ON THE STATISTICAL LOSS OF LONG-PERIOD COMETS FROM THE SOLAR SYSTEM. II

J. M. HAMMERSLEY
OXFORD UNIVERSITY

1. Introduction and summary

This paper deals with the distribution of the lifetimes of comets, and the way in which this distribution of lifetimes affects the total population of observable comets regarded as a function of the age of the solar system.

The energy per unit mass of a comet is $-\gamma M/2a$, where γ is the constant of gravitation, M is the mass of the sun, and a is the semimajor axis of the comet's elliptical orbit. In the present work it is convenient to change the sign of this energy, and to work with a quantity $z = c/a$, where c is a positive constant chosen to simplify the notation. The first part of this paper, by R. A. Lyttleton, explained how the value of z is perturbed by Jupiter each time the comet visits the neighborhood of the sun and planets, and how, when successive perturbations eventually lead to a negative or zero value of z, the comet is lost from the solar system along a hyperbolic or parabolic orbit. The lifetime of a comet, brought into the solar system with an energy $z_0 = x$ and having values of z equal to $z_0, z_1, \cdots, z_{T-1}$ at successive orbits up to the moment of loss, is therefore

$$G(x) = V(z_0) + V(z_1) + \cdots + V(z_{T-1}), \tag{1.1}$$

where $V(z)$ is the time taken to describe an orbit with energy value z. Kepler's third law states that $V(z)$ is proportional to $z^{-3/2}$, and this relation is used in various parts of the work. But in other parts of the paper it causes little extra work to treat an arbitrary nonnegative function $V(z)$. One possible advantage of this extra generality is that it will still permit the theory to be applied if, for example, it should turn out that the influence of stellar perturbations upon comets with very long periods can be approximated by some modification of Kepler's third law. Apart from this possibility, however, no account is taken in the present work of the influence of stellar perturbations.

Section 3 of the paper provides the theory for the distribution of the lifetime $G(x)$, when the distribution of successive perturbations $y_t = z_t - z_{t-1}$ has a given arbitrary form $P(y)$. We permit $P(y)$ to be improper at $-\infty$, and thereby allow for losses of comets due to disintegration. Section 2 provides the analogous theory when the perturbations are replaced by a continuous process, namely

Brownian motion, and the sum in (1.1) is replaced by an integral. Effectively, sections 2 and 3 deal with generalized first-passage theory, and may be read independently of the present application to cometary lifetimes. Indeed the theory has other applications which are briefly mentioned.

Section 4 gives the numerical results on cometary lifetimes obtained by applying sections 2 and 3, and also by utilizing Monte Carlo methods.

Section 5 studies the birth and death process associated with the population of comets, and shows that the size of this population should vary as the cube root of the age of the solar system. One of the main interests of this approach is that it enables one to make an estimate of the total size of this population, which does not depend critically (as other estimates do) upon correctly specifying the period of a typical long-period comet. It appears that the total number of comets at present in the solar system may be somewhere between half a million and forty million, though these figures would probably be reduced by a factor of two or more if losses by stellar perturbations are important.

2. Generalized first-passage theory for Brownian paths

2.1. *Statement of results.* Let $t \geqq 0$ be a time parameter, and let $\Omega = \{\omega\}$ be the space of all real linear separable Brownian paths $X(t, \omega)$ having the normalized form

$$X(0, \omega) = 0, \qquad \int_{\Omega} [X(1, \omega)]^2 \, d\mu(\omega) = 1, \tag{2.1}$$

where $\mu(\omega)$ is Wiener measure on the subsets of Ω. For $x > 0$, let $T(x, \omega)$ be the supremum of all T' such that $x + X(t, \omega) > 0$ whenever $0 \leqq t \leqq T'$. In all that follows we shall tacitly suppose that $X(t, \omega)$ is a continuous function of t and that $T(x, \omega)$ is finite; this is justified inasmuch as the set of ω, for which these suppositions are invalid, has zero μ-measure.

For $z \geqq 0$, let $V(z)$ be any nonnegative Borel-measurable function. We work throughout with a real number system extended to include the value $+\infty$, so that $V(z)$ and other subsequent quantities may be $+\infty$. We shall prove that

$$G(x, \omega) = \int_0^{T(x,\omega)} V[x + X(t, \omega)] \, dt \tag{2.2}$$

exists as a (possibly infinite) random variable, that is, that it is μ-measurable. We shall study its moment-generating function

$$\phi(x, u) = \int_{\Omega} e^{-uG(x,\omega)} \, d\mu(\omega), \qquad u \geqq 0. \tag{2.3}$$

In the particular case when

$$V(z) = z^{\rho-2}, \tag{2.4}$$

where ρ is a positive constant, we shall prove that $2x^{\rho}/\rho^2 G(x, \omega)$ is distributed as a gamma-variate with parameter $1/\rho$; in other words,

$$P\{G(x, \omega) \leqq g\} = \frac{1}{\Gamma(1/\rho)} \int_{2x^\rho/\rho^2 g}^{\infty} v^{1/\rho - 1} e^{-v} \, dv. \tag{2.5}$$

In the special case $\rho = 2$, (2.5) reduces to the well-known first-passage distribution ([1], theorem 42.6)

$$P\{T(x, \omega) \leqq g\} = \pi^{-1/2} \int_{x^2/2g}^{\infty} v^{-1/2} e^{-v} \, dv. \tag{2.6}$$

In the special case $\rho = 1/2$, (2.5) reduces to

$$P\{G(x, \omega) \leqq g\} = \left(1 + \frac{8\sqrt{x}}{g}\right) \exp\left(-\frac{8\sqrt{x}}{g}\right); \tag{2.7}$$

and this equation is relevant to our study of comets.

The general problem, in which (2.4) is not assumed, bears a resemblance to Feynman's treatment of quantum mechanics, where one studies

$$\begin{aligned} Q(x, \tau) &= \lim_{h \to 0} h^{-1} \int\limits_{0 \leqq x + X(\tau, \omega) \leqq h} \exp\left\{-\int_0^\tau V[x + X(t, \omega)] \, dt\right\} d\mu(\omega) \\ &= \lim_{h \to 0} h^{-1} \int\limits_{0 \leqq x + X(\tau, \omega) \leqq h} \exp\left\{-\int_0^\tau \left[\frac{1}{2}\dot{X}^2 + V(x + X)\right] dt\right\} d(\text{path}). \end{aligned} \tag{2.8}$$

For a discussion of this see [2], pp. 165–175. The only essential difference between (2.3) and (2.8) is that in the former we terminate the Brownian path as soon as it reaches the origin, whereas in the latter we terminate it at the origin at a prescribed time τ. In quantum mechanics V is a given potential function; and consequently (2.3) may be relevant in quantum mechanical problems with an absorbing potential barrier. The generalization of (2.3) to Brownian paths in two and three dimensions would be of interest in connection with the loss of particles from thermonuclear plasmas in magnetic mirror machines; but we do not consider this here.

If $M_n(x)$ is the nth moment of G about the origin, namely

$$M_n(x) = \int\limits_{\Omega} [G(x, \omega)]^n \, d\mu(\omega), \qquad M_0(x) = 1, \tag{2.9}$$

then

$$M_n(x) = 2n \int_0^\infty \min(x, z) V(z) M_{n-1}(z) \, dz, \qquad n = 1, 2, \cdots, \tag{2.10}$$

in the sense that both sides of (2.10) are finite or infinite together. In the particular case (2.4)

$$\begin{aligned} M_n(x) &= \left(\frac{2x^\rho}{\rho^2}\right)^n \frac{\Gamma\left(\frac{1}{\rho} - n\right)}{\Gamma(1/\rho)}, \qquad & 0 \leqq n < 1/\rho; \\ M_n(x) &= \infty, & n \geqq 1/\rho. \end{aligned} \tag{2.11}$$

Equation (2.11) follows directly from (2.5), or, if n is an integer, by repetition of (2.10).

In stating the properties of $\phi(x, u)$, we assume $u > 0$, since $\phi(x, 0) = 1$

trivially. These properties depend upon a number ξ, defined as the supremum of all η such that $\int_0^\eta zV(z)\,dz$ is finite. If $0 \leqq \xi \leqq x < \infty$, then $\phi(x, u) = 0$ and $G(x, \omega)$ is infinite with probability 1. If $0 < x < \xi < \infty$, then

$$\lim_{x\to 0} \phi(x, u) = 1, \qquad \lim_{x\to \xi} \phi(x, u) = 0, \tag{2.12}$$

and $G(x, \omega)$ is infinite with probability x/ξ. If $\xi = \infty$, then

$$\lim_{x\to 0} \phi(x, u) = 1, \qquad \lim_{x\to \infty} \frac{\partial\phi(x, u)}{\partial x} = 0, \tag{2.13}$$

and $G(x, \omega)$ is finite with probability 1. If $0 < x < \xi \leqq 0$, then $\partial\phi(x, u)/\partial x$ exists and is a continuous function of x, and

$$\frac{\partial^2\phi(x, u)}{\partial x^2} = 2u\phi(x, u)\,\frac{\partial}{\partial x}\int_c^x V(z)\,dz, \tag{2.14}$$

where c is any constant satisfying $0 < c < \xi$. A familiar issue in pure mathematics is the existence of the solution of a given differential equation. Here we have the harder task of discussing the existence of the equation itself, inasmuch as we have to ask whether ϕ is differentiable. Equation (2.14) is to be understood in the sense that, if either side of (2.14) exists, then the other side exists and the two sides are equal. Notice that the right side of (2.14) exists for almost all x in $0 < x < \xi$, and that the set of x for which (2.14) holds is independent of u. The boundary conditions for the differential equation (2.14) are (2.12) or (2.13) according as $\xi < \infty$ or $\xi = \infty$. In the particular case (2.4)

$$\phi(x, u) = \frac{\pi}{\Gamma(1/\rho)}\left[\frac{(8ux^\rho)^{1/2}}{2\rho}\right]^{1/\rho} \mathrm{Kh}_{1/\rho}\left[\frac{(8ux^\rho)^{1/2}}{\rho}\right], \tag{2.15}$$

where $\mathrm{Kh}_{1/\rho}(z)$ is the Bessel function

$$\mathrm{Kh}_{1/\rho}(z) = \pi^{-1}\left(\frac{1}{2}z\right)^{-1/\rho}\int_0^\infty v^{1/\rho-1}\exp\left(-v - \frac{z^2}{4v}\right)dv. \tag{2.16}$$

2.2. *Proof of results.* We shall always assume that u and x are positive and finite, and that $V(z)$ and $V_n(z)$ are nonnegative Borel-measurable functions for $z \geqq 0$.

The continuity of $X(t, \omega)$ as a function of t ensures that $V[x + X(t, \omega)]$ is a nonnegative Borel-measurable function of t for $0 \leqq t \leqq T(x, \omega)$; and hence (2.2) guarantees the existence of $G(x, \omega) \leqq \infty$ for each fixed ω.

Suppose for the moment that $V(z)$ is continuous for $z \geqq 0$. Let l and m be positive integers. For arbitrary $M \geqq 0$, let $\Omega_{lm}(M)$ be the set of ω such that

$$\begin{aligned} & x + X\left(\frac{k}{m}, \omega\right) > 0, \qquad k = 1, 2, \cdots, l; \\ & x + X\left(\frac{l+1}{m}, \omega\right) \leqq 0; \\ & \frac{1}{m}\sum_{k=1}^{l} V\left[x + X\left(\frac{k}{m}, \omega\right)\right] \leqq M. \end{aligned} \tag{2.17}$$

The continuity of V and the separability of X ensure that $\Omega_{lm}(M)$ is μ-measurable. Hence $\limsup_{m\to\infty} \sum_{l=1}^{\infty} \Omega_{lm}(M)$ is also μ-measurable. But the latter set is, by virtue of the continuity of V and X, the set of ω such that $G(x, \omega) \leqq M$. Thus G is μ-measurable when V is continuous.

Now let V be arbitrary; and let V_n be a sequence of functions satisfying

$$V(z) = \lim_{n\to\infty} V_n(z), \qquad z \geqq 0, \tag{2.18}$$

and such that either (a) V_n is bounded uniformly in n, or (b) $V_{n+1} \geqq V_n$ for each n and all $z \geqq 0$. We assert that

$$G_n(x, \omega) = \int_0^{T(x,\omega)} V_n[x + X(t, \omega)]\, dt \to G(x, \omega) \quad \text{as} \quad n \to \infty. \tag{2.19}$$

In case (a), (2.19) is the consequence of Lebesgue's bounded convergence theorem together with the finiteness of $T(x, \omega)$. In case (b), (2.19) follows from [3], theorem 27.B. From (2.19) and Lebesgue's bounded convergence theorem,

$$\phi_n(x, u) = \int_\Omega e^{-uG_n(x,\omega)}\, d\mu(\omega) \to \phi(x, u) \quad \text{as} \quad n \to \infty. \tag{2.20}$$

Since (2.19) follows from (2.18) in the cases stated, the class of functions V for which G is μ-measurable is closed under pointwise bounded convergence and under pointwise nondecreasing convergence, and it also contains all continuous functions. Hence ([4], pp. 168–170) it contains the Borel-measurable functions.

We shall prove presently that

$$\phi(x, u) + 2u \int_0^\infty \min(x, z)\phi(z, u)V(z)\, dz = 1, \tag{2.21}$$

for all functions $V(z)$ which are continuous for $z \geqq 0$. For the moment, however, we consider the consequences of (2.21).

Consider an arbitrary η, where $0 < \eta < \infty$, and let C_η denote the class of all V such that

$$\int_0^\eta zV(z)dz < \infty. \tag{2.22}$$

Clearly all continuous functions belong to C_η. They also belong to C_η^*, the subclass of C_η for which

$$\phi(y, u) + 2u \int_0^\eta \left\{\min\left(1, \frac{y}{z}\right) - \frac{y}{\eta}\right\} \phi(z, u)zV(z)\, dz \tag{2.23}$$

and

$$= 1 - \frac{y}{\eta}\{1 - \phi(\eta, u)\}, \qquad 0 < y \leqq \eta,$$

because (2.23) is the result of eliminating $\int_\eta^\infty \phi(z, u)V(z)\, dz$ from the pair of equations obtained by putting $x = y$ and $x = \eta$ in (2.21). Suppose that (2.18) holds, and that one or other of the cases (a) and (b) mentioned above is fulfilled. Then (2.20) holds. Suppose that V_n belongs to C_η^* and that V belongs to C_η. Then (2.23) holds with ϕ_n and V_n in place of ϕ and V; and the integrand will either be bounded uniformly in n in case (a), or will be bounded by $zV(z)$ in case (b), since

$$0 \leqq \left\{\min\left(1, \frac{y}{z}\right) - \frac{y}{\eta}\right\} \phi_n(z, u) \leqq 1. \tag{2.24}$$

In either case, we may invoke Lebesgue's bounded convergence theorem, since (2.22) holds; and, when $n \to \infty$, we deduce (2.23). Thus V belongs to C_η^*. Using the closure argument already employed in the proof of the μ-measurability of G, we conclude that $C_\eta^* = C_\eta$. Hence (2.22) implies (2.23).

We shall now prove that $\phi(x, u)$ is a nonincreasing function of x, and is therefore continuous for almost all x. Let $x > z > 0$. Since $X(t, \omega)$ is continuous in t, there exists a smallest root $\tau = \tau(x, z, \omega)$ to the equation $x + X(\tau, \omega) = z$; and this root satisfies $0 < \tau < T(x, \omega)$. Moreover, the identity

$$x + X(t + \tau, \omega) = x + X(\tau, \omega) + X(t, \omega^*) = z + X(t, \omega^*), \qquad t \geqq 0, \tag{2.25}$$

is a measure-preserving transformation from ω to ω^*. Hence

$$\begin{aligned} G(z, \omega^*) &= \int_0^{T(z,\omega^*)} V[z + X(t, \omega^*)]\, dt = \int_\tau^{T(x,\omega)} V[x + X(t, \omega)]\, dt \\ &\leqq \int_0^{T(x,\omega)} V[x + X(t, \omega)] dt = G(x, \omega); \end{aligned} \tag{2.26}$$

and

$$\begin{aligned} \phi(z, u) &= \int_\Omega e^{-uG(z,\omega^*)}\, d\mu(\omega^*) = \int_\Omega e^{-uG(z,\omega^*)}\, d\mu(\omega) \\ &\geqq \int_\Omega e^{-uG(x,\omega)}\, d\mu(\omega) = \phi(x, u), \end{aligned} \tag{2.27}$$

as required.

We now embark on the proof of (2.21); and, until further notice, we shall suppose that $V(z)$ is continuous for $z \geqq 0$. We shall prove first that

$$\phi(x, u) \to 1 \quad \text{as} \quad x \to 0. \tag{2.28}$$

$K = \sup_{0 \leqq z \leqq 1} V(z)$ is a finite nonnegative constant. Suppose $0 < x < 1$; and let Δ be the set of all ω such that both $T(x, \omega) \leqq x$ and $\sup_{0 \leqq t \leqq x}[x + X(t, \omega)] \leqq 1$. Then, if ω belongs to Δ,

$$G(x, \omega) \leqq \int_0^{T(x,\omega)} K\, dt \leqq Kx; \tag{2.29}$$

and therefore,

$$\begin{aligned} 1 \geqq \phi(x, u) &\geqq \int_\Delta e^{-uG(x,\omega)}\, d\mu(\omega) \geqq e^{-uKx} \int_\Delta d\mu(\omega) \\ &\geqq e^{-uKx}[P\{T(x, \omega) \leqq x\} - P\{\sup_{0 \leqq t \leqq x} X(t, \omega) > 1 - x\}] \\ &= e^{-uKx}[P\{T(x, \omega) \leqq x\} - P\{T(1 - x, \omega) \leqq x\}] \\ &= e^{-uKx}\pi^{-1/2} \int_{x/2}^{(1-x)^2/2x} v^{-1/2} e^{-v}\, dv \to 1 \quad \text{as} \quad x \to 0, \end{aligned} \tag{2.30}$$

by (2.6). This proves (2.28).

Next let a and y be prescribed numbers satisfying $0 < a < y < \infty$, and suppose $a \leqq x \leqq y$. Let δ be any positive number small enough to satisfy $0 < \epsilon = (-2\delta \log \delta)^{1/2} < a$. In the following work we shall let $\delta \to 0$, and there will be several terms which are $o(\delta)$ as $\delta \to 0$. It will be important to notice that every one of these $o(\delta)$ is uniform in x for $a \leqq x \leqq y$, even though, for the sake of brevity, we shall often omit to say so explicitly hereafter.

Let $S = X(\delta, \omega)$. Then the identity

$$X(t + \delta, \omega) = S + X(t, \omega_2), \qquad t \geqq 0, \tag{2.31}$$

establishes a measure-preserving transformation from ω to ω_2. Let $\theta = \theta(\omega)$ denote the equivalence class of all ω_1 such that $X(t, \omega_1) = X(t, \omega)$ for $0 \leqq t \leqq \delta$; and let $\nu(\theta)$ be the marginal distribution induced by $\mu(\omega)$ on the space $\Theta = \{\theta\}$. Write $\Theta(\epsilon)$ for the subset of Θ such that $\theta(\omega)$ belongs to $\Theta(\epsilon)$ if and only if $\sup_{0 \leqq t \leqq \delta}|X(t, \omega)| \leqq \epsilon$. Then, from (2.6),

$$\int_{\Theta - \Theta(\epsilon)} d\nu(\theta) \leqq 2P\{T(\epsilon, \omega) \leqq \delta\} = 2\pi^{-1/2} \int_{-\log \delta}^{\infty} v^{-1/2} e^{-v} dv = o(\delta). \tag{2.32}$$

Hence

$$\begin{aligned} \phi(x, u) &= \int_{\Theta} d\nu(\theta) \int_{\Omega} d\mu(\omega_2) e^{-uG(x,\omega)} \\ &= \int_{\Theta(\epsilon)} d\nu(\theta) \int_{\Omega} d\mu(\omega_2) e^{-uG(x,\omega)} + o(\delta). \end{aligned} \tag{2.33}$$

When $\theta(\omega)$ belongs to $\Theta(\epsilon)$, then $T(x, \omega) > \delta$ since $x \geqq a > \epsilon$; and so, by (2.31),

$$G(x, \omega) = \int_0^{\delta} V[x + X(t, \omega)]\, dt + \int_0^{T(x+S,\omega_2)} V[x + S + X(t, \omega_2)]\, dt. \tag{2.34}$$

In the first integral in (2.34), $X(t, \delta) \leqq \epsilon$; and $\epsilon \to 0$ as $\delta \to 0$. Hence the first integral is $\delta V(x) + o(\delta)$; and the $o(\delta)$ is uniform in x because $V(z)$, being continuous for $z \geqq 0$, is uniformly continuous in the closed interval $0 \leqq z \leqq a + y$. Thus, when $\theta(\omega)$ belongs to $\Theta(\epsilon)$,

$$G(x, \omega) = \int_0^{T(x+S,\omega_2)} V[x + S + X(t, \omega_2)]\, dt + \delta V(x) + o(\delta). \tag{2.35}$$

Inserting (2.35) into (2.33) and performing the integration with respect to ω_2, we have for fixed u

$$\phi(x, u) = \int_{\Theta(\epsilon)} \{1 - u\delta V(x)\} \phi(x + S, u)\, d\nu(\theta) + o(\delta). \tag{2.36}$$

Hence, from (2.32) and (2.36), we have, taking $\phi(x + S, u) = 1$ formally for $x + S \leqq 0$,

$$\phi(x, u) = \{1 - u\delta V(x)\} \int_{\Theta} \phi(x + S, u)\, d\nu(\theta) + o(\delta). \tag{2.37}$$

In (2.37), the integrand depends on θ only through S; so we may integrate out onto the marginal distribution of S, which is normal with mean zero and variance δ. This gives

$$\phi(x, u) = \{1 - u\delta V(x)\} \int_{-\infty}^{\infty} \phi(x + S, u) \frac{e^{-S^2/2\delta}}{(2\pi\delta)^{1/2}} dS + o(\delta). \tag{2.38}$$

In this we write $z = S/\sqrt{\delta}$ and

$$P(z) = (2\pi)^{-1/2} \int_{-\infty}^{z} e^{-v^2/2}\, dv \tag{2.39}$$

for the standardized normal distribution. After rearranging terms a little, we get

$$\begin{aligned}\int_{-\infty}^{\infty} \{\phi(x + z\sqrt{\delta}, u) - \phi(x, u)\}\, dP(z) \\ = u\delta V(x) \int_{-\infty}^{\infty} \phi(x + z\sqrt{\delta}, u)\, dP(z) + o(\delta).\end{aligned} \tag{2.40}$$

Since the $o(\delta)$ in (2.40) is uniform in x for $a \leqq x \leqq y$, we deduce

$$\begin{aligned}\int_{a}^{y} dx(y - x)^3 \int_{-\infty}^{\infty} \{\phi(x + z\sqrt{\delta}, u) - \phi(x, u)\}\, dP(z) \\ = \int_{a}^{y} dx(y - x)^3 u\delta V(x) \int_{-\infty}^{\infty} \phi(x + z\sqrt{\delta}, u)\, dP(z) + o(\delta).\end{aligned} \tag{2.41}$$

Divide (2.41) by δ and let $\delta \to 0$. On the left side we may invoke Fubini's theorem to invert the order of integration, since the integrand consists of the difference of two nonnegative bounded terms and P is a probability measure. On the right side we may use Lebesgue's bounded convergence theorem to bring the limit with respect to δ inside both integrals, provided that $\lim_{\delta \to 0} \phi(x + z\sqrt{\delta}, u)$ exists for almost all x. This proviso is met, and this limit is $\phi(x, u)$, since $\phi(x, u)$ is continuous almost everywhere for fixed u. We thus get

$$\begin{aligned}\lim_{\delta \to 0} \int_{-\infty}^{\infty} dP(z) \int_{a}^{y} dx(y - x)^3 \{\phi(x + z\sqrt{\delta}, u) - \phi(x, u)\}/\delta \\ = u \int_{a}^{y} (y - x)^3 V(x)\phi(x, u)\, dx.\end{aligned} \tag{2.42}$$

