AN APPLICATION OF THE CENTRAL LIMIT THEOREM

YU. A. ROZANOV
STEKLOV MATHEMATICAL INSTITUTE, MOSCOW

1. Introduction

The limit theorems established for the classical case of sums of independent quantities were not adequate for those questions which arose both in the theory of probability itself and in its applications.

As far back as the time of Bernstein's work [1], attempts were made to extend these theorems to the case of dependent quantities. The most definitive results [2] to [5] in this direction were, of course, obtained for quantities connected in a Markov chain.

It is abundantly clear that it would be desirable to establish some general limit theorems at least for quantities which are, in some sense, weakly dependent. The concept of m-dependent random quantities, to which the results of the classical case of independent quantities can be fairly easily generalized [6] to [8], arose in a natural way.

Recently somewhat different conditions for weak dependence appeared, the use of which led to the establishment [9] to [11] of a series of new limit theorems. By far the widest of these conditions was formulated by Rosenblatt [9] for the case of a stationary sequence $\xi(t)$. It consists of the requirement that

$$(1) \qquad |P(AB) - P(A)P(B)| \leqq \alpha(\tau),$$

where $A \in M^{t}_{-\infty}$, $B \in M^{\infty}_{t+\tau}$ and M^{t}_{s} is the σ-algebra generated by the events of the form $\{\xi(u) < x\}$ for $s \leqq u \leqq t$ and $\alpha(\tau) \to 0$ when $\tau \to \infty$.

In this form condition (1), which, following Rosenblatt [9], we shall call the *strong mixing condition*, was applied to the arbitrary random process $\xi(t)$ and generally to some family of σ-algebras M^{t}_{s} of ω-sets in a space Ω with a probability measure $P(d\omega)$.

The strong mixing condition (1) is satisfied in the broad class of ergodic Markov processes and also in Gaussian processes. In [12] it was established that for a stationary Gaussian process the strong mixing property is associated with the smoothness of its spectral density; for example, in the case of discrete time it is always satisfied if the spectral density is continuous and never vanishes.

For quantities $\xi(t)$ satisfying the strong mixing condition (1), the central limit theorem itself was obtained in [11] together with more precise details [17] including asymptotic expansions and some results concerned with large deviations.

Another condition of weak dependence was used by Ibragimov [10], also with reference to a stationary sequence. This condition is more restrictive than condition (1) and is that, with probability one

$$\sup_{A \in M_{t+\tau}^{\infty}} |P(A|M_{-\infty}^{t}) - P(A)| \leqq \beta(\tau) \downarrow 0, \qquad \tau \to \infty. \tag{2}$$

The use of this condition yielded a new derivation of the limit theorems for Markov processes [4] and also for some special processes which are of interest in the theory of numbers [15].

2. The central limit theorem for additive random functions

Let $H^{s,t}(\omega)$ be a family of random quantities additively dependent upon the interval (s, t), so that, for all $s \leqq u \leqq t$,

$$H^{s,u}(\omega) + H^{u,t}(\omega) = H^{s,t}(\omega) \tag{3}$$

with probability one, let

$$m(s, t) = E\{H^{s,t}\}, \tag{4}$$

and let $\sigma^2(s, t)$ be the variance of $H^{s,t}$.

The most important examples of such quantities are those of the form $H^{s,t} = \sum_{s<k\leqq t}\xi(k)$ and $H^{s,t} = \int_s^t \xi(u)\, du$ where $\xi(t)$ is a random process. Let

$$\eta^{s,t}(\omega) = \frac{H^{s,t}(\omega) - m(s, t)}{\sigma(s, t)} \tag{5}$$

and let $F_\eta^{s,t}(x)$ be the distribution function of the random variable $\eta^{s,t}$. We shall be interested in the conditions under which

$$F_\eta^{s,t}(x) \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2}\, du, \qquad t - s \to \infty \tag{6}$$

uniformly over the whole family.

We shall say that the random quantities $\eta^{s,t}(\omega)$ satisfy the condition L if, for any $\epsilon > 0$, numbers N_ϵ and T_ϵ can be found such that

$$\int_{|x|>N_\epsilon} x^2\, dF_\eta^{s,t}(x) \leqq \epsilon, \qquad t - s \geqq T_\epsilon. \tag{7}$$

It is easily seen that condition (7) is always necessary for the distribution functions $F_\eta^{s,t}(x)$ to converge uniformly to some continuous probability law with zero mean and unit variance. Under certain conditions of weak dependence and regularity of growth of the variance $\sigma^2(s, t)$, condition (7) turns out to be also sufficient for the convergence of $F_\eta^{s,t}(x)$ when $t - s \to \infty$. Furthermore, the limiting distribution is normal. Before formulating the relevant theorem [14], we introduce the symbol $\sigma^2(s, t) \asymp t - s$ to mean

