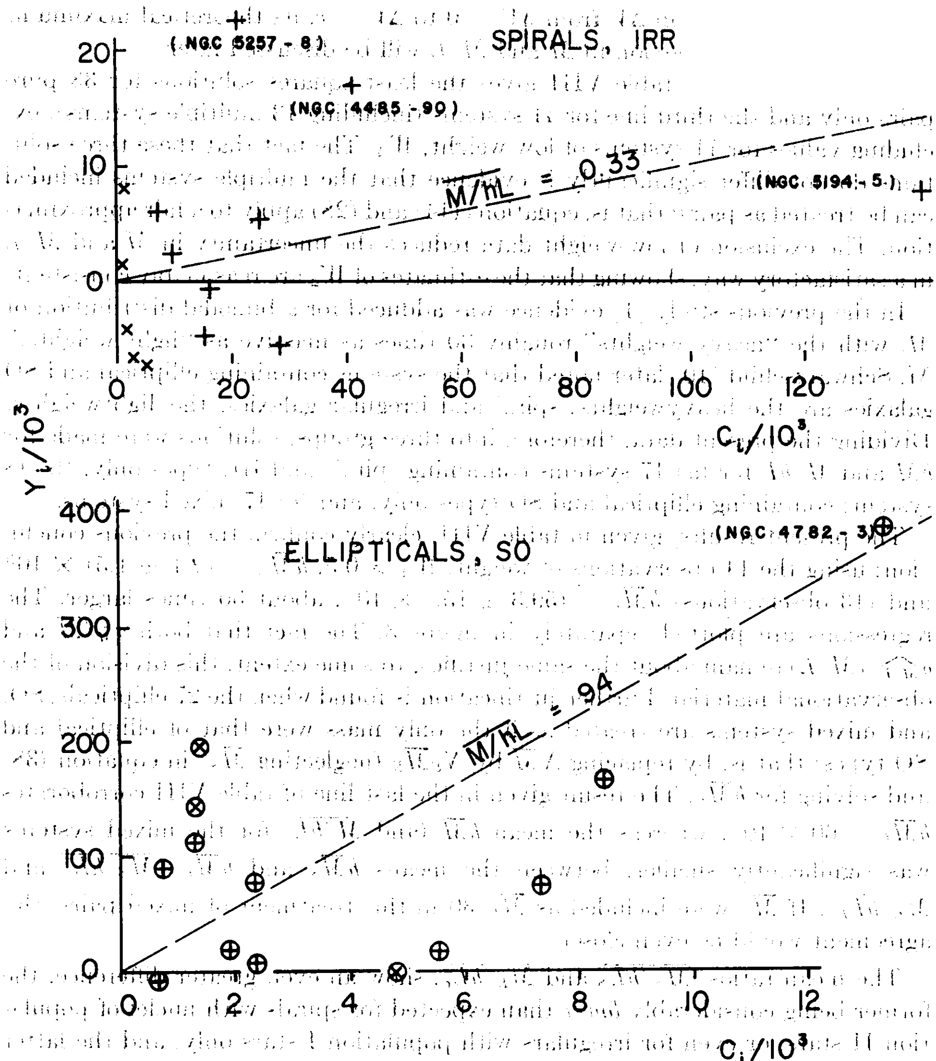


FIGURE 4

The regression $Y_i = (\overline{M/\ })C_i$ for spirals and Irr, and for ellipticals and SO. The crosses and circled crosses refer to pure pairs only; the $x$'s and circled $x$'s to multiple systems. The dashed lines represent the least squares solutions, omitting systems of weight $W_\Delta < 0.5$ and $(10^4/SV) + 0.19 > 4.75$ (not plotted). Note the difference in scales.

for ellipticals. The negative values of $Y_i$ (and of $Y_i/N_i$ in figure 3) are of course due to overcorrection of the bias term $\sigma_\Delta^2/W_\Delta$ in equations (31) and (43).

Since ellipticals and spirals differ so widely in mass, any investigation of the inherent variance in $M$ must consider the two classes separately, which reduces the sample sizes to 14 systems of spirals and 13 systems of ellipticals, too small to justify the application of the complex formula for $\sigma_M$. (A trial with the present data yielded slightly negative values of $\sigma_M$ for the 14 systems of spirals and the 13 systems of ellipticals.)