Next write $\phi(x, u) = \phi_0(x, u)$, and defined for $n = 1, 2, \cdots$

$$\begin{aligned}\phi_n(x, u) &= \int_{a}^{x} \phi_{n-1}(v, u)\, dv, \\ \psi_n(a, \delta, u) &= \int_{-\infty}^{\infty} dP(z)\phi_n(a + z\sqrt{\delta}, u)/\delta.\end{aligned} \tag{2.43}$$

Integrate the inner integral on the left of (2.42) three times by parts, and express the result in terms of (2.43). We find

$$\begin{aligned}\lim_{\delta \to 0} \int_{-\infty}^{\infty} dP(z)\{\phi_4(y + z\sqrt{\delta}, u) - \phi_4(y, u)\} \\ = \frac{1}{6} u \int_{a}^{y} (y - x)^3 V(x)\phi(x, u)\, dx + \lim_{\delta \to 0} \sum_{n=0}^{3} \frac{(y - a)^n}{n!} \psi_n(a, \delta, u),\end{aligned} \tag{2.44}$$

provided the left side of (2.44) exists. Since $dP(z) = dP(-z)$, the left side of (2.44) is

$$\frac{1}{2}\lim_{\delta\to 0}\int_{-\infty}^{\infty} dP(z)\{\phi_4(y + z\sqrt{\delta}, u) - 2\phi_4(y, u) + \phi_4(y - z\sqrt{\delta}, u)\}/\delta \tag{2.45}$$

$$= \frac{1}{2}\lim_{\delta\to 0}\int_{-\infty}^{\infty} dP(z)\{z^2\phi_2(y, u) + \frac{1}{6}z^3\delta^{1/2}[\phi_1(y + \theta_1 z\sqrt{\delta}, u) + \phi_1(y - \theta_2 z\sqrt{\delta}, u)]\},$$

where $|\theta_i| < 1$, $i = 1, 2$, because the second and third derivatives of ϕ_4 are ϕ_2 and ϕ_1; and ϕ_1, being an integral, is continuous. Also, by (2.43), $\phi_1(y \pm \theta_i z\sqrt{\delta}, u)$ is $O(z)$ as $z \to \pm\infty$; and $\int_{-\infty}^{\infty} z^4\, dP(z)$ is finite. Hence the right side of (2.45) exists and equals

$$\frac{1}{2}\phi_2(y, u)\int_{-\infty}^{\infty} z^2\, dP(z) = \frac{1}{2}\phi_2(y, u). \tag{2.46}$$

Thus, from (2.44), whose left side has been shown to exist by the above argument,

$$\frac{1}{2}\phi_2(y, u) - \frac{1}{6}u\int_a^y (y - x)^3 V(x)\phi(x, u)\, dx = \lim_{\delta\to 0}\sum_{n=0}^{3}\frac{(y-a)^n}{n!}\psi_n(a, \delta, u). \tag{2.47}$$

We can choose a sequence $\{\delta_k\}$, where $\delta_k \to 0$ as $k \to \infty$, such that

$$\chi_n(a, u) = \lim_{k\to\infty}\psi_n(a, \delta_k, u) \tag{2.48}$$

exists for each $n = 0, 1, 2, 3$. At first sight, it might appear that, to be able to assert the existence of each χ_n, we should have to admit the possibility that $\chi_n = \pm\infty$. However, this is unnecessary; for, if any χ_n were infinite, we could, by taking k large enough, show that the right side of (2.47) was arbitrarily close to a cubic polynomial in y with at least one arbitrarily large coefficient; and this would contradict the fact that the left side of (2.47) is bounded in any closed interval $a \leqq y \leqq b < \infty$. Hence $\chi_n(a, u)$ is finite, and

$$\phi_2(y, u) = \frac{1}{3}u\int_a^y (y - x)^3 V(x)\phi(x, u)\, dx + \sum_{n=0}^{3}\frac{2(y-a)^n}{n!}\chi_n(a, u). \tag{2.49}$$

Now the right side of (2.49) is clearly a thrice differentiable function of y; so the third derivative of the left side exists and is

$$\frac{\partial\phi(y, u)}{\partial y} = 2u\int_a^y V(x)\phi(x, u)\, dx + 2\chi_3(a, u). \tag{2.50}$$

Now (2.50) holds for arbitrary $y > a$. Also $\phi(x, u)$ is a bounded nonincreasing function of x. Hence the left side of (2.50) tends to zero as $y \to \infty$. Thus

$$0 = 2u\int_a^{\infty} V(x)\phi(x, u)\, dx + 2\chi_3(a, u). \tag{2.51}$$

Since $\chi_3(a, u)$ is finite, the integral in (2.51) is also finite; and we may therefore subtract (2.51) from (2.50) to get

$$\frac{\partial\phi(y, u)}{\partial y} = -2u \int_y^\infty V(z)\phi(z, u)\, dz. \tag{2.52}$$

Equation (2.52) holds for $y > a > 0$, and hence for $y > 0$, since the arbitrary number a does not appear in (2.52). If we integrate (2.52) over all positive $y \leqq x$ and use (2.28), we get

$$\begin{aligned} \phi(x, u) &= 1 - 2u \int_0^x dy \int_y^\infty V(z)\phi(z, u)\, dz \\ &= 1 - 2u \int_0^\infty \min(x, z)\phi(z, u)V(z)\, dz, \end{aligned} \tag{2.53}$$

since the inner integral in (2.53) is finite and we may invert the order of integration. This completes the proof of (2.21).

Now relax the condition that $V(z)$ be continuous for $z \geqq 0$. For arbitrary $V(z)$, let $V_0(z)$ be any function, which is continuous for $z \geqq 0$ and satisfies $0 \leqq V_0(z) \leqq V(z)$. Abandoning the notation of (2.43), we now write ϕ_0 for the moment-generating function associated with V_0 (in the same way as ϕ is associated with V). From (2.21) applied to ϕ_0, we have

$$\begin{aligned} 1 &\geqq \phi_0(x, u) + 2u \int_0^x \min(x, z) V_0(z)\phi_0(z, u)\, dz \\ &\geqq \phi_0(x, u)\left\{1 + 2u \int_0^x zV_0(z)\, dz\right\}, \end{aligned} \tag{2.54}$$

since $\phi_0(x, u)$ is a nonincreasing function of x. Hence

$$0 \leqq \phi(x, u) \leqq \phi_0(x, u) \leqq \left\{1 + 2u \int_0^x zV_0(z)\, dz\right\}^{-1}. \tag{2.55}$$

Let ξ be the supremum of all η for which (2.22) is true. Consider first the case $\xi = 0$. Then, for any given $x > 0$, we can choose $V_0(z)$ so that the right side of (2.55) is arbitrarily small. This proves that $\phi(x, u) = 0$, and hence $\phi(x, 0+) = 0$. Hence $G(x, \omega)$ is infinite with probability 1. There is nothing further to be said about the case $\xi = 0$; and hereafter we shall always suppose that $\xi > 0$.

Next suppose $0 < \xi < \infty$. Consider an arbitrary finite positive M, and then choose $x(M)$, satisfying $0 < x(M) < \xi$, such that $\int_0^{x(M)} zV(z)\, dz \geqq M$. Next choose $V_0(z)$ such that $\int_0^{x(M)} zV_0(z)\, dz \geqq M/2$. By (2.55), we have $0 \leqq \phi\{x(M), u\} \leqq (1 + uM)^{-1}$. Since M is arbitrary and $\phi(x, u)$ is nonincreasing, we conclude that $\phi(x, u) \to 0$ as $x \to \xi$, and that $\phi(x, u) = 0$ for $x \geqq \xi$. Hence $\phi(x, 0+) = 0$ for $x \geqq \xi$, and $G(x, \omega)$ is infinite with probability 1 if $x \geqq \xi$.

It only remains to consider the case $0 < x < \xi \leqq \infty$. Let η satisfy $x < \eta < \xi$; so that (2.22), and therefore (2.23) hold. Put $y = x$ in (2.23) and let $u \to 0$. The integral in (2.23) is finite, since the integrand is bounded by $zV(z)$ uniformly in u. We conclude that

(2.56) $$\phi(x, 0+) = 1 - \frac{x}{\eta}\{1 - \phi(\eta, 0+)\},$$

since $\phi(x, u)$ is clearly a nonincreasing function of u, by (2.3), and $\phi(x, 0+)$ consequently exists. Once more return to (2.23) with $y = x$. Fix x and let $\eta \to \xi$. The integrand in (2.23) is a nondecreasing function of η, and the range of integration increases with η. On the right side, $\phi(\eta, u) \to 0$ if $\xi < \infty$; while if $\xi = \infty$, $x\{1 - \phi(\eta, u)\}/\eta \leqq x/\eta \to 0 = x/\xi$, formally. Hence, we may write

(2.57) $$\phi(x, u) + 2u \int_0^{\xi} \left\{\min\left(1, \frac{x}{z}\right) - \frac{x}{\xi}\right\} \phi(z, u) z V(z)\, dz = 1 - \frac{x}{\xi}, \qquad 0 < x < \xi \leqq \infty.$$

The integrand in (2.57) is nonnegative, and hence

(2.58) $$\phi(x, u) \leqq 1 - \frac{x}{\xi}, \qquad 0 < x < \xi \leqq \infty.$$

Since (2.58) holds for all $x < \xi$, we may replace x by η in (2.58) and then let $u \to 0$ to give

(2.59) $$\phi(\eta, 0+) \leqq 1 - \frac{\eta}{\xi}.$$

From (2.56) and (2.59),

(2.60) $$\phi(x, 0+) \geqq 1 - \frac{x}{\eta}\frac{\eta}{\xi} = 1 - \frac{x}{\xi}.$$

Now let $u \to 0$ in (2.58) and compare the result with (2.60). We get

(2.61) $$\phi(x, 0+) = 1 - \frac{x}{\xi};$$

and this proves that $G(x, \omega)$ is infinite with probability x/ξ, when $\xi < \infty$, and is finite with probability 1, when $\xi = \infty$. To prove that $\phi(x, u) \to 1$ as $x \to 0$, let $y = x \to 0$ in (2.23).

From (2.57) we have

(2.62) $$1 - \frac{x}{\xi} \geqq 2u \int_0^{x} \left\{\min\left(1, \frac{x}{z}\right) - \frac{x}{\xi}\right\} \phi(z, u) z V(z)\, dz = \left(1 - \frac{x}{\xi}\right) 2u \int_0^x \phi(z, u) z V(z)\, dz;$$

and hence

(2.63) $$\int_0^x \phi(z, u) z V(z)\, dz \leqq \frac{1}{2u}.$$

Letting $x \to \xi$ in (2.63), we see that $\int_0^{\xi} \phi(z, u) z V(z)\, dz$ is finite; and this allows us to rewrite (2.57) in the form

(2.64)
$$\begin{aligned}\phi(x, u) &= 1 - \frac{x}{\xi} + \frac{2ux}{\xi}\int_0^\xi \phi(z, u)zV(z)\,dz - 2u\int_0^\xi \min(x, z)\phi(z, u)V(z)\,dz \\ &= 1 - \frac{x}{\xi} + \frac{2ux}{\xi}\int_0^\xi \phi(z, u)zV(z)\,dz - 2u\int_0^x dy\int_y^\xi dz\,\phi(z, u)V(z).\end{aligned}$$

The right side of (2.64) is a differentiable function of x; so

$$\frac{\partial\phi(x, u)}{\partial x} = -\frac{1}{\xi} + \frac{2u}{\xi}\int_0^\xi \phi(z, u)zV(z)\,dz - 2u\int_x^\xi \phi(z, u)V(z)\,dz \tag{2.65}$$

exists and is a continuous function of x for $0 < x < \xi \leqq \infty$. A fortiori, $\phi(x, u)$ is continuous in $0 < x < \xi \leqq \infty$; and therefore (2.14) is an immediate consequence of differentiating (2.65) whenever either side is differentiable. The second equation of (2.13) also follows at once from (2.65).

If $\xi < \infty$, there is a positive probability that $G(x, \omega) = \infty$. Hence $M_n(x)$ is infinite for $n = 1, 2, \cdots$. This confirms the result given by (2.10). Hence we have only to prove (2.10) in the case $\xi = \infty$. In this case, (2.57) reduces to (2.21). We write (2.21) in the form

$$\begin{aligned}2\int_0^\infty dz\min(x, z)V(z)\int_\Omega d\mu(\omega)e^{-uG(z,\omega)} &= 2\int_0^\infty \min(x, z)V(z)\phi(z, u)\,dz \\ = \frac{1 - \phi(x, u)}{u} = \int_\Omega d\mu(\omega)\,\frac{1 - e^{-uG(x,\omega)}}{u} &= \int_\Omega d\mu(\omega)\int_0^{G(x,\omega)} e^{-uz}\,dz.\end{aligned} \tag{2.66}$$

Let v be any complex number whose real part is strictly greater than a prescribed $\epsilon > 0$. Write $f(v)$ for the left side of (2.66) when u is replaced by v. We have

$$f(v) = \int_{R\times\Omega} d\lambda e^{-vG} = \int_{R\times\Omega} d\lambda\,\frac{1}{2\pi i}\oint_\Gamma \frac{e^{-wG}}{w - v}\,dw, \tag{2.67}$$

where R is the set of positive finite real numbers, λ is a measure on the product space $R \times \Omega$, and Γ is a circle centered at v and of radius $\Re(v) - \epsilon$. For fixed integers k and m, $x + X(k/m, \omega)$ is a measurable function on $R \times \Omega$ since x is a measurable function on R and therefore on $R \times \Omega$, and since $X(k/m, \omega)$ is a measurable function on Ω and therefore on $R \times \Omega$. Hence the set of (x, ω) for which (2.17) holds is a measurable subset of $R \times \Omega$ when V is continuous. Using this set in place of $\Omega_{lm}(M)$, we can now follow through the argument, already used to prove that $G(x, \omega)$ is μ-measurable for fixed x, to prove that G is also measurable on $R \times \Omega$. It follows that the real and imaginary parts of the integrand on the right of (2.67) are measurable on $R \times \Omega \times R'$, where R' is the whole real line (covered by the real or imaginary coordinates of w). The real and imaginary parts of the integrand are also absolutely integrable, since

$$\int_{R\times\Omega} d\lambda\,\frac{1}{2\pi}\int_\Gamma \frac{|e^{-wG}|}{|w - v|}\,|dw| \leq \int_{R\times\Omega} d\lambda e^{-\epsilon G} = \frac{1 - \phi(x, \epsilon)}{\epsilon} < \infty. \tag{2.68}$$

Hence we may apply Fubini's theorem to (2.67) with the result

$$f(v) = \frac{1}{2\pi i} \oint\limits_{\Gamma} \frac{dw}{w - v} \oint\limits_{R\times\Omega} d\lambda e^{-wG} = \frac{1}{2\pi i} \oint\limits_{\Gamma} \frac{f(w)\,dw}{w - v}. \tag{2.69}$$

But (2.69) shows that $f(v)$ is analytic when $\Re(v) > \epsilon$. Hence, if $u > \epsilon$, we may differentiate $f(u)$ as many times as we please, and

$$\begin{aligned} \frac{d^{n-1}f(u)}{du^{n-1}} &= \frac{(n-1)!}{2\pi i} \oint\limits_{\Gamma} \frac{f(w)\,dw}{(w-u)^n} \\ &= \frac{(n-1)!}{2\pi i} \oint\limits_{\Gamma} \frac{dw}{(w-u)^n} \int\limits_{R\times\Omega} d\lambda e^{-wG} \\ &= \int\limits_{R\times\Omega} d\lambda \frac{(n-1)!}{2\pi i} \oint\limits_{\Gamma} \frac{e^{-wG}\,dw}{(w-u)^n} \\ &= \int\limits_{R\times\Omega} d\lambda(-G)^{n-1}e^{-uG}\,; \end{aligned} \tag{2.70}$$

for we may apply Fubini's theorem in (2.70) for the same reasons as in (2.68). Since $\epsilon > 0$ is arbitrary, (2.70) holds for all $u > 0$. Thus we may differentiate the left side of (2.66) with respect to u under the integral signs. A similar argument applies to the right side of (2.66); and we deduce

$$\begin{aligned} 2\int_0^\infty dz \min(x, z)V(z) \int\limits_{\Omega} d\mu(\omega)[G(z, \omega)]^{n-1}e^{-uG(z,\omega)} \\ = \int\limits_{\Omega} d\mu(\omega) \int_0^{G(x,\omega)} z^{n-1}e^{-uz}\,dz. \end{aligned} \tag{2.71}$$

The integrands on each side are nonnegative and are nondecreasing as u decreases to zero. Hence we may let $u \to 0$ under the integral signs, to obtain

$$\begin{aligned} 2\int_0^\infty \min(x, z)V(z)M_{n-1}(z)\,dz \\ &= 2\int_0^\infty dz \min(x, z)V(z) \int\limits_{\Omega} d\mu(\omega)[G(z, \omega)]^{n-1} \\ &= \int\limits_{\Omega} d\mu(\omega) \int_0^{G(x,\omega)} z^{n-1}\,dz \\ &= \frac{1}{n}\int\limits_{\Omega} d\mu(\omega)[G(x, \omega)]^n = \frac{M_n(x)}{n}, \end{aligned} \tag{2.72}$$

which establishes (2.10).

When (2.4) holds, (2.14) reduces to Riccati's equation ([5], pp. 195–196),

$$\frac{\partial^2\phi(x, u)}{\partial x^2} = 2ux^{\rho-2}\phi(x, u); \tag{2.73}$$

and the solution of this can be written in terms of Bessel functions. For the relevant boundary conditions (2.13), we obtain (2.15). The Bessel function $\mathrm{Kh}_{1/\rho}(z)$ can be written in the form (2.16) ([6], p. 548); and so (2.15) and (2.16) give

$$\int_0^\infty e^{-ug} d_g P\{G(x, \omega) \leqq g\} = \phi(x, u) = \frac{1}{\Gamma(1/\rho)} \int_0^\infty v^{1/\rho - 1} e^{-v} e^{-u(2x^\rho/\rho^2 v)} \, dv. \tag{2.74}$$

Since the inverse Stieltjes-Laplace transform is unique, (2.5) is an immediate consequence of writing $g = 2x^\rho/\rho^2 v$ in (2.74).

3. Generalized first-passage theory for walks

3.1. *Statement of results.* Let $P(y)$ be a given cumulative distribution function, which is proper at $+\infty$ though it may be improper at $-\infty$; that is to say, $P(y)$ is a nondecreasing function which is continuous on the right and satisfies

$$\lim_{y \to +\infty} P(y) = 1, \qquad \lim_{y \to -\infty} P(y) \geqq 0. \tag{3.1}$$

Let $y_1, y_2, \cdots$ be an infinite sequence of mutually independent random variables, each distributed according to $P(y)$. For given $x > 0$, define

$$z_0 = x, \qquad z_t = z_{t-1} + y_t, \qquad t = 1, 2, \cdots. \tag{3.2}$$

Thus the infinite sequence $z_0, z_1, z_2, \cdots$ is a random walk starting at x, and we denote it by W^x. Let T be the smallest integer such that $z_T \leqq 0$, with the understanding that $T = +\infty$ if $z_t > 0$ for all $t = 1, 2, \cdots$. Thus T is the first-passage time for W^x. We shall write W_0^x for the truncated walk $z_0, z_1, \cdots, z_T$, it being understood that $W_0^x = W^x$ if $T = +\infty$.

Let $V(z)$ be a given nonnegative Borel-measurable function such that (a) for each given $\zeta > 0$

$$\sum_{n=0}^{\infty} \sup_{n < z \leqq n+1} V(z + \zeta) < \infty; \tag{3.3}$$

and (b) there exist constants a and ρ satisfying $a > 0$ and $\rho < 1$ for which

$$V(z) \sim \left(\frac{z}{a}\right)^{\rho - 2} \quad \text{as} \quad z \to 0+. \tag{3.4}$$

For the walk W^x, specified by (3.2), we define the random variable

$$G(x) = \sum_{0 \leqq t < T} V(z_t). \tag{3.5}$$

We are interested in the distribution of G, and our main result here is the asymptotic relation

$$\lim_{g \to \infty} g^{1/(2-\rho)} P\{G(x) > g\} = \frac{aq(x)}{R_0^2} \sum_{n=0}^{\infty} (n+1)^{1/(2-\rho)} \left(1 - \frac{1}{R_0}\right)^n, \tag{3.6}$$

where $q(x)$ and R_0 are quantities defined below. There is one trivial exception to (3.6), which arises when, with probability 1, all the y_t are zero. In this case, of course, $G(x) = 0$ or $+\infty$, with probability 1, according as $V(x) = 0$ or $V(x) > 0$. This trivial exception only occurs if $P(y) = H(y)$, where $H(y)$ is the step function

$$H(y) = \begin{cases} 1, & y \geqq 0, \\ 0, & y < 0. \end{cases} \tag{3.7}$$

Hence, throughout our work, we assume that $P(y) \not\equiv H(y)$.

To specify the quantities $q(x)$ and R_0, we define $Q(z, x)$ to be the expected number of visits by W_0^x to the semiclosed interval $(0, z]$: that is to say, $Q(z, x)$ is the expected number of distinct integers t such that $0 < z_t \leqq z$ and $0 \leqq t \leqq T$. We shall prove that

$$Q(z, x) = H(z - x) + \int_0^\infty Q(z, y)\, dP(y - x), \tag{3.8}$$

where H is given by (3.7). In general, the integral equation (3.8) has infinitely many solutions; but, among these solutions, there is a unique Neumann solution, that is to say, the series solution obtained by iterating

$$Q_n(z, x) = H(z - x) + \int_0^\infty Q_{n-1}(z, y)\, dP(y - x), \qquad n = 1, 2, \cdots, \tag{3.9}$$

starting from $Q_0(z, x) = 0$. We shall prove $Q(z, x)$, defined as the expected number of visits by W_0^x to $(0, z]$, is the Neumann solution of (3.8), and that it has the following two properties: for fixed $z > 0$

$$Q(z, x) = O(1) \quad \text{as} \quad x \to \infty, \tag{3.10}$$

while for fixed $x > 0$,

$$Q(z, x) = O(z) \quad \text{as} \quad z \to \infty. \tag{3.11}$$

The two properties (3.10) and (3.11) will often be useful in picking out the Neumann solution when the general solution of (3.8) is available. The quantity R_0 is defined by

$$R_0 = \lim_{z \to 0+} Q(2z, z); \tag{3.12}$$

and it satisfies $1 \leqq R_0 < \infty$, so the series on the right of (3.6) is indeed convergent. The limit in (3.12) always exists. We shall also prove that, for almost all $x > 0$, there exists

$$q(x) = \lim_{z \to 0+} \frac{Q(z, x)}{z}, \tag{3.13}$$

and (3.13) provides the definition of the function $q(x)$ used in (3.6). The limit on the left side of (3.6) exists if the limit on the right side of (3.13) exists; and therefore in general (3.6) is only true for almost all x.

The two conditions (3.3) and (3.4) on $V(z)$ are sufficient but not necessary conditions for (3.6). At the expense of additional complications in the proof, condition (3.3) could be weakened quite considerably. On the other hand, only a relatively trivial weakening of (3.4) would be possible without destroying (3.6).