(8) $$0 < \liminf_{t-s\to\infty} \frac{\sigma^2(s,t)}{t-s} \leqq \limsup_{t-s\to\infty} \frac{\sigma^2(s,t)}{t-s} < \infty .$$

THEOREM 1. *Let the quantities $H^{s,t}(\omega)$ be the functionals defined on trajectories of a certain random process $\xi(t)$ satisfying the strong mixing condition* (1), *let*

(9) $$E\{[H^{s,t} - E(H^{s,t}|M_{s-u}^{t+u})]^2\} \leqq C(t-s)\phi(u)$$

where C is a certain constant and $\phi(u) \to 0$ when $u \to \infty$ and, in addition, suppose that

(10) $$\sigma^2(s,t) \asymp t-s.$$

Then condition (7) *is not only necessary but also sufficient for the uniform asymptotic normality* [see (6)] *of the quantities $\eta^{s,t}(\omega)$.*

The qualitatively simple condition (7) is, unfortunately, often difficult to establish. If the exponent $\alpha(t)$ in the strong mixing condition (1) decreases sufficiently fast, precisely if

(11) $$\alpha(\tau) = O[\tau^{-1-\epsilon}], \qquad \tau \to \infty,$$

then the asymptotic normality of the quantities $\eta^{s,t}(\omega)$ follows [13] from the existence of moments of a sufficiently high order.

THEOREM 2. *Let the quantities $H^{s,t}(\omega)$ be measurable with respect to the σ-algebras M_s^t and let them satisfy conditions* (10) *and* (11). *Furthermore suppose that, for a certain $\Delta > 0$ we have*

(12) $$E\{|H^{s,t} - m(s,t)|^{2+\delta}\} \leqq E_0 < \infty$$

where $t - s = \Delta$ and $\delta > 2/\epsilon$. Then the quantities $H^{s,t}(\omega)$ are asymptotically normal in the sense of (6).

It should be noticed that the above results are easily extended (compare [14]) to the multidimensional case.

Analogous results hold also in the case of the "sequence scheme," that is, when we have a family of random functions depending on a parameter n, with $n \to \infty$, and we study the behavior of the distribution function of the quantities $\eta_n^{s,t}(\omega)$ where $s = s_n$, $t = t_n$, and $t_n - s_n \to \infty$ when $n \to \infty$ (see section 6).

3. The locally Gaussian nature of spectral measures for stationary processes

Many papers in radiotechnology assume that after the transmission of a stationary process through a narrow linear filter it becomes almost Gaussian. This fact can be given a rigorous mathematical foundation.

For the sake of simplicity we shall restrict ourselves to the case when the frequency characteristic $\phi_n(\lambda)$ of the linear filter, transmitting frequencies near λ_0, has the form

(13) $$\phi_n(\lambda) = \frac{1}{2}\{\phi[(\lambda - \lambda_0)n] + \phi[(\lambda + \lambda_0)n]\}\sqrt{n}$$

where the function $\phi(\lambda)$ is such that $\phi(-\lambda) = \overline{\phi(\lambda)}$ and $\int_{-\infty}^{\infty} |\phi(\lambda)|^2 \, d\lambda < \infty$ and its Fourier transform is uniformly continuous almost everywhere. Let the real process $\xi(t)$, stationary in the wide sense, be defined by

$$\xi(t) = \int_{-\infty}^{\infty} e^{i\lambda t} \, \Phi(d\lambda), \tag{14}$$

and suppose that $\xi(t)$ has a bounded spectral density $f(\lambda)$ such that

$$\inf_{|\lambda - \lambda_k| < \epsilon} f(\lambda) > 0, \tag{15}$$

where the infimum is taken over a certain neighborhood of points λ_k, with $k = 1, \cdots, m$.

Let us study the random quantities

$$\eta_n^{(k)} = \frac{1}{\sigma_{kn}} \int_{-\infty}^{\infty} e^{i\lambda t_k} \phi_n^{(k)}(\lambda) \Phi(d\lambda), \tag{16}$$

where $\phi_n^{(k)}(\lambda)$ is a frequency characteristic of the form (13) with $\lambda_0 = \lambda_k$ and σ_{kn}^2 the variance of

$$\int_{-\infty}^{\infty} e^{i\lambda t_k} \phi_n^{(k)}(\lambda) \Phi(d\lambda). \tag{17}$$

From theorem 2 we can deduce (compare [11])

THEOREM 3. *Let the stationary process $\xi(t)$ possess the strong mixing property* (1), *where in addition*

$$\alpha(\tau) = O[\tau^{-1-\epsilon}] \tag{18}$$

and for some $\delta > 2/\epsilon$ we have

$$E[|\xi(t)|^{2+\delta}] \leqq E_0 < \infty \tag{19}$$

for all t. Suppose further that

$$\lim_{n \to \infty} E[\eta_n^{(k)} \eta_n^{(j)}] = b_{kj} \qquad k, j = 1, \cdots, m. \tag{20}$$

Then the joint distribution function of the quantities $\eta_n^{(1)}, \cdots, \eta_n^{(m)}$ converges to a normal law with variance-covariance matrix $[b_{kj}]$.