## 8. Validity of the results and alternative interpretations

The four major assumptions underlying the circular-orbit model are

(a) tidal effects are negligible in the dynamical system of two galaxies moving in closed orbits;

(b) there is no correlation between the mass of a system $N\overline{M}$ and its physical separation $r$;

(c) the density of intergalactic matter is zero;

(d) the orbits of double galaxies are closed and circular.

In addition, the significance of the results depend on whether the double galaxies—more properly, this particular set of double galaxies—are representative of field galaxies, or average galaxies in the observable universe. A detailed discussion of selection is beyond the scope of this paper, but it is clear that the galaxies for which spectra are available are selected for high surface brightness. This is reflected in the fact that Irr. types of low surface brightness are almost lacking from the sample (table IX). In general, however, Holmberg's catalogue

TABLE IX

TYPES OF GALAXIES IN THE SAMPLE

Excluding 14 systems with too small projected separation

| Set | $E$ | SO | $S$ | Irr. | Unknown | Total |
|---|---|---|---|---|---|---|
| All | 41 | 20 | 51 | 2 | 2 | 116 |
| Pure pairs | 24 | 12 | 28 | 1 | 1 | 66 |
| $W_\Delta > 0.5$ | 29 | 15 | 43 | 1 | 2 | 90 |

[7] shows that the double galaxies appear to be fairly similar to the field galaxies in their morphological types and sizes. The effect of selecting brighter than average pairs probably makes $\hat{M}$ too high an estimate.

Tidal effects, while undoubtedly present, probably cannot account for a major error in $\overline{M}$. A closely associated effect, probably more serious, would seem to be spurious orbital velocities, $\Delta V$, introduced by large internal motions in each single galaxy.

Depending on the mechanism of formation of the double galaxies, the possibility exists that $M$ depends on $r$, in which case equation (16) does not hold, and equation (18) includes $p_{M|r}$. For instance, it might be that the process of formation of the double galaxies results in larger masses at larger separations so that, for a given $M$, we have $E[(\Delta V_o)^2]$ decreasing less rapidly with $r$ than predicted by equation (28), and $\hat{M}$ is an *underestimate.*

Closely associated with this correlation is the case of an intergalactic density $\rho_I$ such that the mass included within an orbit of radius $r$ is

$$M_r = \frac{4}{3}\pi\rho_I r^3 + N\overline{M}. \tag{57}$$

The effect of this correlation would also be to increase $\Delta V_0$ for large $r$, given $M$, or to reduce the slope of the regression $Y_i$ on $A_i$, equation (30), yielding too low an estimate of $\hat{M}$. This might possibly account for the low value of $\hat{M}_S$.

If the density $\rho_I$ is uniform, equation (14) is changed by substituting equation (57), and equation (28) becomes

$$E\{(\Delta V)^2|a\} - \frac{\sigma_\Delta^2}{W_\Delta} = (9.48\pi)^2\left[\frac{N\overline{M}}{2}\frac{I_1}{aI_0} + \frac{2\pi}{3}\rho_I a^2\right] \tag{58}$$

or

$$Y_i = A_i M + (9.48\pi)^2\frac{2\pi}{3}\rho_I a^2. \tag{59}$$

The slope of this curve near $A_i = 1.2 \times 10^{-7}$, or $a = 3 \times 10^9/h$ a.u. has been called $\hat{M}$. Therefore, using the approximation of equation (29),

$$\begin{aligned}\frac{dY}{dA} = \hat{M} &= \overline{M} - \frac{4}{3}\frac{\pi\rho_1 a^3}{0.4N/2}\\ &= \overline{M} - 10.5\rho_I a^3, \qquad N = 2.\end{aligned} \tag{60}$$

Thus, if the true average mass of a galaxy $\overline{M}$ differs from the estimate $\hat{M}$ (in suns), the intergalactic density must be

$$\begin{aligned}\rho_I &\doteq \frac{\overline{M} - \hat{M}}{10.5a^3}\ \text{suns}/(\text{a.u.})^3\\ &= 2.1 \times 10^{-36}\, h^3(\overline{M} - \hat{M})\text{gm/cm}^3,\end{aligned} \tag{61}$$

and any significant correction to $\hat{M}$ (by, say, $10^{10}$ suns) would require an intergalactic density of $10^{-26}$gm/cm$^3$ or more. And if the spirals were not really of lesser mass than the ellipticals, $\overline{M} - \hat{M} \geqq (60 \mp 2) \times 10^{10}/h$ and $\rho_I \doteq 1.2 \times 10^{-24}$gm/cm$^3$, a density comparable to the internal densities of spirals.