Equation (3.6) shows that G, defined for a walk by (3.5), behaves quite differently from the analogous G, defined for a Brownian path by (2.2). For example, consider the special case when $V(z)$ is given by (2.4). Then, with probability 1, G for a walk is finite for all $\rho < 1$, whereas G for a Brownian path is finite if and only if $0 < \rho < 1$. Further, the expected value of G for a walk is infinite for all $\rho < 1$, whereas it is finite for a Brownian path if and only if $0 < \rho < 1$. The present work is therefore an interesting addition to the relatively few known cases in the literature, in which results for Brownian paths cannot be regarded as limiting cases of results for walks.

It is worth noting from (3.6) that the asymptotic behavior of large G depends upon V only through the multiplier a and the coefficients $(n + 1)^{1/(2-\rho)}$. The functions R_0 and q, on the other hand, depend only on the distribution $P(y)$. Further we shall show that, if $P(y)$ is continuous, then $R_0 = 1$, and the right side of (3.6) reduces to the simple expression $aq(x)$. At the other extreme, if $P(y)$ is a lattice distribution, then $q(x) = 0$ for almost all x; and (3.6) provides little useful information in such a case. It is natural to ask whether one cannot obtain an integral equation for $q(x)$ itself, for instance by differentiating (3.8) under the integral sign. It is fairly easy to construct counterexamples to show that such a procedure is not in general justifiable, as indeed might have been expected from the fact that $q(x)$ may not exist on a set of measure zero. However, in the special case when $P(y)$ is convex for all sufficiently large negative y and possesses a bounded derivative $p(y)$ for all y, then we shall show that $q(x)$ exists for all positive x, satisfies the integral equation

$$q(x) = p(-x) + \int_0^\infty q(y)p(y - x)\, dy, \tag{3.14}$$

and is bounded as $x \to \infty$. There are heuristic reasons for believing that, when the distribution P possesses a variance σ^2 and satisfies certain additional conditions, then $q(x) \to 2^{1/2}/\sigma$ as $x \to \infty$. But I have not succeeded in discovering what these additional conditions are, though it is clear from the foregoing that the mere existence of σ is insufficient to provide a limit for $q(x)$, inasmuch as σ may exist even though $p(-x)$ does not tend to a limit.

As an illustration of the multiplicity of solutions of (3.8), we may take the example

$$P(y) = \begin{cases} \kappa^2 + \frac{1}{2}(1 - \kappa^2)e^{y}, & y < 0, \\ 1 - \frac{1}{2}(1 - \kappa^2)e^{-y}, & y \geqq 0, \end{cases} \qquad 0 \leqq \kappa \leqq 1. \tag{3.15}$$

Then we can solve (3.8) by differentiating twice to obtain an ordinary differential equation, whose solution can be substituted into (3.8) to obtain the relevant boundary conditions. For details of the procedure see the companion paper [13] by D. G. Kendall. The resulting general solution of (3.8) is

$$(3.16)\qquad Q(z, x) = (1-\kappa)\left(\frac{\cosh \kappa z + \kappa \sinh \kappa z - 1}{\kappa^2}\right)e^{-\kappa x} + \left\{\frac{1-(1-\kappa^2)\cosh \kappa(z-x)}{\kappa^2}\right\}H(z-x) + \left(\frac{\kappa \cosh \kappa x + \sinh \kappa x}{\kappa}\right)C(z),$$

where $C(z)$ is an arbitrary function of z for $0 \leqq \kappa < 1$, but $C(z) = 0$ for $\kappa = 1$. Thus there are infinitely many solutions or just a unique solution according as $0 \leqq \kappa < 1$, or $\kappa = 1$. In either case, the Neumann solution is the one obtained by putting $C(z) = 0$. When $\kappa = 0$, the expressions in (3.16) are to be interpreted as their limiting values as $\kappa \to 0+$. It follows from (3.16) that, when (3.15) holds, then

$$(3.17)\qquad q(x) = (1-\kappa)e^{-\kappa x}.$$

3.2. *Proof of results.* We shall first show that there exist positive numbers c and η, depending on x but independent of z, such that

$$(3.18)\qquad \alpha(z) \leqq c,$$

where $\alpha(z)$ is the expected number of visits by W_0^x to the closed interval $[z, z+\eta]$ for $z \geqq 0$.

Since P is not of the form (3.7) it belongs to one of the following three types.

Type I. $P(y) = 0$ for all $y < 0$, while $P(y) < 1$ for some $y > 0$. In this case we choose $\eta > 0$ such that $P(\eta) < 1$.

Type II. $P(0) = 1$, while $P(y) > 0$ for some $y < 0$. In this case we choose $\eta > 0$ such that $P(-2\eta) > 0$.

Type III. $P(0) < 1$ and $P(y) > 0$ for some $y < 0$. In this case we choose numbers η, η_0, η_1, ν, and ξ in the following manner, which, it should be noted, does not involve z. First, since $P(0) < 1 = \lim_{y\to+\infty} P(y)$ and $P(y)$ is nondecreasing, we may choose $\eta_1 > 0$ such that

$$(3.19)\qquad P(y) < P(\eta_1), \qquad 0 < y < \eta_1.$$

Next we choose η to satisfy simultaneously the three relations

$$(3.20)\qquad P(-4\eta) > 0, \qquad 0 < 2\eta < \eta_1, \qquad 3\eta < x,$$

this being possible because $P(y) > 0$ for some $y < 0$ and $P(y)$ is nondecreasing. Then we choose a positive integer ν such that

$$(3.21)\qquad \nu\eta_1 > x + 3\eta.$$

Finally we choose η_0 such that

$$(3.22)\qquad 0 < \max\left(2\eta, \eta_1 - \frac{\eta}{\nu}\right) < \eta_0 < \eta_1,$$

this being possible by virtue of (3.20). These choices lead to the following consequences. By (3.19) and (3.22),

$$(3.23)\qquad P(\eta_1) - P(\eta_0) > 0.$$

By (3.22),

$$0 < \nu(\eta_1 - \eta_0) < \eta. \tag{3.24}$$

By (3.21) and (3.24),

$$\xi = x - \nu\eta_0 < -2\eta, \tag{3.25}$$

where ξ is defined by (3.25).

Let $\beta(z, \zeta) = 1 - \beta^*(z, \zeta)$ denote the probability that W_0^ζ does not visit $[z, z + \eta]$ for any $t > 0$, though it may visit this interval for $t = 0$. Define

$$\beta(z) = \inf_{z \leqq \zeta \leqq z+\eta} \beta(z, \zeta). \tag{3.26}$$

If P is of type I, and if $y_1 > \eta$ and $z \leqq \zeta = z_0 \leqq z + \eta$, then W_0^ζ cannot visit $[z, z + \eta]$ for $t > 0$; and consequently

$$\beta(z) \geqq 1 - P(\eta) > 0. \tag{3.27}$$

If P is of type II, and if $y_1 \leqq -2\eta$ and $z \leqq \zeta = z_0 \leqq z + \eta$, then W_0^ζ cannot visit $[z, z + \eta]$ for $t > 0$; and consequently

$$\beta(z) \geqq P(-2\eta) > 0. \tag{3.28}$$

Finally, if P is of type III, let μ be the least integer exceeding $(z + \eta)/4\eta$. Then, if $z \leqq \zeta = z_0 \leqq z + \eta$ and $y_j \leqq -4\eta$ for $j = 1, 2, \cdots, \mu$, we see that W_0^ζ cannot visit $[z, z + \eta]$ for $t > 0$; and consequently

$$\beta(z) \geqq [P(-4\eta)]^\mu > 0. \tag{3.29}$$

Thus, whatever the type of P, we have $\beta(z) > 0$ and by virtue of (3.26) we can find a number σ satisfying $0 \leqq \sigma \leqq \eta$ such that

$$0 < \beta(z) \leqq \beta(z, z + \sigma) = \beta_0(z) \leqq 2\beta(z), \tag{3.30}$$

where $\beta_0(z)$ is defined by (3.30).

Since a walk starting at an arbitrary point of $[z, z + \eta]$ has a probability at most $[1 - \beta(z)]^k$ of visiting $[z, z + \eta]$ at least k times before visiting $[-\infty, 0]$, we have

$$\alpha(z) \leqq \sum_{k=0}^{\infty} [1 - \beta(z)]^k = \frac{1}{\beta(z)} \leqq \frac{2}{\beta_0(z)}, \tag{3.31}$$

by (3.30). If P is of type I or II, (3.18) follows from (3.31) and (3.27) or (3.28), respectively. Hence, in the remainder of the proof of (3.11), we may and do assume that P is of type III. We may also assume that $z > x$; for, if $z \leqq x$, we have $\mu \leqq 5/4 + x/4\eta$ and (3.11) follows from (3.29) and (3.31). When $z > x$, the probability that W_0^x visits $[z, z + \eta]$ at least k times is not greater than $\beta^*(z, x)[1 - \beta(z)]^{k-1}$; and therefore

$$\alpha(z) \leqq \sum_{k=1}^{\infty} \beta^*(z, x)[1 - \beta(z)]^{k-1} \leqq \frac{2\beta^*(z, x)}{\beta_0(z)}. \tag{3.32}$$

If $\beta^*(z, x) = 0$, then $\alpha(z) = 0$ and (3.11) is trivial. Hence we may and do assume that

$$\beta^*(z, x) > 0. \tag{3.33}$$

Now let γ denote the probability that W^0 never visits $[-\eta, \eta]$ for $t > 0$. Clearly γ is independent of z. By translating both the walk and the interval through a distance $z + \sigma$, we see that γ is also the probability that $W^{z+\sigma}$ never visits $[z + \sigma - \eta, z + \sigma + \eta]$ for $t > 0$. This process of bodily translation of a walk and boundaries will often be invoked in what follows; and we shall simply use the phrase "by translation" to indicate this method of obtaining a conclusion. Since $[z, z + \eta]$ lies within $[z + \sigma - \eta, z + \sigma + \eta]$, γ does not exceed the probability that $W^{z+\sigma}$ never visits $[z, z + \eta]$ for $t > 0$. Moreover, this last probability does not exceed $\beta_0(z)$. Thus $\gamma \leqq \beta_0(z)$; and (3.11) follows from (3.31) if $\gamma > 0$. Hence we may and do assume that $\gamma = 0$. Consequently W^0 revisits $[-\eta, \eta]$ with probability 1; and hence by translation W^ξ revisits $[\xi - \eta, \xi + \eta]$ with probability 1.

Let $\beta_1(z)$ denote the probability that W^ξ visits $[z, z + 2\eta]$ before it revisits $[\xi - 3\eta, \xi + 2\eta]$. The validity of this definition follows from (3.25) and $z > x$. In particular, consider the case when W^ξ starts with ν steps satisfying $\eta_0 < y_j \leqq \eta_1$ for $j = 1, 2, \cdots, \nu$, and then continues with any sequence of steps according to which a W^x would visit $[z, z + \eta]$ before entering $[-\infty, 0]$. The probability of such a particular W^ξ is $[P(\eta_1) - P(\eta_0)]^\nu \beta^*(z, x)$. Now, by (3.22), $\eta_0 > 2\eta$; so the first step of this particular W^ξ carries it clear of $[\xi - 3\eta, \xi + 2\eta]$. By (3.24) and (3.25), the first ν steps of this W^ξ will bring it to some point of $[x, x + \eta]$; and it will not revisit $[\xi - 3\eta, \xi + 2\eta]$ in the course of these ν steps since $\eta_0 > 0$. The succeeding steps of this W^ξ will carry it to some point of $[z, z + 2\eta]$ without entering $[-\infty, 0]$, and therefore before revisiting $[\xi - 3\eta, \xi + 2\eta]$ because $\xi + 2\eta < 0$ by virtue of (3.25). Consequently this particular W^ξ visits $[z, z + 2\eta]$ before it revisits $[\xi - 3\eta, \xi + 2\eta]$. Hence

$$\beta_1(z) \geqq \beta^*(z, x)[P(\eta_1) - P(\eta_0)]^\nu > 0, \tag{3.34}$$

the last part of this inequality being a consequence of (3.23) and (3.33). Let $\beta_2(z)$ denote the probability that W^ξ visits $[z + \sigma - 3\eta, z + \sigma + 2\eta]$ before it revisits $[\xi - 3\eta, \xi + 2\eta]$. Since $[z, z + 2\eta]$ is contained in $[z + \sigma - 3\eta, z + \sigma + 2\eta]$, we we have $\beta_1(z) \leqq \beta_2(z)$; and (3.34) gives

$$0 < \beta^*(z, x)[P(\eta_1) - P(\eta_0)]^\nu \leqq \beta_2(z). \tag{3.35}$$

Notice that $3\eta < x < z \leqq z + \sigma$ and $\xi \leqq -2\eta$ by (3.20) and (3.25), so that the intervals $[\xi - 3\eta, \xi + 2\eta]$ and $[z + \sigma - 3\eta, z + \sigma + 2\eta]$ do not overlap; and this validates the definition of $\beta_2(z)$ given above.

Now, as already noted, W^ξ has probability 1 of revisiting $[\xi - \eta, \xi + \eta]$; and, according to (3.34), it has positive probability $\beta_1(z)$ of visiting $[z, z + 2\eta]$ before it revisits $[\xi - 3\eta, \xi + 2\eta]$. A fortiori, W^ξ has positive probability of visiting $[z, z + 2\eta]$ before it revisits $[\xi - \eta, \xi + \eta]$. If, for every point $z + \lambda$ in $[z, z + 2\eta]$, $W^{z+\lambda}$ had a positive probability of never revisiting $[\xi - \eta, \xi + \eta]$, W^ξ would have a positive probability of never revisiting $[\xi - \eta, \xi + \eta]$. This contradicts the fact that W^ξ revisits $[\xi - \eta, \xi + \eta]$ with probability 1. Hence there is at least one point $z + \lambda$ in $[z, z + 2\eta]$ such that $W^{z+\lambda}$ visits $[\xi - \eta, \xi + \eta]$

with probability 1. By translation, $W^{z+\sigma}$ has probability 1 of visiting $[\xi + \sigma - \lambda - \eta, \xi + \sigma - \lambda + \eta]$. Since $0 \leqq \sigma \leqq \eta$ and $0 \leqq \lambda \leqq 2\eta$, $W^{z+\sigma}$ has probability 1 of visiting $[\xi - 3\eta, \xi + 2\eta]$. Let $\beta_3(z)$ be the probability that $W^{z+\sigma}$ visits $[\xi - 3\eta, \xi + 2\eta]$ before it revisits $[z + \sigma - 3\eta, z + \sigma + 2\eta]$. Let $W^{z+\sigma}(r)$ denote a $W^{z+\sigma}$, which starts with a step $y_1 \leqq -4\eta$ and which visits $[\xi - 3\eta, \xi + 2\eta]$ after precisely r steps. Because of (3.20) and the fact that $W^{z+\sigma}$ visits $[\xi - 3\eta, \xi + 2\eta]$ with probability 1, there exists a positive integer m such that $W^{z+\sigma}$ is a $W^{z+\sigma}(m)$ with some strictly positive probability π_m. Let $W_*^{z+\sigma}(m)$ denote a $W^{z+\sigma}(m)$ in which the first m steps are in increasing order of magnitude algebraically, that is, the most negative values of y come earliest. Then $W_*^{z+\sigma}(m)$ visits $[\xi - 3\eta, \xi + 2\eta]$ before it revisits $[z + \sigma - 3\eta, z + \sigma + 2\eta]$; and there is a positive probability at least equal to $\pi_m/m!$ that $W^{z+\sigma}$ is a $W_*^{z+\sigma}(m)$. This proves that

$$\beta_3(z) > 0. \tag{3.36}$$

Since $[z + \sigma - 3\eta, z + \sigma + 2\eta]$ contains $[z, z + \eta]$, and since $[-\infty, 0]$ contains $[\xi - 3\eta, \xi + 2\eta]$, we have

$$0 < \beta_3(z) \leqq \beta_0(z). \tag{3.37}$$

From (3.32), (3.35), and (3.37) we deduce

$$\alpha(z) \leqq 2[P(\eta_1) - P(\eta_0)]^{-\nu} \frac{\beta_2(z)}{\beta_3(z)}. \tag{3.38}$$

Next, for brevity, write I_ξ and $I_{z+\sigma}$ for the intervals $[\xi - 3\eta, \xi + 2\eta]$ and $[z + \sigma - 3\eta, z + \sigma + 2\eta]$ respectively. Let Ω denote a walk which starts from $z_0 = \xi$ and continues indefinitely with independent steps, each distributed according to $P(y)$, each step being taken from the point reached by the previous step *except* that, whenever Ω visits a point of I_ξ or of $I_{z+\sigma}$, the next step is taken from the point ξ or $z + \sigma$ respectively. Evidently Ω coincides with a W^ξ up to its first visit to $I_{z+\sigma}$ or its first revisiting of I_ξ. After a visit to $I_{z+\sigma}$ or I_ξ, Ω for its next sequence of steps until a fresh visit to one of these intervals coincides with a $W^{z+\sigma}$ or a W^ξ respectively. Now W^ξ revisits I_ξ with probability 1, and similarly by translation $W^{z+\sigma}$ revisits $I_{z+\sigma}$ with probability 1. If Ω had both a last visit to I_ξ and also a last visit to $I_{z+\sigma}$, it would, after the later of these last visits, coincide with either a W^ξ not revisiting I_ξ or a $W^{z+\sigma}$ not revisiting $I_{z+\sigma}$, and it would therefore have probability zero. Hence, with probability 1, Ω makes infinitely many visits to the union of I_ξ and $I_{z+\sigma}$. Further, whenever Ω visits I_ξ it has a positive probability $\beta_2(z)$ of next visiting $I_{z+\sigma}$ before it revisits I_ξ; and whenever Ω visits $I_{z+\sigma}$ it has a positive probability $\beta_3(z)$ of next visiting I_ξ before it revisits $I_{z+\sigma}$. It follows that, with probability 1, Ω visits I_ξ infinitely often and also, with probability 1, Ω visits $I_{z+\sigma}$ infinitely often. The probability that, following upon a visit to I_ξ, Ω will make at least k visits to $I_{z+\sigma}$ before it revisits I_ξ equals $\beta_2(z)[1 - \beta_3(z)]^{k-1}$; and hence the expected number of times Ω visits $I_{z+\sigma}$ between two successive visits to I_ξ is

$$\sum_{k=1}^{\infty} \beta_2(z)[1 - \beta_3(z)]^{k-1} = \frac{\beta_2(z)}{\beta_3(z)}. \tag{3.39}$$

Since Ω visits I_ξ infinitely often with probability 1, we may regard Ω as having started in the indefinitely distant past. Thus (3.39) also gives the expected number of visits by Ω to $I_{z+\sigma}$ between a given visit to I_ξ and the preceding visit to I_ξ. Hence (3.39) represents the expected number of visits to $I_{z+\sigma}$ between successive visits to I_ξ when Ω is described *backward*.

We now set up the following mechanism for describing Ω backward, starting from a visit to I_ξ. Draw a sequence of independent random numbers $y_1, y_2, \cdots$ from the common distribution $P(y)$. We say that event A_r occurs if

$$-3\eta \leqq \sum_{j=1}^{r} y_j \leqq 2\eta, \tag{3.40}$$

and that event B_r occurs if

$$\xi - 3\eta - z - \sigma \leqq \sum_{j=1}^{r} y_j \leqq \xi + 2\eta - z - \sigma. \tag{3.41}$$

Since I_ξ and $I_{z+\sigma}$ do not overlap, the events A_r and B_r are exclusive for each given r. If either A_1 occurs or there is an integer $r > 1$ such that A_r occurs although none of $B_1, B_2, \cdots, B_{r-1}$ occurs, we say that event A occurs. Similarly if B_1 occurs or there is an integer $r > 1$ such that B_r occurs although none of $A_1, A_2, \cdots, A_{r-1}$ occurs, we say that event B occurs. Evidently events A and B are exclusive. Since the steps of a W^ξ or a $W^{z+\sigma}$ are independent, and may therefore be taken in reverse order, we see that, if A eventually occurs, then no visits to $I_{z+\sigma}$ occurred between the given visit to I_ξ and the previous visit to I_ξ; whereas, if B eventually occurs, then some visit to $I_{z+\sigma}$ occurred between the given visit to I_ξ and the previous visit to I_ξ. However, one and the same translation will carry ξ and I_ξ into $z + \sigma$ and $I_{z+\sigma}$ respectively. Hence, if we do not take the y_j in reverse order, the events A and B will respectively tell us whether, in describing Ω *forward* from a given visit to $I_{z+\sigma}$, Ω does or does not revisit $I_{z+\sigma}$ before it visits I_ξ. Consequently, events A and B are exhaustive, apart from events of zero probability; and event B must have probability $\beta_3(z)$ and event A must have probability $1 - \beta_3(z)$. A similar argument shows that, if we describe Ω backward starting from a visit to $I_{z+\sigma}$, then the probability that no visits to I_ξ occurred between the given visit to $I_{z+\sigma}$ and the previous visit to $I_{z+\sigma}$ is $\beta_2(z)$. Hence, in tracing Ω backward, the expected number of visits to $I_{z+\sigma}$ between successive visits to I_ξ is

$$\sum_{k=1}^{\infty} \beta_3(z)[1 - \beta_2(z)]^{k-1} = \frac{\beta_3(z)}{\beta_2(z)}. \tag{3.42}$$

Since, as already remarked, the expectations in (3.39) and (3.42) are equal and reciprocals of one another, each equals 1. Thus (3.38) reduces to

$$\alpha(z) \leqq 2[P(\eta_1) - P(\eta_0)]^{-r}, \tag{3.43}$$

and this completes the proof of (3.18).

Now define $Q(z, x)$ to be the expected number of visits by W_0^x to the semiclosed interval $(0, z]$. From (3.18) we have

$$Q(n\eta + \eta, x) - Q(n\eta, x) \leqq \alpha(n\eta) \leqq c \tag{3.44}$$

for $n = 0, 1, \cdots, N - 1$. Summing (3.44) over these values of n and noting that $Q(0, x) = 0$, we get

$$Q(N\eta, x) \leqq cN. \tag{3.45}$$

Since $Q(z, x)$ is a nondecreasing function of z for fixed x, (3.11) follows at once from (3.45).

Next we shall prove the assertions connected with (3.12). For this and for subsequent work we require an extension of the notation W_0^x. We write W_ξ^x for the truncated sequence $z_0 = x, z_1, z_2, \cdots, z_t$ where t is the smallest nonnegative integer such that $z_t \leqq \xi$. This notation carries the gloss that $W_\xi^x = W^x$ if $z_t > \xi$ for all t. For $z > 0$ we define

$$R(z) = Q(2z, z). \tag{3.46}$$

Thus $R(z)$ is the expected number of visits by W_0^z to $(0, 2z]$, and this does not exceed the expected number of visits by W_0^z to $(\zeta, 2z - \zeta]$ if $0 < \zeta \leqq z$. The latter expected number does not exceed the expected number of visits by W_ζ^z to $(\zeta, 2z - \zeta]$, which in turn by translation does not exceed the expected number of visits by $W_0^{z-\zeta}$ to $(0, 2z - 2\zeta]$. Hence

$$R(z) \geqq R(z - \zeta); \tag{3.47}$$

so that $R(z)$ is a nondecreasing function, which establishes the existence of

$$R_0 = \lim_{z \to 0+} R(z). \tag{3.48}$$

Since $z_0 = z < 2z$, we have $R(z) \geqq 1$; and therefore $R_0 \geqq 1$. The inequality (3.45) shows that $Q(z, x)$ is finite for every given z and x; so $R(z)$ is finite for given z; and now $R_0 < \infty$ follows from the nondecreasing character of R.