In particular, condition (20) is satisfied when all the frequency characteristics $\phi_n^{(k)}(\lambda)$ correspond to the same function $\phi(\lambda)$ appearing in formula (13). Also, condition (20) is satisfied if all the points $\lambda_1, \lambda_2, \cdots, \lambda_m$ are different. In this case the matrix $||b_{kj}||$ is the unit matrix. Another case of convergence to the normal law with the unit variance-covariance matrix is that of the joint distribution of the random variables

$$\eta_n^{(k)} = \frac{2\mathcal{R}\{\Phi(\Delta_n^k)\}}{[F(\Delta_n^k)]^{1/2}} \quad \text{and} \quad \zeta_n^{(k)} = \frac{2\mathcal{I}\{\Phi(\Delta_n^k)\}}{[F(\Delta_n^k)]^{1/2}} \tag{21}$$

where $\Delta_n^k = (\lambda_k - n^{-1}, \lambda_k + n^{-1})$ for $k = 1, \cdots, m$, where the points $\lambda_1, \cdots, \lambda_m$ are all different, and $F(\Delta) = E[|\Phi(\Delta)|^2]$ is the spectral measure of the process $\xi(t)$.

4. Applicability of the central limit theorem to the logarithm of the likelihood ratio

Let the probability measure $P(d\omega)$ corresponding to some random process $\xi(t)$ depend on a parameter which takes its value from a certain interval, so that

$$P(d\omega) = P_\theta(d\omega). \tag{22}$$

We shall denote by $P^{s,t}$ the probability measure corresponding to the process $\xi(t)$ considered only over the interval $[s, t]$. Thus, $P^{s,t}$ is defined over the σ-algebra M_s^t and on sets $A \in M_s^t$ coincides with the measure P, so that $P^{s,t}(A) = P(A)$.

Let $m(d\omega)$ be some measure, not necessarily a probability measure, defined on the σ-algebra $M_{-\infty}^{\infty}$ such that $P^{s,t}$ is absolutely continuous with respect to $m^{s,t}$ for all s and t and let

$$p^{s,t}(\omega, \theta) = \frac{P_\theta^{s,t}(d\omega)}{m^{s,t}(d\omega)} \tag{23}$$

be the probability density (the "likelihood ratio").

An important property [14] of (23) is the asymptotic normality where $t - s \to \infty$, of the quantities

$$L^{s,t}(\omega, \theta) = \log p^{s,t}(\omega, \theta) \tag{24}$$

and of their derivatives $(\partial/\partial\theta)[L^{s,t}(\omega, \theta)]$.

We note that, as a rule, it is possible to define the conditional measures $P^{s,t}(d\omega|M_t^s)$ in such a way that $P^{s,t}(d\omega|M_{s-v}^s)$ will be absolutely continuous with respect to $P^{s,t}(d\omega|M_{s-u}^s)$ whenever $u \leqq v$ for almost all ω.

Let us write

$$\pi_\theta^{s,t}(u, v) = \frac{P_\theta^{s,t}(d\omega|M_{s-v}^s)}{P_\theta^{s,t}(d\omega|M_{s-u}^s)} \tag{25}$$

and similarly

$$\mu^{s,t}(u, v) = \frac{m^{s,t}(d\omega|M_{s-v}^s)}{m^{s,t}(d\omega|M_{s-u}^s)}. \tag{26}$$

Further, let

$$l^{s,t}(\omega, \theta) = \frac{L^{s,t}(\omega, \theta) - E\{L^{s,t}\}}{\sigma\{L^{s,t}\}}. \tag{27}$$

THEOREM 4. *Let the strong mixing property* (1) *be satisfied uniformly with respect to the parameter* θ, *let*

$$\sigma^2[L^{s,t}(\omega, \theta)] \asymp t - s \tag{28}$$

and let

$$\sigma^2\left[\log \frac{\pi_\theta^{s,t}(u, v)}{\mu^{s,t}(u, v)}\right] \leqq C(t - s)\phi(u) \tag{29}$$

where C *is a certain constant and* $\phi(u) \to 0$ *when* $u \to \infty$. *Then for the asymptotic normality* (*uniform with respect to* s, t *and the parameter* θ) *of the quantities* $l^{s,t}(\omega, \theta)$

when $t - s \to \infty$, *it is necessary and sufficient that they should satisfy condition* (7).