It seems highly unlikely that $\rho_I$ can change $\overline{M}$ by more than a factor of two.

The assumption of closed, circular orbits has already been discussed in section 2. However, in addition to the possibility that they are in hyperbolic orbits due to chance passages, there is the possibility, suggested by the work of Ambartzumian [5] and the Burbidges [6], that the double galaxies have positive energy and are flying apart due to some unspecified "explosion." It is difficult to imagine a mechanism whereby this could occur. The energy required to accelerate $10^{10}$ suns $= 2 \times 10^{43}$ gm to a speed of 100 km/sec is $10^{59}$ ergs—more than that available from the complete conversion of an average star's mass to energy. Moreover, there are two further difficulties: how a galaxy can be "pushed," and how the push can be directed so as to split a protogalaxy into two parts. However, it may be worthwhile to examine the consequences of the extreme case, *radial motion*, as such consequences might be used to interpret the motions of double galaxies.

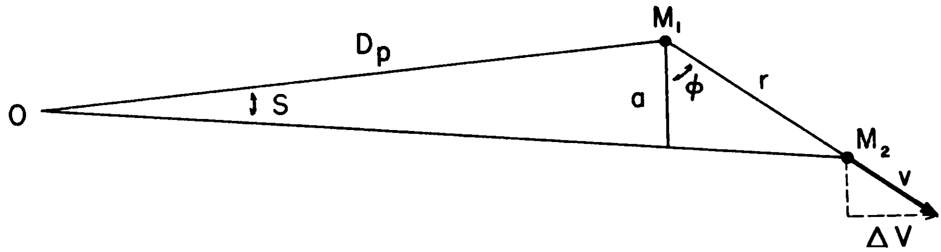


FIGURE 5

Radial-motion model.
The galaxies $M_1$ and $M_2$, separated by $r$ astronomical units, are seen from 0 at distance $D_p$ parsecs. The relative velocity, $v$, is parallel to $r$ in this case.

The geometry of this radial-motion model is illustrated in figure 5. As before,

$$a = r\cos\phi \tag{62}$$

but, in contrast to equation (13),

$$V = 4.74v\sin\phi \tag{63}$$

and, in contrast to equation (14),

$$v^2 = 4\pi^2\left(\frac{2NM}{r} + D\right), \tag{64}$$

where $D$, related to the total energy per unit mass, is greater than zero if the double galaxies are unstable, and may have a spectrum of values if explosions of various sizes account for their origin. Equation (20) takes the form

$$E\{(\Delta V_0)^2|a, M\} = (9.48\pi)^2\left[2NM\frac{J_{12}(a)}{aI_0(a)} + D\frac{J_{02}}{I_0(a)}\right], \tag{65}$$

in which Holmberg's distribution, equation (2), has been used, assuming that $p_{r,m}(r, M) = p_r(r)$; that is, that $p_r(r)$ is independent of $NM$, and

$$\frac{J_{12}(a)}{K} = \frac{1}{3}\sin^3\alpha - \sin\alpha\cos^2\alpha - \alpha\cos^3\alpha, \tag{66}$$

$$\frac{J_{02}(a)}{K} = \frac{1}{2}\alpha - \frac{5}{2}\sin\alpha\cos\alpha - \cos^3\alpha\log\frac{1+\sin\alpha}{\cos\alpha}, \tag{67}$$

where $I_0$ is given by equation (21) as before, and $\cos\alpha = a/r_m$. Over the range $0.03 \leqq a/r_m \leqq 0.3$ (in which most of the observations lie), the quantity $J_{12}/I_0$