If $0 < y \leqq z$, $Q(z, y)$ is the expected number of visits by W_0^y to $(0, z]$, which does not exceed the expected number of visits by W_{y-z}^y to $(y - z, z]$, which equals $Q(2z - y, z)$ upon translation. Hence

$$Q(z, y) \leqq Q(2z - y, z) \leqq Q(2z, z), \tag{3.49}$$

since $Q(z, x)$ is a nondecreasing function of its first argument. From (3.46) and (3.49),

$$Q(z, y) \leqq R(z). \tag{3.50}$$

For any $x > 0$, we have

$$Q(z, x) \leqq \sup_{0 < y \leqq z} Q(z, y). \tag{3.51}$$

Hence (3.50) holds for all $y > 0$; and (3.10) follows at once.

Next we shall prove (3.13). Define $L(z, x)$ to be the probability that W_0^x visits $(0, z]$. For arbitrary $h > 0$ we have

(3.52) $L(z + h, z + x) - L(z, z + x)$

$= P\{W_0^{z+x} \text{ visits } (z, z + h] \text{ but does not visit } (0, z]\}$

$= P\{W_{-z}^{x} \text{ visits } (0, h] \text{ but does not visit } (-z, 0]\}$

$= P\{W^{x} \text{ visits } (0, h] \text{ but not } (-z, 0] \text{ before entering } [-\infty, -z]\}$

$\leqq P\{W^{x} \text{ visits } (0, h] \text{ before entering } [-\infty, 0]\}$

$= P\{W_0^{x} \text{ visits } (0, h]\} = L(h, x).$

Define $K(z)$ to be the probability that W_0^z does not visit the open interval $(0, z)$. Since $R(z - \zeta)$ is the expected number of visits by $W_0^{z-\zeta}$ to $(0, 2z - 2\zeta]$, which equals on translation the expected number of visits by W_ζ^z to $(\zeta, 2z - \zeta]$, we have on letting $\zeta \to z - 0$ that R_0 is the expected number of visits by W^z to the point z before entering $[-\infty, z)$. Thus

(3.53) $R(z) - R_0$

= expected number of visits by W_0^z to $(0, 2z]$ excluding visits to the point z

$\geqq$ expected number of visits by W_0^z to $(0, z)$

$\geqq P\{W_0^z \text{ visits } (0, z)\} = 1 - K(z).$

Now consider the product $L(h, x)K(z + h)$. The first term of this product is the probability that W_0^x visits $(0, h]$. Conditional upon W_0^x having visited $(0, h]$ at some point ξ, say, $K(z + h)$ is the conditional probability that the ensuing part of W^x does not visit $(\xi - z - h, \xi)$ before entering $[-\infty, \xi - z - h]$, because this ensuing part is independent of the part of W^x up to and including the first visit to $(0, h]$. Thus

(3.54) $L(h, x)K(z + h)$

$= P\{W^x$ visits $(0, h]$ before entering $[-\infty, 0]$, and having first visited $(0, h]$ at some point ξ, say, does not thereafter visit $(\xi - z - h, \xi)$ before entering $[-\infty, \xi - z - h]\}$

$\leqq P\{W_{-z}^{x} \text{ visits } (0, h] \text{ but does not visit } (-z, 0]\}$

$= L(z + h, z + x) - L(z, z + x).$

In (3.54) the last step follows from the first two steps of (3.52); and the penultimate step of (3.54) holds because $0 < \xi \leqq h$ implies $\xi - z - h \leqq -z$. Assembling (3.52), (3.53), and (3.54) we deduce

(3.55) $$[1 + R_0 - R(z + h)]L(h, x) \leqq K(z + h)L(h, x) \leqq L(z + h, z + x) - L(z, z + x) \leqq L(h, x).$$

Suppose $0 < z < x$, and write $x - z$ in place of x in (3.55). We get

(3.56)
$$[1 + R_0 - R(z + h)]L(h, x - z) \leqq L(z + h, x) - L(z, x) \leqq L(h, x - z).$$

For fixed x, $L(z, x)$ is a nondecreasing function of z by virtue of the definition of L, and is therefore differentiable for almost all z, say for all z belonging to some set $\mathfrak{Z}_x$ depending on x. Suppose z belongs to $\mathfrak{Z}_x$. Then there exists

$$l(z, x) = \lim_{h \to 0+} h^{-1}\{L(z + h, x) - L(z, x)\}. \tag{3.57}$$

From (3.56) we get

$$[1 + R_0 - R(z + 0)] \limsup_{h \to 0+} h^{-1}L(h, x - z) \leqq l(z, x) \leqq \liminf_{h \to 0+} h^{-1}L(h, x - z), \tag{3.58}$$

provided z belongs to $\mathfrak{Z}_x$. By (3.48) we deduce that

$$(1 - m^{-1}) \limsup_{h \to 0+} h^{-1}L(h, x - z) \leqq \liminf_{h \to 0+} h^{-1}L(h, x - z), \tag{3.59}$$

provided that z both belongs to $\mathfrak{Z}_x$ and also satisfies $0 < z < \delta(m)$, where m is a positive integer, and $\delta(m)$ is determined by (3.48) and is therefore independent of x. Since $\mathfrak{Z}_x$ comprises almost all z, (3.59) holds for almost all z in $0 < z < \delta(m)$. By writing $x = (1/2)n\delta(m)$ in (3.59) and combining the results for $n = 1, 2, \cdots$ we deduce that

$$(1 - m^{-1}) \limsup_{h \to 0+} h^{-1}L(h, x) \leqq \liminf_{h \to 0+} h^{-1}L(h, x) \tag{3.60}$$

holds for almost all $x > 0$. Note that the x of (3.60) is an $x - z$ of (3.59). Since m is an integer and takes only countably many values, we can let $m \to \infty$ in (3.60) and deduce that

$$\limsup_{h \to 0+} h^{-1}L(h, x) \leqq \liminf_{h \to 0+} h^{-1}L(h, x) \tag{3.61}$$

holds for almost all $x > 0$. Consequently

$$l(x) = \lim_{h \to 0+} h^{-1}L(h, x) \tag{3.62}$$

exists for almost all $x > 0$.

Now $Q(z, y)$ is at least equal to the expected number of visits by W^y to the point y before entering $[-\infty, y)$, and this is R_0. So, by (3.50),

$$R_0 \leqq Q(z, y) \leqq R(z). \tag{3.63}$$

By the definitions of Q and L, we have

$$Q(z, x) = \int_0^z Q(z, y)\, dL(y, x), \tag{3.64}$$

y being the first point (if any) at which W_0^x enters $(0, z]$. From (3.63) and (3.64),

$$R_0L(z, x) = R_0 \int_0^z dL(y, x) \leqq \int_0^z Q(z, y)\, dL(y, x) = Q(z, x) \leqq R(z) \int_0^z dL(y, x) = R(z)L(z, x). \tag{3.65}$$

Divide (3.65) by z and let $z \to 0+$. By (3.48) we conclude that

$$q(x) = \lim_{z \to 0+} z^{-1}Q(z, x) = R_0l(x) \tag{3.66}$$

exists if and only if $l(x)$ exists. In particular, by (3.62), $q(x)$ exists for almost all $x > 0$. This completes the proof of (3.13).

We now turn to our main topic of discussing the moment-generating function

$$\phi(x, u) = Ee^{-uG(x)}, \tag{3.67}$$

where $u > 0$, and $G(x)$ is the random variable defined in (3.5). For typographical convenience we shall write

$$v(z) = e^{-uV(z)}. \tag{3.68}$$

We define

$$\psi(z) = 1 - \phi(z, u); \tag{3.69}$$

and for $t = 1, 2, \cdots$ we write

$$A_t(z_0) = \left\{\prod_{s=1}^{t} \int_0^\infty v(z_{s-1})d_sP(z_s - z_{s-1})\right\} \{1 - v(z_t)\}, \tag{3.70}$$

$$B_t(z_0) = \left\{\prod_{s=1}^{t} \int_0^\infty v(z_{s-1})d_sP(z_s - z_{s-1})\right\} \psi(z_t). \tag{3.71}$$

In (3.70) and (3.71) the symbol d_s indicates that the Stieltjes integration is with respect to z_s, and the products indicate multiple integrals. From (3.5), (3.67), and (3.68) we have

$$\phi(z_0, u) = v(z_0) \left\{\int_{-\infty}^{0} d_1P(z_1 - z_0) + \int_0^\infty \phi(z_1, u)d_1P(z_1 - z_0)\right\}. \tag{3.72}$$

Substituting (3.69) into (3.72) and using (3.71) we obtain

$$\begin{aligned}\psi(z_0) &= 1 - v(z_0) + v(z_0) \int_0^\infty \psi(z_1)d_1P(z_1 - z_0) \\ &= 1 - v(z_0) + B_1(z_0).\end{aligned} \tag{3.73}$$

Now write z_t and z_{t+1} for z_0 and z_1 in (3.73) and substitute into (3.71). By (3.70) we obtain

$$B_t(z_0) = A_t(z_0) + B_{t+1}(z_0); \tag{3.74}$$

and hence from (3.73) and (3.74),

$$\psi(z_0) = 1 - v(z_0) + \sum_{s=1}^{t} A_s(z_0) + B_{t+1}(z_0). \tag{3.75}$$

Next define

$$Q_0(z, z_0, u) = H(z - z_0); \tag{3.76}$$

and, for $t = 1, 2, \cdots$,

$$Q_t(z, z_0, u) = \int_0^\infty d_1P(z_1 - z_0)Q_{t-1}(z, z_1, u)v(z_0). \tag{3.77}$$

Repeated application of (3.77) gives

$$(3.78)\qquad Q_t(z, z_0, u) = \left\{\prod_{s=1}^{t} \int_0^\infty v(z_{s-1}) d_s P(z_s - z_{s-1})\right\} H(z - z_t);$$

and therefore, by (3.70),

$$(3.79)\qquad A_t(z_0) = \int_0^\infty [1 - v(z)] dQ_t(z, z_0, u).$$

From (3.75) and (3.79) we deduce

$$(3.80)\qquad \psi(z_0) = \int_0^\infty [1 - v(z)] d\left[\sum_{s=0}^{t} Q_s(z, z_0, u)\right] + B_{t+1}(z_0).$$

We shall now prove that

$$(3.81)\qquad \lim_{t\to\infty} B_t(z_0) = 0.$$

In proving (3.81) it is convenient to adopt the convention that $V(z) = 0$ when $z \leqq 0$. We also write Z_t for the smallest of $z_0, z_1, \cdots, z_t$. Since u, G, and V are all nonnegative, we know that $0 \leqq v(z) \leqq 1$ and $0 \leqq \psi(z) \leqq 1$ for all z. Thus

$$(3.82)\qquad 0 \leqq B_t(z_0) \leqq \prod_{s=1}^{t} \int_0^\infty d_s P(z_s - z_{s-1}) = P\{Z_t > 0\}.$$

If X is any nonnegative random variable and $\mathfrak{e}$ is any event, then

$$(3.83)\qquad E(X|\mathfrak{e})P\{\mathfrak{e}\} \leqq E(X|\mathfrak{e})P\{\mathfrak{e}\} + E(X|\mathfrak{e}')P\{\mathfrak{e}'\} = EX,$$

where $E(X|\mathfrak{e})$ is the conditional expectation of X given $\mathfrak{e}$, and $\mathfrak{e}'$ is the event complementary to $\mathfrak{e}$. In (3.83) we take $\mathfrak{e}$ to be the event $Z_t > 0$ and also take $X = \psi(z_t) \prod_{s=1}^{t} v(z_{s-1})$. From (3.82) and (3.83) we have

$$(3.84)\qquad [B_t(z_0)]^2 \leqq B_t(z_0)P\{Z_t > 0\} \leqq \left\{\prod_{s=1}^{t} \int_{-\infty}^\infty v(z_{s-1}) d_s P(z_s - z_{s-1})\right\} \psi(z_t).$$

If $z_t \leqq 0$, then $\psi(z_t) = 0$. If $z_t > 0$, we define τ to be the largest s such that $z_t, z_{t+1}, \cdots, z_s$ are all positive ($\tau \leqq \infty$). Then, whatever the sign of z_t, we have

$$(3.85)\qquad \begin{aligned}\psi(z_t) &\leqq \left\{\prod_{s=t+1}^{\infty} \int_{-\infty}^\infty d_s P(z_s - z_{s-1})\right\}\left\{1 - \exp\left[-u \sum_{s=t}^{\tau} V(z_s)\right]\right\} \\ &\leqq \left\{\prod_{s=t+1}^{\infty} \int_{-\infty}^\infty d_s P(z_s - z_{s-1})\right\}\left\{1 - \exp\left[-u \sum_{s=t}^{\infty} V(z_s)\right]\right\},\end{aligned}$$

since the middle term of (3.85) may be trivially omitted when $z_t \leqq 0$. On substituting (3.85) into (3.84) we find

$$(3.86)\qquad [B_t(z_0)]^2 \leqq E\left[\left\{\prod_{s=0}^{t-1} v(z_s)\right\}\left\{1 - \prod_{=t}^{\infty} v(z_s)\right\}\right].$$

The expression in square brackets on the right of (3.86) lies between 0 and 1 inclusive; and hence, by Lebesgue's bounded convergence theorem, we have

$$(3.87) \quad E\left[\limsup_{t\to\infty}\left\{\prod_{s=0}^{t-1} v(z_s)\right\}\left\{1-\prod_{s=t}^{\infty} v(z_s)\right\}\right] = \limsup_{t\to\infty} E\left[\left\{\prod_{s=0}^{t-1} v(z_s)\right\}\left\{1-\prod_{s=t}^{\infty} v(z_s)\right\}\right] \geqq \limsup_{t\to\infty}\,[B_t(z_0)]^2.$$

However,

$$(3.88) \quad \limsup_{t\to\infty}\left\{\prod_{s=0}^{t-1} v(z_s)\right\}\left\{1-\prod_{s=t}^{\infty} v(z_s)\right\} = 0,$$

since the two expressions in braces in (3.88) lie between 0 and 1 inclusive, and as $t \to \infty$ either the first or the second expression tends to zero according as the series of nonnegative terms $\sum_{s=0}^{\infty} V(z_s)$ diverges or converges. Consequently (3.81) follows from (3.87) and (3.88). Finally, by (3.80) and (3.81),

$$(3.89) \quad \psi(x) = \lim_{t\to\infty}\int_0^\infty [1-v(z)]d\left[\sum_{s=0}^{t} Q_s(z,x,u)\right].$$

From (3.78) with $u = 0$, we have

$$(3.90) \quad Q_t(z,x,0) = P\{0 < Z_t \leqq z_t \leqq z\},$$

where the probability is taken over the walk W^x with $x = z_0$. By definition, $Q(z, x)$ is the expected number of visits by W_0^x to $(0, z]$, and we deduce from (3.90) that

$$(3.91) \quad \sum_{t=0}^{\infty} Q_t(z,x,0) = Q(z,x).$$

From (3.76) and (3.77) by induction on $t = 1, 2, \cdots$ we see that $Q_t(z, z_0, u)$ is a nonnegative nonincreasing function of u for fixed z and z_0. Hence there exists

$$(3.92) \quad Q(z,x,u) = \sum_{t=0}^{\infty} Q_t(z,x,u) \leqq Q(z,x).$$

From (3.89) and (3.92), we deduce

$$(3.93) \quad \psi(x) \leqq \int_0^\infty [1-v(z)]\,dQ(z,x,u),$$

since, for each fixed x and u, $Q_t(z, x, u)$ is a nonnegative nondecreasing function of z. On the other hand, since $B_t(z) \geqq 0$, we have from (3.80)

$$(3.94) \quad \psi(x) \geqq \int_0^Z [1-v(z)]d\left[\sum_{s=0}^{t} Q_s(z,x,u)\right],$$

for arbitrary $Z > 0$ and $t > 0$. Since $Q_s(0, x, u) = 0$ by (3.78), and since $0 \leqq 1 - v(z) \leqq 1$, we have from (3.94)

$$(3.95)\qquad \begin{aligned}\psi(x) &\geqq \int_0^Z [1 - v(z)]\, dQ(z, x, u) \\ &\qquad - \int_0^Z [1 - v(z)] d\left[Q(z, x, u) - \sum_{s=0}^{t} Q_s(z, x, u)\right] \\ &\geqq \int_0^Z [1 - v(z)]\, dQ(z, x, u) - \int_0^Z d\left[Q(z, x, u) - \sum_{s=0}^{t} Q_s(z, x, u)\right] \\ &= \int_0^Z [1 - v(z)]\, dQ(z, x, u) - \left[Q(Z, x, u) - \sum_{s=0}^{t} Q_s(Z, x, u)\right].\end{aligned}$$

In (3.95) fix Z and let $t \to \infty$, with the result

$$(3.96)\qquad \psi(x) \geqq \int_0^Z [1 - v(z)]\, dQ(z, x, u),$$

by virtue of (3.92). In (3.96) Z is arbitrary; so letting $Z \to \infty$ we conclude

$$(3.97)\qquad \psi(x) \geqq \int_0^\infty [1 - v(z)]\, dQ(z, x, u).$$

Finally, by (3.93) and (3.97), we obtain

$$(3.98)\qquad \psi(x) = \int_0^\infty [1 - v(z)]\, dQ(z, x, u).$$

So far we have made no use of the special properties of $V(z)$, other than the fact that it is nonnegative. But now we shall embark on a study of the asymptotic behavior of $\psi(x)$ as $u \to 0$; and for this we shall require (3.3) and (3.4). Indeed, instead of (3.4) we shall temporarily and until further notice make the stronger assumption that there exists a fixed $\zeta > 0$ such that

$$(3.99)\qquad V(z) = \left(\frac{z}{a}\right)^{\rho-2}, \qquad 0 < z \leqq \zeta.$$

Let $V_0(z)$ be any nonnegative Borel-measurable function which satisfies (3.3). We shall show that there exists a finite function $M(x)$, which is independent of u, such that

$$(3.100)\qquad \int_\zeta^\infty V_0(z)\, dQ(z, x, u) \leqq M(x).$$

If $0 < z < z'$, we have by (3.78)

$$(3.101)\qquad \begin{aligned} &Q_t(z', z_0, u) - Q_t(z, z_0, u) \\ &\qquad = \left\{\prod_{s=1}^{t} \int_0^\infty v(z_{s-1}) d_s P(z_s - z_{s-1})\right\} \left\{H(z' - z_t) - H(z - z_t)\right\}.\end{aligned}$$

The right side of (3.101) is nonnegative and a nonincreasing function of u. Consequently, for any t, $Q_t(z, z_0, u)$ regarded as a function of z induces a measure which is dominated by the corresponding measure induced by $Q_t(z, z_0, u')$ for any $u' < u$, and in particular by the measure induced by $Q_t(z, z_0, 0) = Q_t(z, z_0)$. Since this holds for each t, similar statements hold for the measures induced by

$Q(z, z_0, u)$ and $Q(z, z_0)$. In the sequel these facts will be used freely without further explicit mention. In particular, we have

$$\int_\zeta^\infty V_0(z)\, dQ(z, x, u) \leqq \int_\zeta^\infty V_0(z)\, dQ(z, x). \tag{3.102}$$

By (3.18) we have

$$Q(\zeta + n + 1, x) - Q(\zeta + n, x) \leqq c(1 + \eta^{-1}) \tag{3.103}$$

for any nonnegative integer n. Hence

$$\int_\zeta^\infty V_0(z)\, dQ(z, x) \leqq c(1 + \eta^{-1}) \sum_{n=0}^{\infty} \sup_{n < z \leqq n+1} V_0(z + \zeta); \tag{3.104}$$

and now (3.100) follows from (3.3), (3.102), and (3.104). Since $\rho < 1$, we have $1/(2 - \rho) < 1$ and therefore, by (3.100),

$$\begin{aligned} 0 &\leqq u^{-1/(2-\rho)} \int_\zeta^\infty \{1 - e^{-uV_0(z)}\}\, dQ(z, x, u) \\ &\leqq u^{-1/(2-\rho)} \int_\zeta^\infty uV_0(z)\, dQ(z, x, u) \\ &\leqq u^{1-1/(2-\rho)} M(x) \to 0 \quad \text{as} \quad u \to 0. \end{aligned} \tag{3.105}$$

Taking $V_0(z)$ to be either $V(z)$ or $(z/a)^{\rho-2}$, both of which satisfy (3.3), we deduce from (3.105) that

$$\lim_{u\to0} u^{-1/(2-\rho)} \int_\zeta^\infty [1 - v(z)]\, dQ(z, x, u) = 0 \tag{3.106}$$

and

$$\lim_{u\to0} u^{-1/(2-\rho)} \int_\zeta^\infty \left\{1 - \exp\left[-u\left(\frac{z}{a}\right)^{\rho-2}\right]\right\} dQ(z, x, u) = 0. \tag{3.107}$$

By (3.98), (3.99), (3.106), (3.107), we conclude that

$$\lim_{u\to0} u^{-1/(2-\rho)} \left[\psi(x) - \int_0^\infty \left\{1 - \exp\left[-u\left(\frac{z}{a}\right)^{\rho-2}\right]\right\} dQ(z, x, u)\right] = 0. \tag{3.108}$$

Now $Q(0, x, u) = 0$ by (3.78) and (3.92). Also, by (3.11),

$$0 \leqq Q(z, x, u) \leqq Q(z, x) = O(z) \quad \text{as} \quad z \to \infty; \tag{3.109}$$

and

$$0 \leqq 1 - \exp\left[-u\left(\frac{z}{a}\right)^{\rho-2}\right] \leqq u\left(\frac{z}{a}\right)^{\rho-2} = o(z^{-1}) \quad \text{as} \quad z \to \infty. \tag{3.110}$$

Thus, integrating (3.108) by parts and using (3.109) and (3.110) to dispose of the integrated part, we find

$$\lim_{u\to0} u^{-1/(2-\rho)} \left[\psi(x) - \int_{z=0}^\infty Q(z, x, u) d\left\{\exp\left[-u\left(\frac{z}{a}\right)^{\rho-2}\right]\right\}\right] = 0. \tag{3.111}$$

In (3.111) we make the substitution

$$t = u\left(\frac{z}{a}\right)^{\rho-2} \tag{3.112}$$

with the result

$$\lim_{u\to 0}\left\{\frac{\psi(x)}{u^{1/(2-\rho)}} - a\int_0^\infty \frac{Q\left[a\left(\frac{u}{t}\right)^{1/(2-\rho)}, x, u\right]}{a\left(\frac{u}{t}\right)^{1/(2-\rho)}} t^{-1/(2-\rho)}e^{-t}\,dt\right\} = 0. \tag{3.113}$$

The next step is to prove the existence of

$$S(t, x) = \lim_{u\to 0} \frac{Q\left[a\left(\frac{u}{t}\right)^{1/(2-\rho)}, x, u\right]}{a\left(\frac{u}{t}\right)^{1/(2-\rho)}} \tag{3.114}$$

for each fixed $t > 0$ and to obtain an explicit expression for $S(t, x)$. By (3.112) and (3.99), (3.114) is equivalent to

$$S(t, x) = \lim_{z\to 0} \frac{1}{z} Q\left[z, x, \frac{t}{V(z)}\right]; \tag{3.115}$$

and we shall deal with (3.115) instead of (3.114). Throughout the argument $t > 0$ will be fixed. Let $z < x$. By (3.64),

$$Q(z, x) \leqq L(z, x) \sup_{0<y\leqq z} Q(z, y). \tag{3.116}$$

Prescribe ϵ satisfying $0 < \epsilon < 1$. Then, by (3.63), (3.116), and (3.48), we can find $\delta_0(\epsilon) > 0$ such that

$$0 \leqq Q(z, x) - R_0L(z, x) \leqq [R(z) - R_0]L(z, x) \leqq R(z) - R_0 \leqq \epsilon \tag{3.117}$$

provided $0 < z \leqq \delta_0(\epsilon)$. In (3.117), $Q(z, x) - R_0L(z, x)$ is not less than the expected number of visits by W_0^x to $(0, z]$ excluding any visits to the particular point of $(0, z]$ at which W_0^x first visits $(0, z]$ if any such first visit occurs. Hence, if $0 < \delta \leqq \delta_0(\epsilon)$,

$$P\{W_0^x \text{ visits more than one point of } (0, \delta]\} \leqq \epsilon. \tag{3.118}$$

Of course, we have to remember that even when W_0^x only visits one point of $(0, z]$ the expected number of visits to this particular point is at least R_0. We now choose a fixed $\delta > 0$ such that

$$\delta \leqq \delta_0(\epsilon), \qquad \delta \leqq \zeta, \qquad V(\delta) \geqq \sup_{z\geqq\delta} V(z), \tag{3.119}$$

as we may clearly do in view of (3.3) and (3.99).

For $0 < y \leqq \delta$, let $\pi(y, \delta)$ denote the conditional probability that W_0^y revisits the point y, given that W_0^y does not visit any point of $(0, \delta]$ except y. We have

(3.120) $P\{W_0^y \text{ revisits } y\}$

$$= \pi(y, \delta)P\{W_0^y \text{ visits no point of } (0, \delta] \text{ except } y\} + P\{W_0^y \text{ revisits } y \text{ given that it visits some other point of } (0, \delta]\}$$
$$P\{W_0^y \text{ visits some point of } (0, \delta] \text{ other than } y\}$$
$$\leqq \pi(y, \delta) + P\{W_0^y \text{ visits some point of } (0, \delta] \text{ other than } y\}$$
$$\leqq \pi(y, \delta) + \{\text{Expected number of visits by } W_0^y \text{ to points of } (0, \delta] \text{ other than } y\}$$
$$\leqq \pi(y, \delta) + Q(\delta, y) - R_0 \leqq \pi(y, \delta) + R(\delta) - R_0$$
$$\leqq \pi(y, \delta) + \epsilon,$$

because of (3.50), (3.117), and (3.119). Also, if π_0 denotes the probability that W_0^y revisits y before it visits $[-\infty, y)$, then

(3.121) $$R_0 = \frac{1}{1 - \pi_0}$$

and

(3.122) $$\pi_0 \leqq P\{W_0^y \text{ revisits } y\}.$$

From (3.120), (3.121), and (3.122), we deduce

(3.123) $$\pi(y, \delta) \geqq 1 - R_0^{-1} - \epsilon.$$

Let k denote the number of times W_0^y revisits the point y; and let E^* denote the conditional expectation operator given that W_0^y does not visit any point of $(0, \delta]$ except y. For $0 < \theta \leqq 1$ define

(3.124) $$J(\theta) = E^*\left\{\sum_{j=0}^{k} \exp\left[-jt\theta^{\rho-2}\right]\right\}.$$

By the definition of $\pi(y, \delta)$, we have

(3.125) $$J(\theta) = \sum_{j=0}^{\infty} [\pi(y, \delta)]^j \exp\left[-jt\theta^{\rho-2}\right]$$
$$= \{1 - \pi(y, \delta) \exp\left[-t\theta^{\rho-2}\right]\}^{-1}$$
$$\geqq \{1 - (1 - R_0^{-1} - \epsilon) \exp\left[-t\theta^{\rho-2}\right]\}^{-1}$$

by (3.123). Also let $k(n)$ denote the number of times W_0^y revisits the point y within the first n steps of W^y. Define

(3.126) $$J_n(\theta) = E^*\left\{\sum_{j=0}^{k(n)} \exp\left[-jt\theta^{\rho-2}\right]\right\}.$$

As $n \to \infty$, $k(n) \to k$. Also the series in (3.126) is dominated by the convergent geometric series obtained by replacing $k(n)$ by ∞. Hence, for $n = 1, 2, \cdots$, $J_n(\theta)$ is a nondecreasing sequence such that $J_n(\theta) \to J(\theta)$ as $n \to \infty$. By (3.125) and (3.126),

(3.127) $$\lim_{n\to\infty} J_n(\theta) \geqq \{1 - (1 - R_0^{-1} - \epsilon) \exp\left[-t\theta^{\rho-2}\right]\}^{-1}.$$

We recall that Z_s is defined to be min $(z_0, z_1, \cdots, z_s)$; and we define

$$(3.128)\qquad I(z, z_s, Z_s) = \begin{cases} 1 & \text{if } z \geqq z_s \geqq Z_s > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Then, by (3.78) and (3.92) we have, for $x = z_0 > z$,

$$(3.129)\qquad Q\left[z, x, \frac{t}{V(z)}\right] = E\left[\sum_{s=1}^{\infty} I(z, z_s, Z_s) \exp\left\{-\frac{t}{V(z)}\sum_{r=0}^{s-1} V(z_r)\right\}\right].$$

Next define

(3.130)

$$Q^*\left[z, x, \frac{t}{V(z)}\right] = E\left[\sum_{s=1}^{\infty} I(z, z_s, Z_s) \exp\left\{-\frac{t}{V(z)}\sum_{r=0}^{s-1} H(\delta - Z_r)V(z_r)\right\}\right].$$

For $z < \delta$ we have

$$\begin{aligned}(3.131)\qquad & z^{-1}\left\{Q^*\left[z, x, \frac{t}{V(z)}\right] - Q\left[z, x, \frac{t}{V(z)}\right]\right\} \\ &= E\left[z^{-1}\sum_{s=1}^{\infty} I(z, z_s, Z_s) \exp\left\{-\frac{t}{V(z)}\sum_{r=0}^{s-1} H(\delta - Z_r)V(z_r)\right\}\right. \\ &\qquad\qquad \left.\left(1 - \exp\left\{-\frac{t}{V(z)}\sum_{r=0}^{s-1}[1 - H(\delta - Z_r)]V(z_r)\right\}\right)\right] \\ &\leqq E\left[z^{-1}\sum_{s=1}^{\infty} I(z, z_s, Z_s)\left(1 - \exp\left\{-\frac{t}{V(z)}\sum_{r=0}^{s-1}[1 - H(\delta - Z_r)]V(z_r)\right\}\right)\right] \\ &\leqq E\left[\frac{t}{zV(z)}\sum_{s=1}^{\infty} I(z, z_s, Z_s)\sum_{r=0}^{s-1}\{1 - H(\delta - Z_r)\}V(z_r)\right] \\ &= ta^{\rho-2}z^{1-\rho}E\left[\sum_{s=1}^{\infty} I(z, z_s, Z_s)\sum_{r=0}^{s-1}\{1 - H(\delta - Z_r)\}V(z_r)\right],\end{aligned}$$

by (3.99) and (3.119). Also, since $z < \delta$,

$$\begin{aligned}(3.132)\qquad & E\left[\sum_{s=1}^{\infty} I(z, z_s, Z_s)\sum_{r=0}^{s-1}\{1 - H(\delta - Z_r)\}V(z_r)\right] \\ &\leqq E\left[\sum_{s=1}^{\infty} I(\delta, z_s, Z_s)\sum_{r=0}^{s-1}\{1 - H(\delta - Z_r)\}V(z_r)\right] \\ &= E_\delta\left[\sum_{s=1}^{\infty} I(\delta, z_s, Z_s)\sum_{r=0}^{s-1}\{1 - H(\delta - Z_r)\}V(z_r)\right]L(\delta, x),\end{aligned}$$

where E_δ denotes the conditional expectation that W_0^x visits $(0, \delta]$ at least once. The last step in (3.132) follows from the consideration that, if W_0^x never visits $(0, \delta]$, then every factor $I(\delta, z_s, Z_s)$ is zero. We propose to prove that the right side of (3.132) is finite; and accordingly we may suppose without loss of generality that

$$(3.133)\qquad L(\delta, x) > 0.$$

Given that W_0^x visits $(0, \delta]$ at least once, write $y = z_\sigma$ for the first visit to $(0, \delta]$.

Then

$$\sum_{r=0}^{s-1} \{1 - H(\delta - Z_r)\} V(z_r) = \sum_{r=0}^{\sigma-1} V(z_r), \qquad s \geqq \sigma, \tag{3.134}$$

and

$$I(\delta, z_s, Z_s) = 0, \qquad s < \sigma. \tag{3.135}$$

It follows that

$$E_\delta\left[\sum_{s=1}^{\infty} I(\delta, z_s, Z_s) \sum_{r=0}^{s-1} \{1 - H(\delta - Z_r)\} V(z_r)\right] = E_\delta\left[N(y, \delta) \sum_{r=0}^{\sigma-1} V(z_r)\right], \tag{3.136}$$

where $N(y, \delta)$ is the number of visits to $(0, \delta]$ by W_0^x given that the first visit to $(0, \delta]$ is at y. However,

$$N(y, \delta) \leqq 1 + N^*(\delta), \tag{3.137}$$

where $N^*(\delta)$ is the number of visits to $(y - \delta, y + \delta]$ by $W_{y-\delta}^x$, excluding the first visit which occurred at $z_\sigma = y$. Moreover, $N^*(\delta)$ depends only on $z_{s+1} - z_s$ for $s \geqq \sigma$, and is therefore independent of $\sum_{r=0}^{\sigma-1} V(z_r)$. Consequently,

$$\begin{aligned} E_\delta\left[N(y, \delta) \sum_{r=0}^{\sigma-1} V(z_r)\right] &\leqq E_\delta\left[\{1 + N^*(\delta)\} \sum_{r=0}^{\sigma-1} V(z_r)\right] \\ &= E_\delta [1 + N^*(\delta)] E_\delta\left[\sum_{r=0}^{\sigma-1} V(z_r)\right] \\ &= [L(\delta, x)]^{-2}\{E_\delta[1 + N^*(\delta)] L(\delta, x)\} \left\{E_\delta\left[\sum_{r=0}^{\sigma-1} V(z_r)\right] L(\delta, x)\right\} \\ &\leqq [L(\delta, x)]^{-2}\{E[1 + N^*(\delta)]\} \left\{E\left[\sum_{r=0}^{\sigma-1} V(z_r)\right]\right\}, \end{aligned} \tag{3.138}$$

by two applications of (3.83). From the definition of $N^*(\delta)$, we have

$$E[1 + N^*(\delta)] = Q(2\delta, \delta) = R(\delta). \tag{3.139}$$

When $r \leqq \sigma - 1$, $z_r > \delta$ by the definition of σ. Hence, if

$$V^*(z) = \begin{cases} V(z), & z > \delta, \\ 0, & z \leqq \delta, \end{cases} \tag{3.140}$$

we have

$$\sum_{r=0}^{\sigma-1} V(z_r) \leqq \sum_{r=0}^{\infty} I(+\infty, z_r, Z_r) V^*(z_r). \tag{3.141}$$

Thus

$$(3.142)\qquad E\left[\sum_{r=0}^{\sigma-1} V(z_r)\right] \leqq \int_0^\infty V^*(z)\, dQ(z, x) = \int_\delta^\infty V(z)\, dQ(z, x) \leqq V(\delta)Q(\zeta, x) + M(x)$$

by (3.141), (3.140), (3.119), and (3.100). Collecting (3.131), (3.132), (3.136), (3.138), (3.139), and (3.142), we obtain

$$(3.143)\qquad z^{-1}\left\{Q^*\left[z, x, \frac{t}{V(z)}\right] - Q\left[z, x, \frac{t}{V(z)}\right]\right\} \leqq \begin{cases} 0 & \text{if } L(\delta, x) = 0 \\ \dfrac{ta^{\rho-2}z^{1-\rho}R(\delta)\{V(\delta)Q(\zeta, x) + M(x)\}}{L(\delta, x)} & \text{if } L(\delta, x) > 0. \end{cases}$$

Since $\rho < 1$, we may conclude from (3.143) that

$$(3.144)\qquad \liminf_{z\to 0} z^{-1}Q\left[z, x, \frac{t}{V(z)}\right] \geqq \liminf_{z\to 0} z^{-1}Q^*\left[z, x, \frac{t}{V(z)}\right].$$

Next in (3.83), we take $\mathfrak{e}$ to be the event that W_0^x does not visit more than one point of $(0, \delta]$, and we take X to be the expression in square brackets on the right of (3.130). By (3.118), we get

$$(3.145)\qquad Q^*\left[z, x, \frac{t}{V(z)}\right] \geqq (1 - \epsilon)E'\left[\sum_{s=1}^{\infty} I(z, z_s, Z_s) \exp\left\{-\frac{t}{V(z)}\sum_{r=0}^{s-1} H(\delta - Z_r)V(z_r)\right\}\right],$$

where E' denotes conditional expectation given that W_0^x visits $(0, \delta]$ at most at a single point, say the point $y = \theta z$. Here we may suppose that $0 < \theta \leqq 1$, since the factor $I(z, z_0, Z_s)$ will be zero for all s when $\theta > 1$. The right side of (3.145) will not be increased if we make the following substitutions: (a) $\sum_{s=1}^{\sigma+n}$ for $\sum_{s=1}^{\infty}$; (b) $V(\delta)$ for $V(z_r)$ when $z_r > \delta$; and (c) $V(\delta) + V(y)$ for $V(z_r)$ when $z_r \leqq \delta$, that is, when $z_r = y$. With these substitutions, we get from (3.126) and (3.99),

$$(3.146)\qquad Q^*\left[z, x, \frac{t}{V(z)}\right] \geqq (1 - \epsilon)\int_0^z \exp\left\{-\frac{tnV(\delta)}{V(z)}\right\}J_n(\theta)\, dL(y, x).$$

We have already noted that $J_n(\theta)$ is a bounded function of θ; and (3.126) shows that it is a nondecreasing function of θ. Thus we may integrate the right side of (3.146) by parts to obtain

$$(3.147)\qquad z^{-1}Q^*\left[z, x, \frac{t}{V(z)}\right] \geqq (1 - \epsilon)\exp\left\{-tn\left(\frac{z}{\delta}\right)^{2-\rho}\right\}\left\{\frac{L(z, x)}{z}J_n(1) - \int_{y=0}^{z} \frac{y}{z}\frac{L(y, x)}{y}dJ_n\left(\frac{y}{z}\right)\right\}.$$

In the ensuing analysis we assume that x is such that $q(x)$ exists. When $q(x)$

exists, $l(x)$ exists as already proved. We let $z \to 0$ in (3.147), which entails $y \to 0$ because $0 < y \leqq z$, and invoke (3.62) and Lebesgue's bounded convergence theorem to deduce

$$\liminf_{z\to 0} z^{-1}Q^*\left[z, x, \frac{t}{V(z)}\right] \geqq (1-\epsilon)l(x)\left\{J_n(1) - \int_0^1 \theta dJ_n(\theta)\right\}. \tag{3.148}$$

Next integrate by parts once more to obtain

$$\liminf_{z\to 0} z^{-1}Q^*\left[z, x, \frac{t}{V(z)}\right] \geqq (1-\epsilon)l(x)\int_0^1 J_n(\theta)\, d\theta. \tag{3.149}$$

The left side of (3.149) is independent of n. We may therefore let $n \to \infty$ in (3.149), and by a further use of Lebesgue's bounded convergence theorem conclude that

$$\begin{aligned}\liminf_{z\to 0} z^{-1}Q^*&\left[z, x, \frac{t}{V(z)}\right]\\ &\geqq (1-\epsilon)l(x)\int_0^1 \lim_{n\to\infty} J_n(\theta)\, d\theta\\ &\geqq (1-\epsilon)l(x)\int_0^1 \{1-(1-R_0^{-1}-\epsilon)\exp(-t\theta^{\rho-2})\}^{-1}\, d\theta,\end{aligned} \tag{3.150}$$

by virtue of (3.127). In view of (3.144) we may omit the asterisk on the left side of (3.150), and, when this is done, the left side becomes independent of ϵ. Hence we may let $\epsilon \to 0$ on the right side of (3.150), with the result that

$$\begin{aligned}\liminf_{z\to 0} z^{-1}Q&\left[z, x, \frac{t}{V(z)}\right]\\ &\geqq l(x)\int_0^1 \{1-(1-R_0^{-1})\exp(-t\theta^{\rho-2})\}^{-1}\, d\theta.\end{aligned} \tag{3.151}$$

For $0 < y \leqq \delta/2$, let $\omega(y)$ denote the probability that W_0^y revisits the point y. Then the expected number of visits by W_0^y to the point y, including the visit at $z_0 = y$, is

$$\sum_{j=0}^{\infty} [\omega(y)]^j = [1-\omega(y)]^{-1} \leqq Q(2y, y) = R(y) \leqq R_0 + \epsilon, \tag{3.152}$$

by (3.117) and (3.119). Hence,

$$\omega(y) \leqq 1 - (R_0+\epsilon)^{-1}. \tag{3.153}$$

Suppose $0 < z \leqq \delta/2$. If W_0^x visits $(0, z]$, let σ be the smallest integer such that $0 < Z_\sigma \leqq z$; and put $y = z_\sigma$. We do not decrease the right side of (3.129) if we replace $V(z_r)$ by 0 whenever $z_r \neq y$. Thus,

$$\begin{aligned}z^{-1}Q&\left[z, x, \frac{t}{V(z)}\right]\\ &\leqq \frac{1}{z}\int_0^z \sum_{j=0}^{\infty} [\omega(y)]^j \exp\left\{-\frac{jtV(y)}{V(z)}\right\} dL(y, x)\\ &\leqq \frac{1}{z}\int_0^z \left\{1-\left(1-\frac{1}{R_0+\epsilon}\right)\exp\left[-t\left(\frac{y}{z}\right)^{\rho-2}\right]\right\}^{-1} dL(y, x).\end{aligned} \tag{3.154}$$

On integration by parts,

$$(3.155)\qquad z^{-1}Q\left[z, x, \frac{t}{V(z)}\right] \leqq \frac{L(z, x)}{z}\left\{1 - \left(1 - \frac{1}{R_0 + \epsilon}\right)e^{-t}\right\}^{-1} - \int_{\theta=0}^{1} \theta\,\frac{L(\theta z, x)}{\theta z}\, d\left[\left\{1 - \left(1 - \frac{1}{R_0 + \epsilon}\right)\exp\left(-t\theta^{\rho-2}\right)\right\}^{-1}\right].$$

Letting $z \to 0$ and using (3.62) and Lebesgue's bounded convergence theorem, we obtain

$$(3.156)\qquad \limsup_{z\to 0} z^{-1}Q\left[z, x, \frac{t}{V(z)}\right] \leqq l(x)\left\{1 - \left(1 - \frac{1}{R_0 + \epsilon}\right)e^{-t}\right\}^{-1} - l(x)\int_{\theta=0}^{1} \theta\, d\left[\left\{1 - \left(1 - \frac{1}{R_0 + \epsilon}\right)\exp\left(-t\theta^{\rho-2}\right)\right\}^{-1}\right]$$

$$= l(x)\int_0^1 \left\{1 - \left(1 - \frac{1}{R_0 + \epsilon}\right)\exp\left(-t\theta^{\rho-2}\right)\right\}^{-1} d\theta.$$

Letting $\epsilon \to 0$ in (3.156) and combining the result with (3.151), we obtain the desired result (3.115) with

$$(3.157)\qquad S(t, x) = l(x)\int_0^1 \{1 - (1 - R_0^{-1})\exp(-t\theta^{\rho-2})\}^{-1}\, d\theta.$$

Now, by (3.112),

$$(3.158)\qquad 0 \leqq \frac{Q[a(u/t)^{1/(2-\rho)}, x, u]}{a(u/t)^{1/(2-\rho)}} = z^{-1}Q(z, x, u) \leqq z^{-1}Q(z, x);$$

and, whenever $q(x)$ exists, the right side of (3.158) is bounded by virtue of (3.66) and (3.11). Hence we may apply Lebesgue's bounded convergence theorem to (3.113) to obtain

$$(3.159)\qquad \lim_{u\to 0} u^{-1/(2-\rho)}\psi(x) = a\int_0^\infty S(t, x)t^{-1/(2-\rho)}e^{-t}\, dt,$$

because $\int_0^\infty t^{-1/(2-\rho)}e^{-t}\, dt$ is finite.

We shall now and hereafter relax the condition (3.99). By virtue of (3.4), for prescribed $\epsilon > 0$ we can find a number $\zeta = \zeta(\epsilon)$ and two functions $V_+(z)$ and $V_-(z)$ such that

$$(3.160)\qquad \left(\frac{z}{a - \epsilon}\right)^{\rho-2} = V_-(z) \leqq V(z) \leqq V_+(z) = \left(\frac{z}{a + \epsilon}\right)^{\rho-2}$$

whenever $0 < z \leqq \zeta(\epsilon)$, while

$$(3.161)\qquad V_-(z) = V(z) = V_+(z)$$

whenever $z > \zeta(\epsilon)$. Next we define $G_\pm(x)$, $\phi_\pm(x)$, and $\psi_\pm(x)$ by using $V_\pm(z)$ in place of $V(z)$ in the definitions of $G(x)$, $\phi(x)$, and $\psi(x)$. Clearly, from these definitions

$$G_-(x) \leqq G(x) \leqq G_+(x),$$
$$(3.162) \qquad \phi_-(x) \geqq \phi(x) \geqq \phi_+(x),$$
$$\psi_-(x) \leqq \psi(x) \leqq \psi_+(x).$$

By (3.160) and (3.161), (3.3) and (3.99) apply to $V_\pm(z)$. Hence, by (3.159) and (3.162),

$$(3.163) \qquad (a - \epsilon) \int_0^\infty S(t, x) t^{-1/(2-\rho)} e^{-t}\, dt = \lim_{u \to 0} u^{-1/(2-\rho)} \psi_-(x)$$
$$\leqq \liminf_{u \to 0} u^{-1/(2-\rho)} \psi(x) \leqq \limsup_{u \to 0} u^{-1/(2-\rho)} \psi(x)$$
$$\leqq \lim_{u \to 0} u^{-1/(2-\rho)} \psi_+(x) = (a + \epsilon) \int_0^\infty S(t, x) t^{-1/(2-\rho)} e^{-t}\, dt.$$

On letting $\epsilon \to 0$ in (3.163), we recover (3.159), but this time without the assumption (3.99).