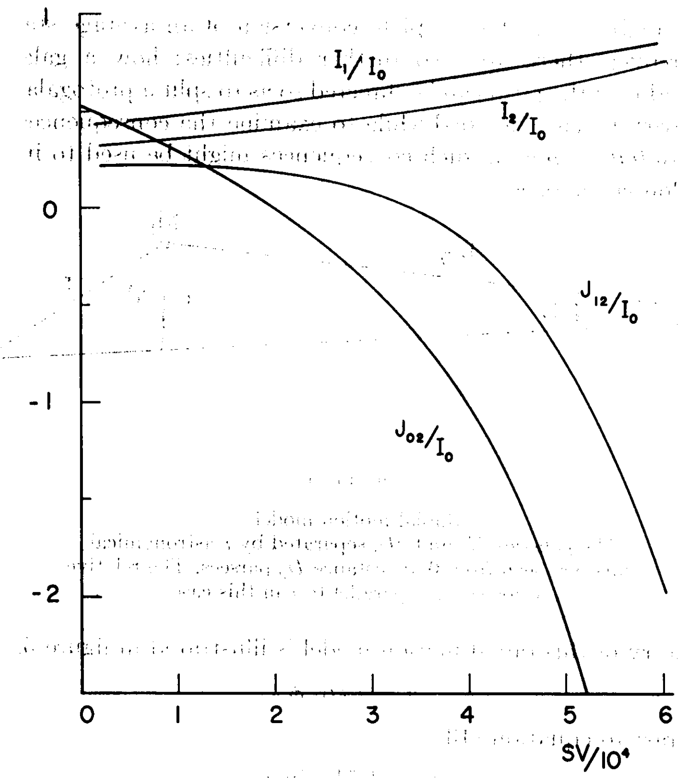


FIGURE 6

Geometrical integrals
[see equations (21) to (29) and (65) to (67)].

is practically constant, as shown by the following values (see, also, figure 6):

| | | | | | |
|---|---|---|---|---|---|
| $\cos\alpha \doteq a/r_m =$ | 0.05 | 0.10 | 0.25 | 0.50 | 0.75 |
| $J_{12}(a)/I_0(a) =$ | 0.2202 | 0.2233 | 0.1875 | −.1749 | −1.8914 |
| $J_{02}(a)/J_0(a) =$ | 0.4246 | 0.3407 | 0.0183 | −.9668 | −3.9566 |

Assuming that the total energy is near 0, then $\bar{D} \doteq 0$ and equation (65) becomes

$$E\{(\Delta V)^2|a\} - \frac{\sigma_\Delta}{W_\Delta} \doteq E_R\{M\}(9.48\pi)^2\, 2N\, \frac{0.2}{a}, \tag{68}$$

which is very similar to equation (28) for the circular-orbit case, using the approximate equation (29), but with

$$E_R\{M\} = \frac{1}{2} E\{M\}. \tag{69}$$

That is, as might be expected, the assumption of radial parabolic motion would approximately halve the mass estimates given in table VIII.

Of course, $\bar{D}$ may differ from 0 or have a distribution depending on $M$, in which case a more elaborate analysis may allow a test for circular versus radial motion.

## 9. Summary

In summary, the available observations of 52 systems of double galaxies, analyzed statistically on the assumption that they may be represented as point masses moving in randomly oriented circular orbits, yield estimates of the average mass and mass-luminosity ration of a galaxy given in table VIII. These estimates, although subject to large errors, are consistent among themselves, but show a surprisingly large difference between the group of 14 systems of spirals and the group of 13 systems of ellipticals. The subsequent discussion is intended to show that the estimates are not likely to be radically changed by consideration of tidal effects or intergalactic material, and that a statistical test of the assumed circular orbits may be possible.

It is a pleasure to acknowledge the helpful advice of Professor Jerzy Neyman.