Next write

$$(3.164) \qquad F(g) = P\{G(x) \leqq g\}.$$

From (3.67)

$$(3.165) \qquad \phi(x, u) = E[e^{-uG(x)}] = \int_{0-}^\infty e^{-ug}\, dF(g);$$

and, by (3.69),

$$(3.166) \qquad \psi(x) = 1 - \int_{0-}^\infty e^{-ug}\, dF(g) = 1 + \int_{0-}^\infty e^{-ug}\, d[1 - F(g)]$$
$$= -\int_{0-}^\infty [1 - F(g)] d(e^{-ug}) = u \int_0^\infty [1 - F(g)] e^{-ug}\, dg$$
$$= u \int_0^\infty e^{-ug}\, d\left[\int_0^g \{1 - F(y)\}\, dy\right].$$

Note that the method of working used in (3.166) uses the fact that $F(+\infty) \leqq 1$, but does not assume that $F(+\infty) = 1$. The lower limit of integration is taken to be $0-$, so that $F(0-) = 0$ because $G(x) \geqq 0$. However, in the penultimate integral, $0-$ may be and has been replaced by 0 without affecting the value of the integral. By (3.159) and (3.166),

$$(3.167) \qquad \lim_{u \to 0} u^{(1-\rho)/(2-\rho)} \int_0^\infty e^{-ug}\, d\left[\int_0^g \{1 - F(y)\}\, dy\right]$$
$$= a \int_0^\infty S(t, x) t^{-1/(2-\rho)} e^{-t}\, dt.$$

We are now in a position to invoke two Tauberian theorems. The first such theorem ([7], theorem 4.3, p. 192) runs as follows. If $\alpha(t)$ is a nondecreasing function of t such that

$$(3.168) \qquad \lim_{s \to 0} s^\gamma \int_0^\infty e^{-st}\, d\alpha(t) = A,$$

where A is a constant and γ is a nonnegative constant, then

$$\lim_{t\to\infty} t^{-\gamma}\alpha(t) = \frac{A}{\gamma!}, \tag{3.169}$$

where $\gamma! = \Gamma(\gamma + 1)$. Since $1 - F(y) \geqq 0$, we see $\int_0^g \{1 - F(y)\}\, dy$ is a nondecreasing function; and we can apply the theorem to (3.167) to obtain

$$\lim_{g\to\infty} g^{-(1-\rho)/(2-\rho)} \int_0^g \{1 - F(y)\}\, dy = \frac{a}{\{(1-\rho)/(2-\rho)\}!} \int_0^\infty S(t,x) t^{-1/(2-\rho)} e^{-t}\, dt. \tag{3.170}$$

The second Tauberian theorem ([8], Hilfssatz 3, p. 517) runs as follows. If $\alpha(t)$ is a nondecreasing function of t such that

$$\lim_{s\to\infty} s^{-\gamma} \int_0^s \alpha(t)\, dt = A, \tag{3.171}$$

where A and γ are arbitrary real constants, then

$$\lim_{t\to\infty} t^{1-\gamma}\alpha(t) = A\gamma. \tag{3.172}$$

If we multiply (3.170) by -1, we can apply this second theorem to obtain

$$\lim_{g\to\infty} g^{1/(2-\rho)}\{F(g) - 1\} = \frac{-a(1-\rho)/(2-\rho)}{\{(1-\rho)/(2-\rho)\}!} \int_0^\infty S(t, x) t^{-1/(2-\rho)} e^{-t}\, dt. \tag{3.173}$$

From (3.164), (3.173), and (3.157) we get

$$\begin{aligned} \lim_{g\to\infty} g^{1/(2-\rho)} P\{G(x) > g\} &= \frac{al(x)}{\{-1/(2-\rho)\}!} \int_0^\infty dt\, e^{-t} t^{-1/(2-\rho)} \int_0^1 \frac{d\theta}{1 - (1 - R_0^{-1}) \exp(-t\theta^{\rho-2})} \\ &= \frac{al(x)}{\{-1/(2-\rho)\}!(2-\rho)} \int_0^\infty dt\, e^{-t} \int_t^\infty \frac{y^{(3-\rho)/(\rho-2)}dy}{1 - (1 - R_0^{-1})e^{-y}}, \end{aligned} \tag{3.174}$$

on making the substitution $y = t\theta^{\rho-2}$. We may invert the order of integration in (3.174) and obtain

$$\begin{aligned} \lim_{g\to\infty} g^{1/(2-\rho)} P\{G(x) > g\} &= \frac{al(x)}{\{-1/(2-\rho)\}!(2-\rho)} \int_0^\infty \frac{y^{(3-\rho)/(\rho-2)}dy}{1 - (1 - R_0^{-1})e^{-y}} \int_0^y e^{-t}\, dt \\ &= \frac{al(x)}{\{-1/(2-\rho)\}!(2-\rho)} \int_0^\infty \frac{y^{(3-\rho)/(\rho-2)}(1 - e^{-y})\, dy}{1 - (1 - R_0^{-1})e^{-y}}. \end{aligned} \tag{3.175}$$

If we write

$$0 \leqq \gamma = 1 - R_0^{-1} < 1, \qquad 0 \leqq \tau = (2-\rho)^{-1} < 1, \tag{3.176}$$

we have

$$\frac{1}{\{-1/(2-\rho)\}!(2-\rho)}\int_0^\infty \frac{y^{(3-\rho)/(\rho-2)}(1-e^{-y})\,dy}{1-(1-R_0^{-1})e^{-y}} \tag{3.177}$$
$$= \frac{\tau}{(-\tau)!}\int_0^\infty \frac{y^{-\tau-1}(1-e^{-y})\,dy}{1-\gamma e^{-y}}$$
$$= \frac{\tau}{(-\tau)!}\sum_{n=0}^{\infty}\int_0^\infty y^{-\tau-1}(1-e^{-y})\gamma^n e^{-ny}\,dy$$
$$= \frac{\tau}{(-\tau)!}\sum_{n=0}^{\infty}\gamma^n\{n^\tau-(n+1)^\tau\}\int_0^\infty Y^{-\tau-1}e^{-Y}\,dY$$
$$= \sum_{n=0}^{\infty}\gamma^n\{(n+1)^\tau-n^\tau\} = (1-\gamma)\sum_{n=0}^{\infty}\gamma^n(n+1)^\tau$$
$$= R_0^{-1}\sum_{n=0}^{\infty}(n+1)^{1/(2-\rho)}(1-R_0^{-1})^n.$$

The required result (3.6) now follows from (3.66), (3.175), and (3.177).

It remains to discuss equations (3.8) and (3.14). Both these equations are of the form

$$\chi(z,x) = \zeta(z,x) + \int_0^\infty \chi(z,y)\,dP(y-x), \tag{3.178}$$

where ζ is a nonnegative function. For brevity, we write (3.178) in the form

$$\chi = \zeta + \Lambda\chi \tag{3.179}$$

and study (3.179). The essential feature of the linear operator Λ is that it maps nonnegative functions into nonnegative functions. If χ is any nonnegative solution of (3.179), either finite or formally infinite, then by successive substitution,

$$\chi = \sum_{r=0}^{n}\Lambda^r\xi + \Lambda^{n+1}\chi \geqq \sum_{r=0}^{n}\Lambda^r\xi. \tag{3.180}$$

Since (3.180) holds for all n, we have

$$\chi \geqq \sum_{r=0}^{\infty}\Lambda^r\xi, \tag{3.181}$$

where the sum on the right, being composed of nonnegative terms, is either convergent or formally equal to $+\infty$. The inequality (3.181) shows that the Neumann solution of (3.179) is the minimal nonnegative solution.

Consider next the special case when P is such that with probability 1 there are only finitely many steps in $W_0^{\overline{z}}$. In particular, any symmetrical P satisfies this condition. Then (3.179) can only have at most one bounded solution. For, if χ_1 and χ_2 are two bounded solutions,

$$|\chi_1-\chi_2| = |\Lambda(\chi_1-\chi_2)| = |\Lambda^n(\chi_1-\chi_2)| \tag{3.182}$$
$$\leqq \{\sup|\chi_1-\chi_2|\}\Lambda^n 1 \to 0 \qquad \text{as} \quad n\to\infty,$$

because $\Lambda^n 1$ is the probability that W_0^z has at least n steps. The same conclusion holds under the weaker assumption that $\lim_{n\to\infty} \Lambda^n 1$ is bounded away from 1 for all x; for we can then take the supremum (over x) of both sides of (3.182), and deduce that $\sup |\chi_1 - \chi_2|$ does not exceed a proper fraction of itself and is therefore zero.

In general, without the assumptions of the previous paragraph, $\Lambda^r H(z - x)$ is the probability that W_0^z will visit $(0, z]$ at the rth step. Hence

$$Q(z, x) = \lim_{n\to\infty} \sum_{r=0}^{n} \Lambda^r H(z - x) = \sum_{r=0}^{\infty} \Lambda^r H(z - x) \tag{3.183}$$

is indeed the Neumann solution of (3.8).

Consider finally the case when $p(x)$ exists for all x, is a bounded function of x, and satisfies

$$P(y) - P(-\infty) = \int_{-\infty}^{y} p(x)\, dx. \tag{3.184}$$

Suppose further that there exist fixed numbers X, M, and ζ such that

$$z^{-1}[P(z - x) - P(-x)] \leqq M\zeta^{-1}[P(\zeta - x) - P(-x)] \tag{3.185}$$

whenever $x \geqq X$ and $0 < z \leqq \zeta$. The condition (3.185) is satisfied in particular if $p(-x)$ is a nonincreasing function of x for all sufficiently large x. Under the above assumptions we shall show that $q(x)$ exists for all $x > 0$ and is a bounded function of x.

Since $z^{-1}H(z - x) = 0$ for $0 < z < x$, we have

$$q(x) = \lim_{z\to 0} z^{-1}Q(z, x) = \lim_{z\to 0} z^{-1}[Q(z, x) - H(z - x)], \tag{3.186}$$

whenever either the second or the third term in (3.186) exists. Also, from (3.8),

$$\begin{aligned} Q(z, x) - H(z - x) &= \Lambda Q(z, x) = \Lambda H(z - x) + \Lambda[Q(z, x) - H(z - x)] \\ &= P(z - x) - P(-x) + \Lambda[Q(z, x) - H(z - x)]. \end{aligned} \tag{3.187}$$

Since $p(x)$ is bounded for all x, there exists a positive number M_0 such that

$$M_0 \geqq \sup_y p(y) \geqq z^{-1} \int_{-x}^{z-x} p(y)\, dy = z^{-1}[P(z - x) - P(-x)] \tag{3.188}$$

and therefore

$$P(z - x) - P(-x) \leqq zM_0 = zM_0H(X - x) \tag{3.189}$$

whenever $x \leqq X$. On the other hand, when $x \geqq X$, we can invoke (3.185). Thus by (3.185) and (3.189)

$$P(z - x) - P(-x) \leqq z\{M_0H(X - x) + M\zeta^{-1}[P(\zeta - x) - P(-x)]\} \tag{3.190}$$

for all x, provided only that $0 < z \leqq \zeta$. Now by (3.187) and (3.190),

$$\begin{aligned} &z^{-1}[Q(z, x) - H(z - x)] \\ &\qquad \leqq M_0H(X - x) + M\zeta^{-1}[P(\zeta - x) - P(-x)] \\ &\qquad\qquad + \Lambda\{z^{-1}[Q(z, x) - H(z - x)]\}. \end{aligned} \tag{3.191}$$

By successive substitution of the left side of (3.191) into the right side of (3.191), we find

$$z^{-1}[Q(z, x) - H(z - x)] \leq \sum_{r=0}^{n} \Lambda^r\{M_0 H(X - x) + M\zeta^{-1}[P(\zeta - x) - P(-x)]\} + \Lambda^{n+1}\{z^{-1}[Q(z, x) - H(z - x)]\}. \tag{3.192}$$

If we fix z and let $n \to \infty$ in (3.192),

$$\Lambda^{n+1}\{z^{-1}[Q(z, x) - H(z - x)]\} \to 0. \tag{3.193}$$

Hence, if $0 < z \leqq \zeta$,

$$\begin{aligned} z^{-1}[Q(z, x) - H(z - x)] &\leqq \sum_{r=0}^{\infty} \Lambda^r\{M_0 H(X - x) + M\zeta^{-1}[P(\zeta - x) - P(-x)]\} \\ &= M_0 Q(X, x) + M\zeta^{-1}[Q(\zeta, x) - H(\zeta - x)] \\ &\leqq M_0 Q(X, x) + M\zeta^{-1} Q(\zeta, x) \\ &\leqq M_0 R(X) + M\zeta^{-1} R(\zeta), \end{aligned} \tag{3.194}$$

by (3.50). Hence the left side of (3.194) is uniformly bounded for all sufficiently small z. Thus, if we divide (3.187) by z and let $z \to 0+$, we may invoke Lebesgue's bounded convergence theorem; and, by virtue of (3.184), the right side of (3.187), after division by z, will tend to the right side of (3.14), since $q(y)$ on the right side of (3.14) exists almost everywhere in y. This convergence of the right side of the divided form of (3.187), however, holds whatever the value of x. Hence the left side converges for all x; and therefore the third member of (3.186) exists for all x. Consequently $q(x)$ exists for all x; and, by letting $z \to 0$ in (3.194), we see that $q(x)$ is bounded.

4. Numerical results on cometary lifetimes

4.1. *General remarks.* Throughout this section of the paper we assume that Kepler's third law

$$V(z) = z^{-3/2} \tag{4.1}$$

holds. We give numerical results for the cumulative distribution function of the lifetime $G(x)$ of a comet with initial energy x, where $G(x)$ is defined by (3.5). We take the normal distribution

$$P(y) = (2\pi)^{-1/2} \int_{-\infty}^{y} e^{-t^2/2}\, dt \tag{4.2}$$

for the distribution of individual energy perturbations. Equations (4.1) and (4.2) prescribe absolute units of time and energy; and the final subsection of this section interprets these units in physical terms. The results, which are displayed in figures 1 to 6, are graphs of

$$F(g) = P\{G(x) > g\} \tag{4.3}$$

for

$$x = \frac{1}{2}, 1, 2, 4, 8. \tag{4.4}$$

In each graph the vertical linear axis is F and the horizontal logarithmic axis is g. Figures 1 to 5 deal respectively with the five individual values of x in (4.4) and exhibit

(a) Monte Carlo estimates of F, which are subject to sampling errors;

(b) the asymptotic form of F, given by (3.6), which only applies when g is large; and

(c) the corresponding results (2.7) for Brownian paths, which are known to be poor approximations for large g, though they should be tolerable approximations when g is small and x is large. Figure 6 consists of freehand curves, drawn from the combined evidence of figures 1 to 5; and thus summarizes the best estimates available to us on the distributions of $G(x)$.

4.2. *Monte Carlo estimates.* The Monte Carlo estimates were obtained by direct simulation on the Ferranti Mercury computer at the United Kingdom Atomic Energy Research Establishment at Harwell. The energy perturbations y_t in (3.2) were generated as pseudorandom normal deviates by the multiplicative congruential process

$$\begin{aligned} y_t &= -6 + 2^{-29} \sum_{j=12t-11}^{12t} \eta_j, \\ \eta_{j+1} &\equiv 3^{17}\, \eta_j \pmod{2^{29}}. \end{aligned} \tag{4.5}$$

In effect, (4.5) makes y_t the sum of 12 variates, each rectangularly distributed between $-1/2$ and $1/2$, which by virtue of the central limit theorem affords observations that are good approximations to (4.2). The quantities z_t and $G(x)$ were calculated according to (3.2), (3.5), and (4.1) inside the machine, which printed out $G(x)$ and T as soon as $z_t \leqq 0$. This done, the machine commenced work on the history of the next comet, using the last value of η in the history just ended to generate the first η of the new history. For the values of x in (4.4), the machine constructed the numbers of histories shown in table I, excluding

TABLE I

Value of x	Number of Histories
1/2	272
1	210
2	226
4	292
8	234
Total	1234

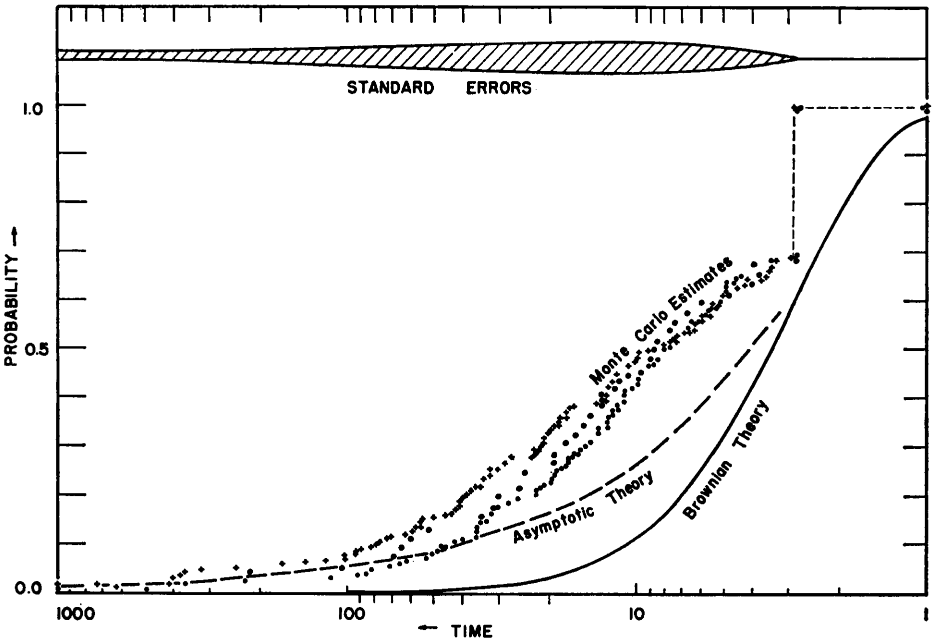

FIGURE 1
Graphs of probability $F(g) = P\{G(x) > g\}$ for $x = 1/2$.

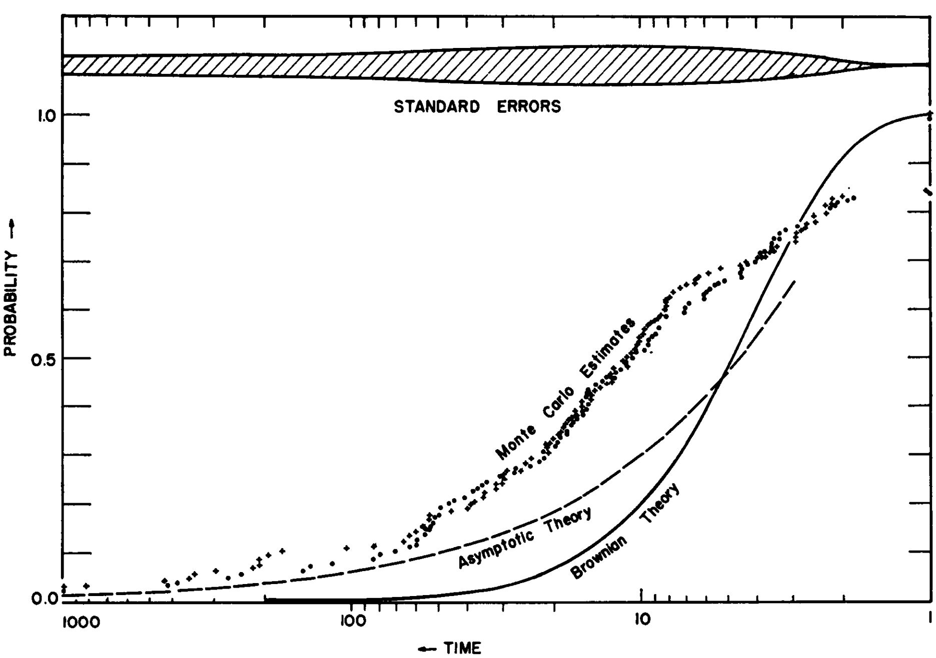

FIGURE 2
Graphs of probability $F(g) = P\{G(x) > g\}$ for $x = 1$.

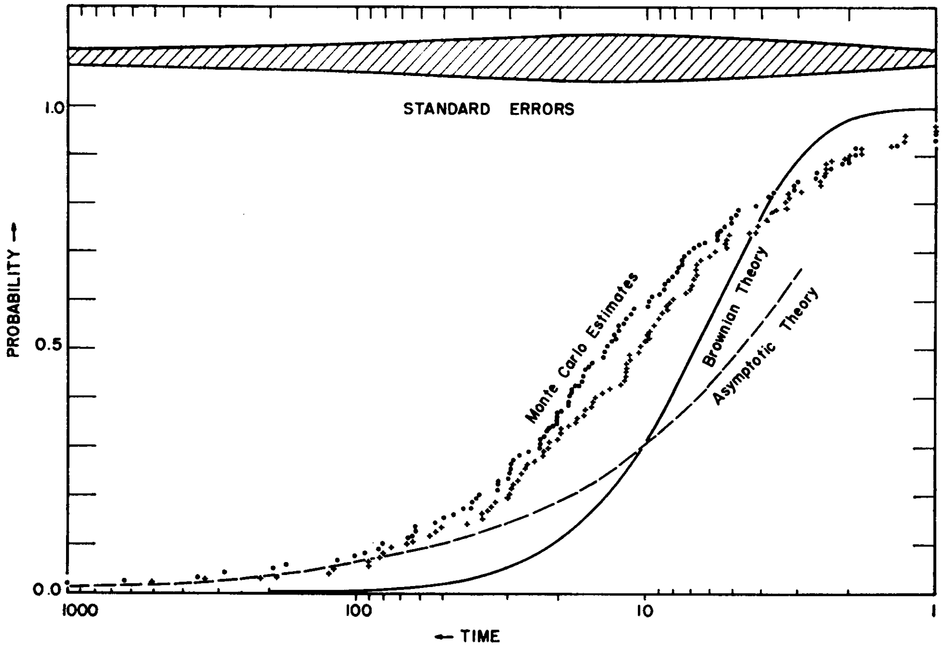

FIGURE 3

Graphs of probability $F(g) = P\{G(x) > g\}$ for $x = 2$.

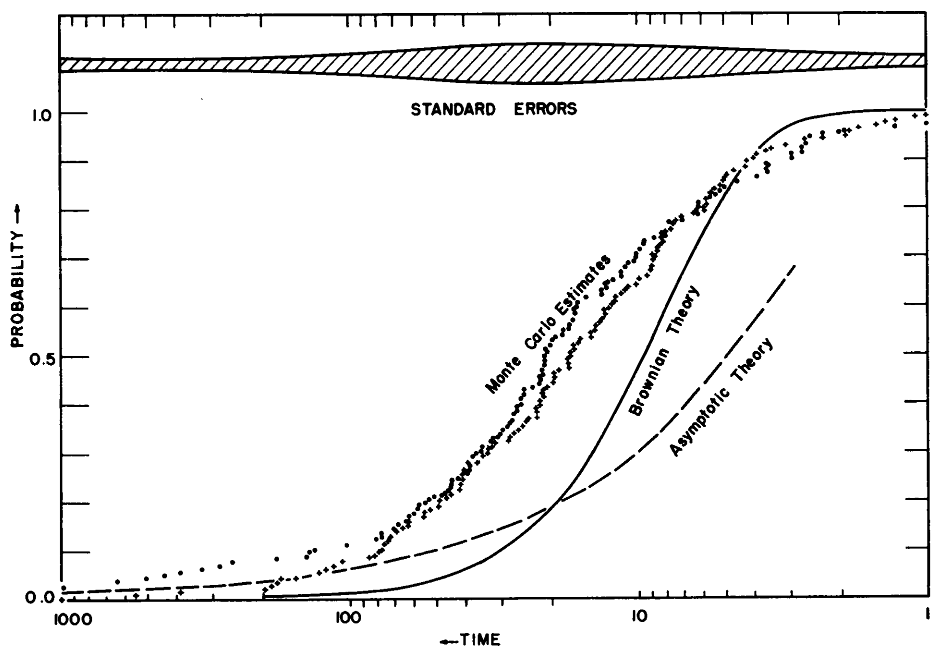

FIGURE 4

Graphs of probability $F(g) = P\{G(x) > g\}$ for $x = 4$.

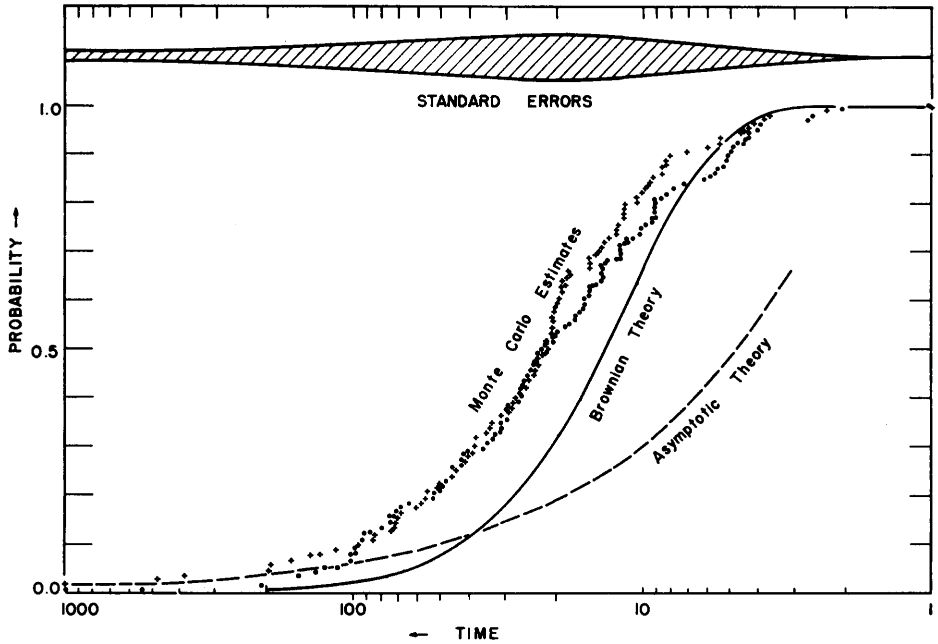

FIGURE 5

Graphs of probability $F(g) = P\{G(x) > g\}$ for $x = 8$.

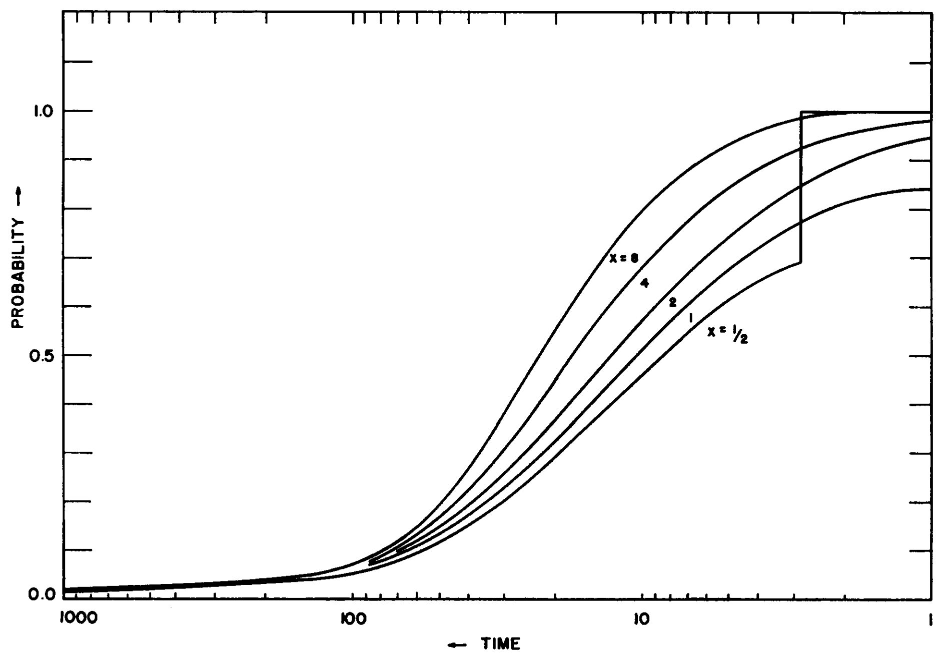

FIGURE 6

Summary curves showing distributions of $G(x)$.

histories for which $T = 1$. The point of excluding histories with $T = 1$ from the Monte Carlo work is that they can be handled analytically instead, thereby increasing the precision of the Monte Carlo results for $T > 1$. In fact, when $T = 1$ the lifetime is simply $x^{-3/2}$ and the probability of $T = 1$ is $P(-x) = 1 - P(x)$. Thus each curve satisfies

$$F(g) = \begin{cases} 1, & g < x^{-3/2}, \\ P(x), & g = x^{-3/2}. \end{cases} \tag{4.6}$$

This discontinuity for $x = 1/2$ is visible in figure 6. For $x = 1$, the discontinuity occurs at the extreme right side of figure 6; while, for larger values of x, the discontinuity is small or very small and occurs to the right of the parts of the curves shown in figure 6.

For each value of x, the histories were divided into two sets, each with equal numbers of histories. Each such set was treated separately to plot a conditional empiric distribution $F(g)$ for $g > x$, given that $F(x^{-3/2}) = P(x)$. One set provides the irregular curve of small crosses, and the other set the irregular curve of solid dots, on each of figures 1 to 5. The accuracy of the estimation may be judged partly from the discrepancy between the two curves, and partly from the shaded region giving the standard error at the top of each graph. This standard error was calculated simply as a binomial standard error,

$$\delta = \left\{\frac{F(g)[1 - F(g)]}{n}\right\}^{1/2}, \tag{4.7}$$

where n is the number of histories in a set. The curves which bound the shaded area are $\pm\delta$. Thus, for any prescribed value of g, the cross or solid dot curve might be expected not to deviate by more than about 2δ, that is, the whole height of the shaded region at the prescribed value of g, from the true curve $F(g)$. If g is not prescribed, but is instead selected to give, say, maximum discrepancy between the cross and the solid dot curve, a larger discrepancy than that indicated by the shaded region should be expected. Such a discrepancy could, of course, be computed by means of the Smirnov-Kolmogorov formulas; but this has not been done, since in the present case the standard error appears to give an adequate picture of the accuracy of the Monte Carlo estimates.

For $x = 1/2$, figure 1 also exhibits an irregular curve of open circles which is plotted from 35 histories generated by Mr. G. Logerman on the IBM computer at the California Institute of Technology. The work on the IBM machine was done before the other calculations. When the first really long history was encountered, it became apparent that a faster machine would be needed; and the remainder of the work was therefore done at Harwell. This explains why only 35 histories were done on the IBM machine and why values of $x > 1/2$ were not studied. Although the total number of orbits followed on the IBM machine was only 1,782, compared with 1,534,779 on Mercury, the IBM results compare extremely well with the Mercury results. The reason for this is provided by an examination of the Mercury results for long histories, which shows that the correlation between long histories and long lifetimes is negligible. The open

TABLE II

NUMBER OF HISTORIES WITH SPECIFIED VALUES OF T

	$x = \frac{1}{2}$	$x = 1$	$x = 2$	$x = 4$	$x = 8$
$T = 2$	74	25	17	1	–
$2 < T \leqq 4$	53	36	30	9	–
$4 < T \leqq 8$	44	36	29	24	1
$8 < T \leqq 16$	35	26	35	30	7
$16 < T \leqq 32$	25	31	28	63	24
$32 < T \leqq 64$	24	16	26	37	29
$64 < T \leqq 128$	15	11	16	28	34
$128 < T \leqq 256$	9	11	19	32	34
$256 < T \leqq 512$	10	8	7	27	33
$512 < T \leqq 1024$	6	6	8	18	24
$1024 < T \leqq 2048$	4	2	2	6	13
$2048 < T \leqq 4096$	4	1	5	8	12
$4096 < T \leqq 8192$	1	1	1	5	7
$8192 < T \leqq 16384$	2	–	1	1	3
$16384 < T \leqq 32768$	–	–	1	1	3
$32768 < T \leqq 65536$	1	–	1	1	5
$65536 < T \leqq 131072$	–	–	–	1	4
$131072 < T \leqq 262144$	–	–	–	–	1
Total	307	210	226	292	234
$\sum T$	93900	22226	129047	285623	1005765

TABLE III

PERCENTAGE OF COMETS WHICH DESCRIBE MORE THAN N ORBITS

N	$x = \frac{1}{2}$	$x = 1$	$x = 2$	$x = 4$	$x = 8$
1	69	84	98	100	100
2	52	74	90	100	100
4	41	60	77	97	100
8	31	45	65	88	100
16	23	35	50	78	97
32	17	22	38	57	86
64	12	16	26	44	74
128	8	12	19	34	59
256	6	7	11	23	45
512	4	4	8	14	31
1024	3	2	5	8	21
2048	2	1	4	6	15
4096	1	–	2	3	10
8192	1	–	1	1	7
16384	–	–	1	1	6
32768	–	–	–	1	4
65536	–	–	–	–	2
131072	–	–	–	–	–

circle curve provides a useful independent check of the other two curves. We are indebted to Mr. Logerman and Professor John Todd for making these results available to us.

The magnitude of the computation is governed principally by the long tail in the distribution of T, as shown in table II. The last line of table II shows the total number of orbits followed for each value of x. The table includes both the IBM and the Mercury results.

From table II we deduce table III, which estimates the percentage of comets that describe more than N orbits before being lost from the solar system.

The Monte Carlo work is, of course, aimed at producing figure 6; and table III is no more than a rather casual by-product. The standard error of any entry in table III is about seven-tenths of the corresponding standard error in figures 1 to 5; so the entries should usually lie within about ± 4 of the correct value. Thus useful information stems from the upper part of the table, where the percentages are, say, 10 or more. But in the lower part of table III the sampling errors swamp the estimates, as may be seen from the fact that the tabulated percentages do not, in every given row, increase steadily from left to right, as they ought.

In using (4.2) for the Monte Carlo work, we have so far made no allowance for a positive probability of a comet disintegrating at perihelion or being lost from the solar system for reasons other than energy perturbations. When such an allowance is made, the tails of the distributions in table III are sharply depressed. In fact, if κ^2 is the probability per orbit of loss for reasons other than energy perturbations, then each entry in table III must be multiplied by $(1 - \kappa^2)^N$. Tables IV, V, and VI show the result of applying this rule with $\kappa^2 = 0.01, 0.02, 0.04$. The effect on figure 6 of allowing for a positive κ^2 is not so easily assessed. There must be some depression of the tails of the lifetime distributions; but, since long lifetimes show negligible correlation with long histories, the depression will be much less marked than for tables IV, V, and VI when x

TABLE IV

PERCENTAGE OF COMETS WHICH DESCRIBE MORE THAN N ORBITS ($\kappa^2 = 0.01$)

N	$x = \frac{1}{2}$	$x = 1$	$x = 2$	$x = 4$	$x = 8$
1	68	83	97	99	99
2	51	73	88	98	98
4	39	58	74	93	96
8	29	42	60	81	92
16	20	30	43	66	83
32	12	16	28	41	62
64	6	8	14	23	39
128	2	3	5	9	16
256	–	1	1	2	3
512	–	–	–	–	–

TABLE V

Percentage of Comets which Describe more than N Orbits ($\kappa^2 = 0.02$)

N	$x = \frac{1}{2}$	$x = 1$	$x = 2$	$x = 4$	$x = 8$
1	68	82	96	98	98
2	50	71	86	96	96
4	38	55	71	89	92
8	26	38	55	75	85
16	17	25	36	56	70
32	9	12	20	30	45
64	3	4	7	12	20
128	1	1	1	3	4
256	–	–	–	–	–

TABLE VI

Percentage of Comets which Describe more than N Orbits ($\kappa^2 = 0.04$)

N	$x = \frac{1}{2}$	$x = 1$	$x = 2$	$x = 4$	$x = 8$
1	66	81	94	96	96
2	48	68	83	92	92
4	35	51	65	82	85
8	22	32	47	63	72
16	12	18	26	41	50
32	5	6	10	15	23
64	1	1	2	3	5
128	–	–	–	–	–

is small. But, when x is large, most comets will have been lost before they can be perturbed to small energies, and the tail will be considerably diminished. Further remarks on this question appear in the next subsection.

In figure 7 there is a simplified flow diagram of the Monte Carlo calculation. In this figure, primes denote new values of variates. For example, "$t' = 0$" means "set t to zero," while "$G' = G + z^{-3/2}$" means "replace G by $G + z^{-3/2}$." The following is a typical example of a question which could be asked by setting switches on the console to the appropriate question number: "If $z \leqq 8$, what are the current values of z and G and how far has t advanced since this question was last answered? If $z > 8$ and t has advanced by 1000 since this question was last answered, what are the current values of z and G? Otherwise, do not answer this question." With these explanations, the flow diagram should be self-explanatory.

4.3. *Asymptotic theory.* When (4.1) and (4.2) apply, (3.6) and (3.14) give

$$F(g) \sim g^{-2/3}q(x) \quad \text{as} \quad g \to \infty, \tag{4.8}$$

where

$$(2\pi)^{1/2}q(x) = e^{-x^2/2} + \int_0^\infty q(y)e^{-(x-y)^2/2}\,dy. \tag{4.9}$$

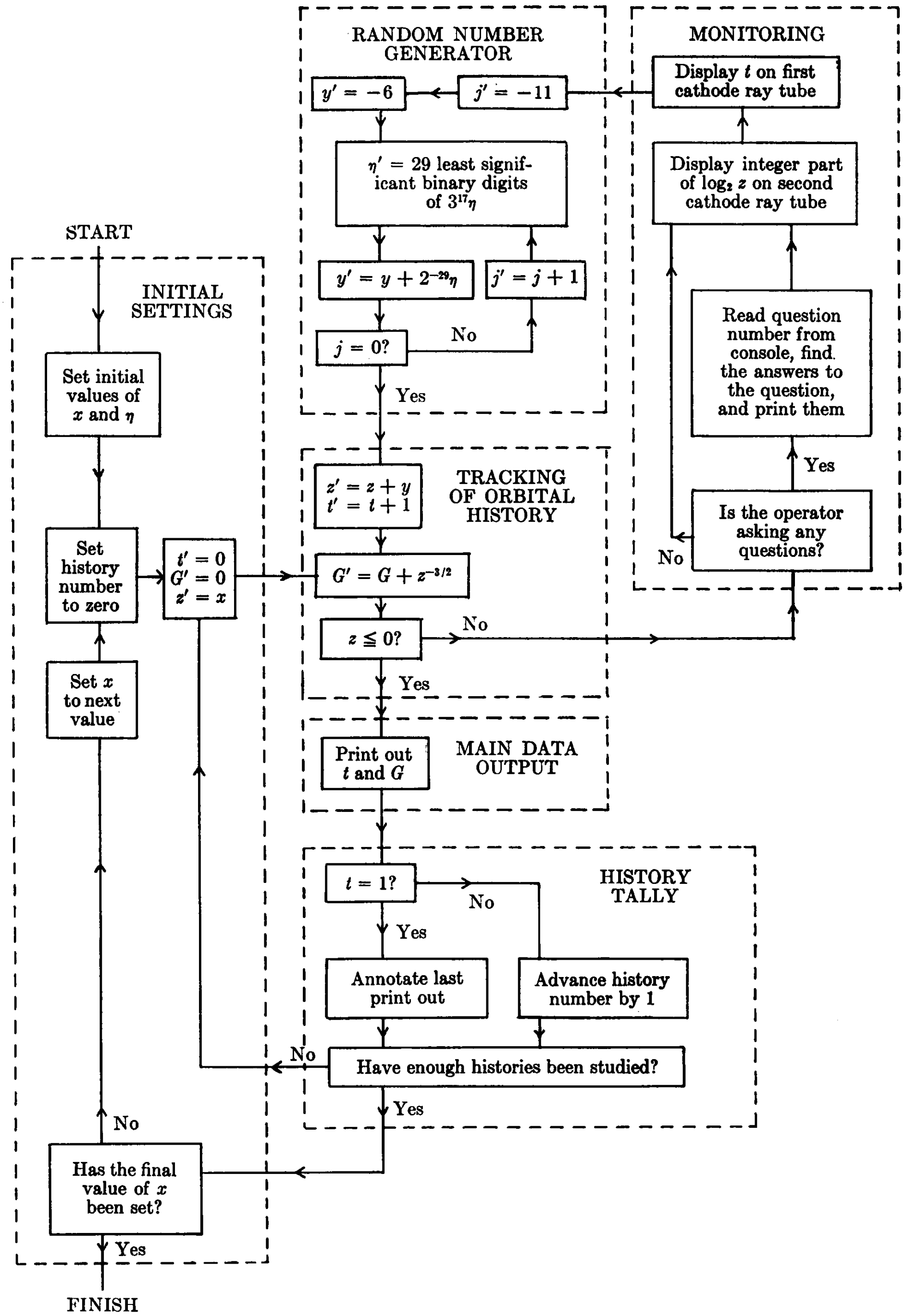

FIGURE 7

Simplified flow diagram of the Monte Carlo calculation.

From section 3 we know that we need a bounded solution of (4.9); and the symmetry of the normal distribution guarantees that there is only one bounded solution to (4.9).

The relevant curves in figures 1 to 5 were calculated from (4.8) after solving (4.9) on the Mercury computer at Oxford University; and we are indebted to Dr. L. Fox for help and advice on the numerical solution of (4.9), and to Mr. K. Wright, for preparing the relevant autocode program. The method of solution is as follows. Write

$$q_j = q(jh), \tag{4.10}$$

where j is a nonnegative integer and $h > 0$ is a fixed interval size. In the belief (now proved by D. G. Kendall) that $q(x)$ will tend to a limit as $x \to \infty$, we assume that for a sufficiently large n

$$q(x) = q_{2n}, \qquad x \geqq 2nh, \tag{4.11}$$

to an adequate degree of approximation. Applying Simpson's rule to (4.9), we can now write down a system of $2n + 1$ simultaneous equations

$$\begin{aligned}(2\pi)^{1/2}q_j &= \exp\left(\frac{-j^2h^2}{2}\right) + \frac{h}{3}\left\{q_0 \exp\left(\frac{-j^2h^2}{2}\right) + 4\sum_{k=1}^{n} q_{2k-1}\exp\left[\frac{-(j-2k+1)^2h^2}{2}\right]\right.\\ &\quad \left. + 2\sum_{k=1}^{n-1} q_{2k}\exp\left[\frac{-(j-2k)^2h^2}{2}\right] + q_{2n}\exp\left[\frac{-(j-2n)^2h^2}{2}\right]\right\}\\ &\quad + q_{2n}\int_{(2n-j)h}^{\infty} e^{-y^2/2}\,dy, \qquad j = 0, 1, \cdots, 2n.\end{aligned} \tag{4.12}$$

The $2n + 1$ possible values of the integral in (4.12), taken from tables [9], were written into the program as constants, and (4.12) was solved by the machine's matrix autocode. As a check on the accuracy of the solution, three different values of $h(= 0.1, 0.2, 0.4)$ were used and various values of $2nh(=2.0, 2.4, 3.2, 4.0, 5.0, 5.2, 6.0)$ were tried. The solution appears essentially stable when $h \leqq 0.2$ and $2nh \geqq 4.0$. Table VII gives the solution for $h = 0.1$ and $2nh = 5.0$.

We feel confident that the entries in table VII are correct to certainly four places of decimals, and probably to five places. The sixth decimal place is suspect; but we have quoted $q(x)$ to six places for reasons which will appear presently. The computation was carried through with 29 binary digits, excluding the sign digit, which is equivalent to about 8 1/2 significant figures in the decimal scale. The equations (4.12) are quite well conditioned, but there are 51 of them. To say that the results are accurate to five places of decimals is thus to assert that only 2 1/2 decimal digits will be lost by round-off and the use of Simpson's rule.

D. G. Kendall has shown that $(1/X)\int_0^X q(x)\,dx \to (2\pi)^{-1/2}\zeta(3/2) = 1.042187\cdots$ as $X \to 0$, where $\zeta(s)$ is the Riemann zeta function. The entry for $q(0)$ in table VII is thus too large by 3 units in the sixth decimal place.

TABLE VII

SOLUTION OF (4.9)

x	$q(x)$	x	$q(x)$	x	$q(x)$
0.0	1.042190	1.7	1.415767	3.4	1.414140
0.1	1.094685	1.8	1.416438	3.5	1.414155
0.2	1.143272	1.9	1.416647	3.6	1.414171
0.3	1.187616	2.0	1.416555	3.7	1.414186
0.4	1.227506	2.1	1.416288	3.8	1.414200
0.5	1.262853	2.2	1.415936	3.9	1.414210
0.6	1.293687	2.3	1.415563	4.0	1.414218
0.7	1.320143	2.4	1.415211	4.1	1.414224
0.8	1.342447	2.5	1.414904	4.2	1.414228
0.9	1.360902	2.6	1.414652	4.3	1.414230
1.0	1.375862	2.7	1.414458	4.4	1.414231
1.1	1.387719	2.8	1.414317	4.5	1.414231
1.2	1.396882	2.9	1.414222	4.6	1.414231
1.3	1.403756	3.0	1.414164	4.7	1.414230
1.4	1.408735	3.1	1.414134	4.8	1.414230
1.5	1.412183	3.2	1.414124	4.9	1.414229
1.6	1.414431	3.3	1.414128	5.0	1.414229

Table VII shows that $q(x)$ is close to $\sqrt{2} = 1.414214\cdots$ when x is large. This might have been expected on the following heuristic grounds. It is easy to see from a linear transformation of (3.14) that, if $p(x)$ is a probability density function with variance σ^2, the solution $q_\sigma(x)$ of (3.14) satisfies

$$\sigma q_\sigma(\sigma x) = q_1(x). \tag{4.13}$$

Hence, from (3.17), when $p(x)$ is a double exponential distribution with unit variance,

$$q(x) = \sqrt{2} \tag{4.14}$$

identically for all x. Now $q(x)$ represents the density of visits by the walk to a thin strip at $z = 0$, when the walk starts from x. If x is large, the walk will have to take many steps before reaching the strip; and the central limit theorem will operate to render unimportant the precise character of the distribution of individual steps. Thus $q(x)$ for large x should be insensitive to the form of $p(x)$; and (4.14) ought to hold asymptotically even when $p(x) = (2\pi)^{-1/2} \exp(-x^2/2)$.

What is at first sight highly surprising about table VII is the damped oscillatory character of $q(x)$. But this, too, can be explained by a heuristic argument, which predicts that the period of the oscillation should be about $(2\pi)^{1/2} = 2.51\cdots$. If the sixth decimal place in table VII can be trusted, at least relatively to neighboring entries if not absolutely, $q(x)$ exhibits maxima at about $x = 1.9$ and $x = 4.5$ together with a minimum at about $x = 3.2$, giving a period of about 2.6 in reasonable agreement with the prediction. We have to thank Mr. D. C. Handscomb for supplying us with the following argument in support of oscillations.

Suppose we approximate to a normal distribution by using a symmetric binomial distribution having steps $y = -n, -n+1, \cdots, n-1, n$. When we replace the integral equation (4.9) by the discrete analogue appropriate to the binomial, we have

$$(4.15) \qquad q_\sigma(x) = 2^{-2n}\left\{\binom{2n}{n-x} + \sum_{y>0}\binom{2n}{n+y-x}q_\sigma(y)\right\}, \qquad x = 1, 2, \cdots,$$

where any binomial coefficient in (4.15) is understood to be zero if the factorial of a negative integer appears formally in its denominator. The solution of the difference equation (4.15) will be of the form

$$(4.16) \qquad q_\sigma(x) = \sum_\theta a_\theta\, \theta^x,$$

and (4.16) will satisfy (4.15) for $x > n$ provided

$$(4.17) \qquad \theta^x = 2^{-2n}\sum_{y=x-n}^{x+n}\binom{2n}{n+y-x}\theta^y = \theta^{x-n}\left(\frac{1}{2}+\frac{1}{2}\theta\right)^{2n}.$$

Hence,

$$(4.18) \qquad 4\theta e^{2\pi i r/n} = (1+\theta)^2, \qquad r = 0, 1, \cdots, n-1.$$

Equation (4.18) for fixed $r > 0$, has two distinct roots θ, one inside the unit circle and one outside. Since $q_\sigma(x)$ is bounded, $a_\theta = 0$ when $|\theta| > 1$. When $r = 0$, (4.18) has a repeated root $\theta = 1$, leading to a term of the form $a + bx$ in (4.16); and here $b = 0$, because $q_\sigma(x)$ is bounded. Thus (4.16) becomes, writing $a = A/\sigma$,

$$(4.19) \qquad q_\sigma(x) = \frac{1}{\sigma}\left(A_0 + \sum_{r=1}^{n-1} A_r\theta_r^x\right),$$

where θ_r is the root of (4.18) satisfying $|\theta_r| < 1$. The coefficients $A_0, A_1, \cdots, A_{n-1}$ in (4.19) are determined in principle by substituting (4.19) into (4.15) and putting $x = 1, 2, \cdots, n$. For the binomial distribution used, $\sigma^2 = n/2$. Hence, by (4.13),

$$(4.20) \qquad q(x) = \sigma q_\sigma(x) = A_0 + \sum_{r=1}^{n-1} A_r\theta_r^{x(n/2)^{1/2}}.$$

We now let $n \to \infty$, so that the binomial distribution approximates to the normal distribution. Since $|\theta_r| < 1$, we know that $\theta_r^{x(n/2)^{1/2}}$ will be negligible unless $|\theta_r|$ is very close to 1, and this cannot occur unless either r is fixed as $n \to \infty$ or $n - r$ is fixed as $n \to \infty$. In these two cases we find, by solution of (4.18),

$$(4.21) \qquad \theta_r \sim \exp\left[2\left(\frac{\pi r}{n}\right)^{1/2}(-1 \pm i)\right] \quad \text{as} \quad n \to \infty.$$

Thus substitution of (4.19) into (4.20) and a little rearrangement of the terms gives for the solution of (4.9)

$$(4.22) \qquad q(x) = \sum_{r=0}^{\infty}\{\alpha_r\cos[(2\pi r)^{1/2}x] + \beta_r\sin[(2\pi r)^{1/2}x]\}\exp[-(2\pi r)^{1/2}x],$$

where α_r, β_r are constants. Equation (4.22) exhibits the damped oscillatory character of $q(x)$, whose dominant oscillation ($r = 1$) has period $(2\pi)^{1/2}$.