◇ ◇ ◇ ◇ ◇

## APPENDIX. FORMULA FOR THE INHERENT VARIANCE IN $M$

The average mass of a galaxy, assuming the circular-orbit model, is determined from equation (30),

$$Y_i = \bar{M}A_i(a), \tag{70}$$

where $Y_i$ and $a$ are observables, $a$ is very nearly exact, $Y_i = (\Delta V_i)^2 - \sigma_\Delta^2/W_\Delta$, and $\Delta V$ is a normal random variable with mean $\Delta V_0$ and variance $\sigma_\Delta^2/W_\Delta$. The weight $W_\Delta$ refers to an observation of $\Delta V$; the weights of the observation equations above are, from equation (34),

$$w_i = \frac{W_\Delta}{N_i A_i(a)}. \tag{71}$$

The function $A_i(a)$ is given in equation (32) and the least squares solution for $\bar{M}$ is

$$\hat{M} = \frac{\sum w_i A_i Y_i}{\sum w_i A_i^2} \tag{72}$$

with mean square error of the estimate given by equation (36).

The inherent variance of $M$, derived from equation (39), is

$$\sigma_M^2 = \frac{S_0^2 - Q(\hat{M})^2 - 4\sigma \hat{M}_\Delta^2 R - 2\sigma_\Delta^4 T}{P} \tag{73}$$

where $S_0^2$ is the sum of least squares and

$$
\begin{aligned}
Q &= \frac{C^2F}{C^2 + D}, \\
C &= \sum w_i A_i^2, \\
D &= \sum w_i^2 A_i^2 (B_i - A_i^2), \\
B_i &= (9.48\pi)^2 \frac{3N^2}{8} \frac{I_2}{a^2 I_0}, \\
F &= \sum w_i \left[1 - \frac{w_i A_i^2}{\sum w_i A_i^2}\right] (B_i - A_i^2), \\
R &= \frac{\sum w_i A_i}{W_\Delta} - 1 - \frac{F \sum w_i^2 A_i^3}{W_\Delta (C^2 + D)}, \\
T &= \sum \left(\frac{w_i}{W_\Delta^2} - \frac{w_i^2 A_i^2}{C W_\Delta^2}\right) - \frac{F}{C^2 + D} \sum \frac{w_i^2 A_i^2}{W_\Delta^2}, \\
P &= \sum \frac{w_i B_i (C^2 - \sum w_i^2 A_i^4)}{C^2 + D},
\end{aligned} \tag{74}
$$

and all summations are from $i = 1$ to $n$, the number of observation equations.

## REFERENCES


[1] T. PAGE, "Radial velocities and masses of double galaxies," *Astrophys. J.*, Vol. 116 (1952), pp. 63–80.

[2] E. HOLMBERG, "On the masses of double galaxies," *Medd. Lunds Astr. Obs., Ser. I*, No. 186 (1954), pp. 1–20.

[3] N. U. MAYALL and L. H. ALLER, "The rotation of the spiral nebula Messier 33," *Astrophys. J.*, Vol. 95 (1942), pp. 5–23.

[4] S. SMITH, "The mass of the Virgo cluster," *Astrophys. J.*, Vol. 83 (1936), pp. 23–30.

[5] V. A. AMBARTZUMIAN, "On the evolution of galaxies," *Armenian Acad. Sci., Math.-Phys. Ser.*, Vol. 11 (1957), pp. 1–20.

[6] G. R. BURBIDGE and E. M. BURBIDGE, "The Hercules cluster of nebulae," *Astrophys. J.*, Vol. 130 (1959), pp. 629–640.

[7] E. HOLMBERG, "A study of double and multiple galaxies," *Ann. Obs. Lund.*, No. 6 (1937), pp. 1–173.

[8] M. L. HUMASON, N. U. MAYALL, and A. R. SANDAGE, "Redshifts and magnitudes of extragalactic nebulae," *Astr. J.*, Vol. 61 (1956), pp. 97–162.

[9] R. MINKOWSKI, "The luminosity function of extragalactic radio sources," *Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability*, Berkeley and Los Angeles, University of California Press, 1961, Vol. 3, pp. 245–259.

[10] M. SCHWARZSCHILD, "Mass distribution and mass-luminosity ratio in galaxies," *Astr. J.*, Vol. 59 (1954), pp. 273–284.

[11] E. HOLMBERG, "A photographic photometry of extragalactic nebulae," *Medd. Lunds Astr. Obs., Ser. II*, No. 136 (1958), 100 pp.

[12] J. L. E. DREYER, "A new and general catalogue of nebulae and clusters of stars," *Mem. Roy. Astr. Soc.*, Vol. 49 (1888), pp. 1–237.