We consider next the appropriate allowance to make when there is probability κ^2 per orbit that the comet will be lost from the solar system owing to disintegration, and so on. Equation (4.8) is still valid; though, to emphasize the dependence upon κ, we write it in the form

$$F(g) \sim g^{-2/3}q^{(\kappa)}(x) \quad \text{as} \quad g \to \infty. \tag{4.23}$$

Instead of (4.9), however, we have

$$(2\pi)^{1/2}(1 - \kappa^2)^{-1}q^{(\kappa)}(x) = e^{-x^2/2} + \int_0^\infty q^{(\kappa)}(y)e^{-(x-y)^2/2}\,dy. \tag{4.24}$$

To see roughly how the solution of (4.24) behaves, write $x + h$ for x in (4.9). This gives after a little rearrangement

$$(2\pi)^{1/2}e^{h^2/2}[e^{hx}q(x + h)] = e^{-x^2/2} + \int_0^\infty [e^{hy}q(y)]e^{-(x-y)^2/2}\,dy. \tag{4.25}$$

Now, from table VII, $q(x)$ is a fairly constant function; so $\exp(hx)q(x + h)$, if h is small, will be approximately equal to $\exp(hx)q(x)$. With this substitution in (4.25), we have by comparison with (4.24)

$$q^{(\kappa)}(x) \cong e^{hx}q(x), \tag{4.26}$$

where

$$e^{h^2/2} = (1 - \kappa^2)^{-1} \cong e^{\kappa^2} \tag{4.27}$$

for small κ. Thus $h \cong \pm\,\kappa\sqrt{2}$; and, since $q^{(\kappa)}(x) \leqq q(x)$, we must choose the lower sign. Thus we expect

$$q^{(\kappa)}(x) \cong e^{-\kappa x\sqrt{2}}q(x) \tag{4.28}$$

for small κ. This approximate relation may be compared with the corresponding exact result for the double-exponential distribution with unit variance, which is available from (3.17) modified according to (4.13),

$$q^{(\kappa)}(x) = e^{-\kappa x\sqrt{2}}(1 - \kappa)\sqrt{2}. \tag{4.29}$$

Thus, except when x is large, disintegration with small κ will have little effect. By substituting (4.14) into (4.28), we see that for the double-exponential distribution (4.28) is correct to within the factor $1 - \kappa$.

The asymptotic theory is, of course, only valid for sufficiently large g. For the values of x studied ($x \leqq 8$), figures 1 to 5 indicate that the asymptotic theory is an adequate approximation when $g \geqq 100$, say.

4.4. *Brownian theory.* The Brownian theory curves in figures 1 to 5 are plotted from a direct evaluation of (2.7) on the Mercury computer at Harwell. We wish to thank Dr. J. Howlett for writing the relevant autocode and furnishing us with the numerical results. As expected, the Brownian theory turns out to be a poor approximation except when g is small and x is large.

4.5. *Physical size of units.* Hitherto in this section the unit of energy has been chosen equal to the standard deviation of an energy perturbation. Then, in terms of this absolute unit of energy, the time scale has been chosen so that the coefficient in Kepler's law (4.1) is 1. There are two advantages in this procedure. The first (and minor) advantage is that it simplifies the calculations and the formulas. The second (and major) advantage is that the calculations are independent of the physical observations relating to comets.

To interpret the results physically, however, we need to estimate the physical sizes of the absolute units. This we do below. The estimation is rather imprecise, because the relevant data on comets is somewhat sketchy, there being a number of practical difficulties against accurately determining the orbital parameters of a comet. If and when better observational data come to hand, one will have to re-estimate the physical sizes of the absolute units; but it will *not* be necessary to recalculate any of the Monte Carlo estimates or any of the asymptotic theory, and so on. In short, what we do now is merely to calibrate the theory.

D. G. Kendall [10] has discussed the magnitude of the energy perturbations, and quotes the following estimates of the standard deviation of the perturbation in $1/a$, where a is the length of the comet's semimajor axis in astronomical units (that is, $a = 1$ for the earth's orbit):

0.00076	(Halley's comet)
0.00079	(27 comets of Fayet)
0.00067	(20 comets of Galibina)
0.00078	(van Woerkom's analysis)

Kendall uses 0.00075 as a convenient single representative of these four figures. We make a slightly different choice, namely 0.000737, which leads to a round number in the time scale. In fact, since $20^{2/3} = 7.37\cdots$, we have $0.000737^{-3/2} = 5 \times 10^4$. Now (4.1) gives $V(z)$ in years when z is in astronomical units. Hence, if 1 absolute unit of z is equal to 0.000737 (astronomical units)$^{-1}$, 1 absolute unit of time will equal 5×10^4 years. Since $0.000737^{-1} = 1360$, a comet with an initial energy $x = 1$ in absolute units, has a semimajor axis of 1360 astronomical units, that is, about a hundred and twenty thousand million miles.

We can summarize the foregoing by saying that the unit of time in figures 1 to 6 is 50,000 years (so that the visible part of the time scale in each of these six figures runs from 50 thousand to 50 million years), and that the lengths of the semimajor axes of comets with specified values of x are

Value of x	*Semimajor axis in astronomical units*
1/2	2700
1	1400
2	680
4	340
8	170

Taking $q(x) = \sqrt{2}$, we see that quite a good rule of thumb for long-period comets is the following: "The probability that a long-period comet will remain in the solar system for at least M million years is about $0.2M^{-2/3}$, provided that $M \geqq 5$." For example, five per cent of comets have lifetimes of at least eight million years. A rough conversion rule for energies is that a comet with an initial energy of x absolute units has a semimajor axis of length $12 \times 10^{10}/x$ miles.

5. The accumulation of comets in the solar system

In this section we make the same assumptions as in section 4; and, except when the contrary is explicitly stated, our units of time and energy are the absolute units defined in section 4. For the next part, we confine our attention to comets with a given initial energy x.

Let θ denote time measured from the formation of the solar system, and let θ_0 be the present time, that is, the present age of the solar system. Suppose that in the time interval $(\theta, \theta + d\theta)$, there is a probability $\lambda(\theta)\, d\theta$ that a comet (with the given initial energy x) will enter the solar system. Thus the input of comets to the solar system is assumed to be describable in terms of a Poisson process with a time-dependent parameter $\lambda(\theta)$. If a comet enters during the interval $(\theta, \theta + d\theta)$, the probability that it will not have been lost from the solar system before θ_0 is

$$F(\theta_0 - \theta) = P\{G(x) > \theta_0 - \theta\}; \tag{5.1}$$

and hence the probability that the solar system now contains a comet which entered during $(\theta, \theta + d\theta)$ is $F(\theta_0 - \theta)\lambda(\theta)\, d\theta$. To get the total number M of comets in the solar system at the present time, we must sum over all θ satisfying $0 \leqq \theta < \theta_0$. It follows that M has a Poisson distribution with parameter

$$m = m(\theta_0) = \int_0^{\theta_0} F(\theta_0 - \theta)\lambda(\theta)\, d\theta. \tag{5.2}$$

Let us consider first the special case when $\lambda(\theta)$ is a constant λ_0. Then

$$m = \lambda_0 \int_0^{\theta_0} F(\theta_0 - \theta)\, d\theta = \lambda_0 \int_0^{\theta_0} F(g)\, dg, \tag{5.3}$$

and this can be evaluated in terms of the information about F provided by section 4. We saw in section 4 that, if $g \geqq 100$, then to an adequate degree of approximation

$$F(g) = q(x)g^{-2/3}. \tag{5.4}$$

Since the age of the solar system is at least 100 absolute units (= 5 million years), we have from (5.3) and (5.4)

$$\begin{aligned} m(\theta_0) &= \lambda_0\left\{3q(x)\theta_0^{1/3} + \int_0^{100} [F(g) - q(x)g^{-2/3}]\, dg\right\} \\ &= \lambda_0\{3q(x)\theta_0^{1/3} + \Delta(x)\}, \qquad \theta_0 \geqq 100, \end{aligned} \tag{5.5}$$

where $\Delta(x)$ is defined by (5.5). By numerical integration of the curves in figure 6 of section 4, we obtain the values for $\Delta(x)$ given in table VIII, the last decimal place being of doubtful significance. Equation (5.5) may be converted into

TABLE VIII

VALUES OF $\Delta(x)$

x	$\frac{1}{2}$	1	2	4	8
$\Delta(x)$	2.56	2.95	4.64	8.07	12.29

physical terms on recalling from section 4 the values of $q(x)$ in table VII and the fact that one absolute unit of time is 50,000 years. In fact, let the age of the solar system be θ_1 thousand million years, let $\lambda_1(x)$ be the average number of comets per annum which enter the solar system with energy x absolute units, and let $m_1(x)$ be the number of comets in millions which are in the solar system today having originally entered it with initial energy x absolute units. Then

$$m_1(x) = \{a(x)\theta_1^{1/3} + b(x)\}\lambda_1(x), \qquad \theta_1 \geqq 0.005, \tag{5.6}$$

where $a(x)$ and $b(x)$ have the values shown in table IX. Then table X gives values

TABLE IX

VALUES OF $a(x)$ AND $b(x)$

x	$\frac{1}{2}$	1	2	4	8
$a(x)$	5.141	5.601	5.767	5.757	5.757
$b(x)$	0.128	0.147	0.232	0.403	0.614

TABLE X

VALUES OF m_1/λ_1

θ_1	$x = \frac{1}{2}$	$x = 1$	$x = 2$	$x = 4$	$x = 8$
3	7.54	8.22	8.55	8.70	8.92
4	8.29	9.04	9.38	9.54	9.75
5	8.92	9.72	10.09	10.25	10.46
6	9.47	10.32	10.71	10.86	11.07

of m_1/λ_1, as a function of x and θ_1. The values in table X make no allowance for a disintegration effect. If κ^2 is the probability of disintegration per orbit, then, as remarked in section 4, $q(x)$ must be multiplied by $\exp(-\kappa x\sqrt{2})$ approximately. Applying this correction to table X we obtain tables XI, XII, and XIII.

In each of tables X to XIII there is only a weak dependence of m_1/λ_1 upon θ_1 and, as already noted in section 4, there is not much dependence on κ^2 unless x is large. If we suppose that the larger values of x are less frequent than the

TABLE XI

VALUES OF m_1/λ_1 WITH $\kappa^2 = 0.01$

θ_1	$x = \frac{1}{2}$	$x = 1$	$x = 2$	$x = 4$	$x = 8$
3	7.03	7.13	6.47	4.94	2.88
4	7.73	7.85	7.07	5.42	3.15
5	8.31	8.44	7.61	5.82	3.38
6	8.83	8.96	8.08	6.17	3.58

TABLE XII

VALUES OF m_1/λ_1 WITH $\kappa^2 = 0.02$

θ_1	$x = \frac{1}{2}$	$x = 1$	$x = 2$	$x = 4$	$x = 8$
3	6.82	6.73	5.73	3.91	1.80
4	7.50	7.40	6.28	4.28	1.97
5	8.07	7.96	6.76	4.60	2.11
6	8.57	8.45	7.18	4.88	2.24

TABLE XIII

VALUES OF m_1/λ_1 WITH $\kappa^2 = 0.04$

θ_1	$x = \frac{1}{2}$	$x = 1$	$x = 2$	$x = 4$	$x = 8$
3	6.54	6.20	4.86	2.81	0.93
4	7.20	6.82	5.33	3.08	1.01
5	7.74	7.33	5.73	3.31	1.09
6	8.22	7.78	6.08	3.51	1.15

smaller values, and such a supposition seems to be in line with the mechanism by which comets enter the solar system, then the present total number of comets (of any initial energy) in the solar system should be between $5\lambda_2$ million and $10\lambda_2$ million, where λ_2 is the annual entry rate of comets irrespective of initial energy.

It remains to make some estimate of λ_2. There are no observations of λ_2 in the remote past; but we are in any case assuming in the present analysis that λ_2 is constant, so we can take λ_2 to be the present rate of entry of comets into the solar system. Next, from (5.6), the net present rate of accumulation of comets in the solar system, is $(1/3)a(x)\theta_1^{-2/3}\lambda_2$ million per thousand million years, that is, $(1/3)a(x)\theta_1^{-2/3}\lambda_2 \times 10^{-3}$ comets per annum. Since this quantity is small in comparison with λ_2 we can say that at the present time the annual number of new comets is effectively equal to the annual loss of comets. Hence we can take λ_2 as the number of comets lost per annum.

Even so, it is not at all easy to observe or decide how many comets on average are lost each year. Nevertheless, the following extremely crude argument might indicate the order of magnitude involved. We know first from Galibina's data that postorbit perturbations have a algebraic mean about equal to their standard deviation, namely +0.0005 (astronomical units)$^{-1}$, the bias being such that, in their onward passage from perihelion, orbits tend to become less hyperbolic. Thus a comet with an elliptic perihelion orbit is unlikely to have a hyperbolic postorbit; and it will more or less suffice to say that a necessary (though not sufficient) condition for loss is a hyperbolic perihelion orbit. Accordingly, we confine attention to the observed hyperbolic perihelion orbits afforded by Strömgren's data. In this data there seems to be a suspicious lack of observations before 1880; and we therefore deal only with the years 1886 to 1936, the latter date being determined by the date of Strömgren's paper. This gives 19 comets in 50 years, as shown in table XIV. This suggests that about 8.7 comets should have been

TABLE XIV

STRÖMGREN'S DATA

Comet	Perihelion Orbit $1/a$	Chance of Hyperbolic Postorbit
1886 I	−.0007	0.66
1886 II	−.0005	0.50
1886 IX	−.0006	0.58
1889 I	−.0007	0.66
1890 II	−.0002	0.27
1897 I	−.0009	0.79
1898 VIII	−.0006	0.58
1902 III	+.0001	0.12
1904 I	−.0005	0.50
1905 VI	−.0001	0.21
1907 I	−.0005	0.50
1908 III	−.0007	0.66
1914 V	−.0001	0.21
1922 II	−.0004	0.42
1925 I	−.0006	0.58
1925 VII	−.0003	0.34
1932 VI	−.0006	0.58
1936 I	−.0005	0.50
		Total 8.66

lost in the course of 50 years, in which case $\lambda_2 = 0.17$. Presumably in these 50 years some comets have gone unobserved (for a variety of reasons ranging from overcast skies to the First World War); so the figure 0.17 will represent a lower bound for λ_2. The rough order of magnitude of λ_2 should, however, be about $\lambda_2 = 1/4$. An upper bound for λ_2 is about 3 or 4, namely the total number of comets observed each year.

We conclude that the number of comets in the present-day solar system, which have perihelion distances sufficiently small for them to become eventually observable, is perhaps about two or three million and in any case is pretty unlikely to be less than half a million or more than forty million. If and when astronomical equipment permits detection of comets at greater perihelion distances than is now possible, the estimate of λ_2 will need to be revised upward to include the new class of observable comets.

The figure of two or three million given above is somewhat greater than previous estimates. For instance (in [11], p. 18), Lyttleton wrote: ". . . by far the majority of comets possess much longer orbital periods . . . , and according to Crommelin the average is about 40,000 years. . . . The average number of comets discovered each year is about six . . . but of these only three or four are really new, the others . . . being returns of earlier discovered comets of moderate periods. This discovery rate means that at least three hundred long-period comets come to perihelion each century, and if we adopt 30,000 to 40,000 years as the average period, we arrive at the amazing but inescapable conclusion that there must be at least 100,000 comets in the solar system." However, the present study of the process by which comets are lost from the solar system shows that comets which have lingered in the solar system from the remote past, have, by being in the tail of the lifetime distribution, energies very near zero and hence immensely long periods. The cube root relation (5.6) indicates that the majority of comets now in the solar system are contributed from the remote past; and accordingly the average period should be considerably greater than Crommelin's figure of 40,000 years. Perhaps an average period of half to one million years is about right at the present day. (It follows from (5.6) that the average period is nearly proportional to the cube root of the age of the solar system.)

The foregoing arguments depend on the assumption that the rate of entry of comets into the solar system is constant. This assumption, however, is rather likely to be invalid. For instance, if comets are born from matter swept up by the sun's gravitational field in traversing intragalactic dust clouds, then $\lambda(\theta)$ will be a fluctuating function of time. We therefore turn to the treatment of variable $\lambda(\theta)$. In this we shall assume that $\lambda(\theta)$ is bounded and possesses a long-term average, that is, that

$$\bar{\lambda} = \lim_{\omega\to\infty} \omega^{-1} \int_0^{\omega} \lambda(\theta)\, d\theta \tag{5.7}$$

exists. The boundedness of $\lambda(\theta)$ provides a Tauberian condition, sufficient (as we shall presently see) for the Tauberian conclusion

$$m(\theta_0) \sim 3\bar{\lambda} q(x)\theta_0^{1/3} \quad \text{as} \quad \theta_0 \to \infty . \tag{5.8}$$

The asymptotic relation (5.8) leads immediately to the required generalization of (5.6), namely

$$m_1(x) \sim a(x)\bar{\lambda}_1(x)\theta_1^{1/3} \quad \text{as} \quad \theta_1 \to \infty , \tag{5.9}$$

where $\bar{\lambda}_1(x)$ is the long-term average number of comets per annum which enter the solar system with energy x absolute units. From the point of view of theory this result may be satisfactory, but there is a practical difficulty in the way of applying it. We can no longer, as we did previously, estimate $\bar{\lambda}_1$ by equating it to the present annual loss of comets without some further assumption or evidence that the present dust-collecting rate of the solar system is typical of its long-term average rate. However, (5.9) could be used the other way around to estimate the long-term dust-collecting rate if we knew the present number of comets in the solar system.

To prove (5.8), take an arbitrary $\epsilon > 0$. Then by results of section 4, we can find $\gamma = \gamma(\epsilon)$ such that

$$F(g) = [q(x) + \epsilon\vartheta]g^{-2/3}, \qquad g \geqq \gamma, \tag{5.10}$$

where ϑ denotes an unspecified number in the interval $(-1, 1)$. By (5.2), we have

$$\begin{aligned} m(\theta_0) - \int_0^{\theta_0} [q(x) + \epsilon\vartheta](\theta_0 - \theta)^{-2/3}\lambda(\theta)\, d\theta \\ = \int_{\theta_0-\gamma}^{\theta_0} \{F(\theta_0 - \theta) - [q(x) + \epsilon\vartheta](\theta_0 - \theta)^{-2/3}\}\lambda(\theta)\, d\theta \\ = O(1) \quad \text{as} \quad \theta_0 \to \infty, \end{aligned} \tag{5.11}$$

by virtue of the boundedness of $\lambda(\theta)$ and $F(g)$. Since ϵ is arbitrary, (5.8) will follow from (5.11) if we can prove that

$$\int_0^{\theta_0} (\theta_0 - \theta)^{-2/3}\lambda^*(\theta)\, d\theta = o(\theta_0^{1/3}) \quad \text{as} \quad \theta_0 \to \infty, \tag{5.12}$$

where

$$\lambda^*(\theta) = \lambda(\theta) - \bar{\lambda}. \tag{5.13}$$

From (5.13) and (5.7) we have

$$\int_0^{\omega} \lambda^*(\theta)\, d\theta = o(\omega) \quad \text{as} \quad \omega \to \infty. \tag{5.14}$$

Finally, (5.14) implies (5.12) on applying the following theorem due to M. Riesz [12].

THEOREM. *Define*

$$\Lambda_r(\omega) = \frac{1}{\Gamma(r)} \int_0^{\omega} (\omega - \theta)^{r-1}\lambda^*(\theta)\, d\theta \tag{5.15}$$

and suppose

$$\lambda^*(\theta) = O(V), \qquad \Lambda_r(\theta) = o(W) \quad \text{as} \quad \theta \to \infty, \tag{5.16}$$

where V and W are nondecreasing functions of θ. Then

$$\Lambda_\alpha(\theta) = o(V^{1-\alpha/r}W^{\alpha/r}) \quad \text{as} \quad \theta \to \infty \tag{5.17}$$

whenever $0 < \alpha < r$.

In this theorem we take $\alpha = 1/3$, $r = 1$, $V = 1$, and $W = \theta$. In general (5.8) will not hold uniformly in x; and we shall not discuss the delicate conditions upon the perturbation distribution $P(y)$ and the distribution of initial energies x, which would be needed before (5.8) could be integrated over initial energies.

REFERENCES

[1] P. Lévy, *Processus Stochastiques et Mouvement Brownien*, Paris, Gauthier-Villars, 1948.
[2] M. Kac, *Probability and Related Topics in Physical Sciences*, London, Interscience, 1959.
[3] P. Halmos, *Measure Theory*, New York, Van Nostrand, 1950.
[4] H. Lebesgue, "Sur les fonctions représentables analytiquement," *J. Math. Pures Appl.*, Vol. 6 (1905), pp. 139–216.
[5] A. R. Forsyth, *A Treatise on Differential Equations*, London, Macmillan, 1948 (6th ed.).
[6] H. Jeffreys and B. S. Jeffreys, *Methods of Màthematical Physics*, Cambridge, Cambridge University Press, 1946.
[7] D. V. Widder, *The Laplace Transform*, Princeton, Princeton University Press, 1941.
[8] G. Doetsch, *Handbuch der Laplace Transformation*, Vol. 1, Basel, Verlag Birkhäuser, 1950.
[9] Committee for the Calculation of Mathematical Tables, *Circular and Hyperbolic Functions*, Vol. 1 of *British Association Mathematical Tables*, Cambridge, Cambridge University Press, 1951 (3rd ed.).
[10] D. G. Kendall, "The distribution of energy perturbations for Halley's and some other comets," *Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability*, Berkeley and Los Angeles, University of California Press, 1961, Vol. 3, pp. 87–98.
[11] R. A. Lyttleton, *The Comets and their Origin*, Cambridge, Cambridge University Press, 1953.
[12] M. Riesz, "Sur un théorème de la moyenne et ses applications," *Acta Litt. Sci. Szeged.*, Vol. 1 (1923), pp. 114–126.
[13] D. G. Kendall, "Some problems in the theory of comets. I," *Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability*, Berkeley and Los Angeles, University of California Press, 1961, Vol. 3, pp. 99–120.