For some problems it is of interest to examine the case in which the quantities $l^{s,t}(\omega, \theta_{s,t})$ are asymptotically normal, where the parameter θ changes as $t - s \to \infty$. For this it is only necessary to require that (28) and (29), instead of being satisfied uniformly in θ, should be satisfied when $\theta = \theta_{s,t}$.

An analogous theorem holds for the quantities

$$\frac{\partial}{\partial\theta} L^{s,t}(\omega, \theta) \tag{30}$$

in which we need only replace conditions (28) and (29) by

$$\sigma^2 \left[\frac{\partial}{\partial\theta} L^{s,t}(\omega, \theta)\right] \asymp t - s \tag{31}$$

$$\sigma^2 \left[\frac{\partial}{\partial\theta} \log \frac{\pi_\theta^{s,t}(u, v)}{\mu^{s,t}(u, v)}\right] \leqq C(t - s)\phi(u). \tag{32}$$

The above facts follow from the application of the general theorem 1 to the quantities

$$H_\tau^{s,t} = L^{\tau,t} - L^{\tau,s} \tag{33}$$

and correspondingly to $(\partial/\partial\theta)[H_\tau^{s,t}(\omega, \theta)]$, depending on the auxiliary parameter τ.

5. A central limit theorem for certain stationary processes

Let the process $\xi(t)$, where t takes integer values, be stationary in the narrow sense and satisfy condition (2). Let the random quantity $\eta(\omega)$ be measurable with respect to the σ-algebra $M_{-\infty}^{\infty}$ and let $\eta(t) = \eta(S_t\omega)$ be the stationary process generated by the random quantity η and the translation S_τ, where

$$S_\tau\xi(t) = \xi(t + \tau). \tag{34}$$

Let η have zero mean and finite variance and suppose

$$\sum_{k=1}^{\infty} [E(\{\eta - E[\eta|M_{-k}^{k}]\}^2)]^{1/2} < \infty. \tag{35}$$

If the function $\beta(\tau)$ appearing in condition (2) decreases sufficiently rapidly, so that

$$\sum_{k=1}^{\infty} \beta^{1/2}(k) < \infty, \tag{36}$$

then

$$\sigma^2 = E[\eta^2] + 2 \sum_{k=1}^{\infty} E[\eta\eta(k)] < \infty. \tag{37}$$

Ibragimov [10] has established

THEOREM 5. *If conditions* (35), (36), *and* (37) *are satisfied, and if* $\sigma \neq 0$, *then*

$$P\left\{\frac{1}{\sigma\sqrt{n}} \sum_{1}^{n} \eta(k) < x\right\} \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2}\, du, \qquad n \to \infty. \tag{38}$$

This result implies the central limit theorem [4] for a Markov chain. In this case, under wide conditions of ergodicity, the function $\beta(\tau)$ decreases exponentially.

Unfortunately condition (2) is satisfied by processes of a class which, though important, is not wide enough. Thus, for example, Gaussian processes are members of this class only if the covariance function $B(s, t)$ is identically zero when $|t - s| \geqq T$ for some finite T. As we have already noted, condition (2) is satisfied by ergodic Markov and also by m-dependent random processes.

Theorem 5 was applied in [15] by Ibragimov to certain special processes and yielded a series of new results which are of interest in number theory (see also [16]).

Let us consider an arbitrary number $x \in [0, 1]$ and its expansion as a dyadic fraction

$$x = \frac{e_1(x)}{2} + \frac{e_2(x)}{4} + \cdots + \frac{e_k(x)}{2^k} + \cdots. \tag{39}$$

If the $e_k(x)$ are treated as random variables, where the sample space Ω is the interval [0, 1] and the probability measure is simply the Lebesgue measure, we find that they are independently and identically distributed and

$$\begin{aligned} P\{e_k(x) = 0\} &= P\{e_k(x) = 1\} = \frac{1}{2}, \\ P\{e_1(x) = i_1, \cdots, e_s(x) = i_s\} &= \frac{1}{2^s}. \end{aligned} \tag{40}$$

Consider the stationary process $\xi(t) = e_t(x)$ for $t \geqq 0$. Obviously, the σ-algebra M_0^k is the algebra generated by the sets $A_{jk} = [(j-1)/2^k, j/2^k]$ for $j = 1, \cdots, 2^k$.

Let $f(x)$ be an arbitrary function of x such that

$$\int_0^1 f(x)\, dx = 0, \qquad \int_0^1 f^2(x)\, dx < \infty. \tag{41}$$

It is easy to see that, for $x \in A_{jk}$, we have

$$[f]_k(x) = E\{f|M_0^k\} = 2^k \int_{A_{jk}} f(x)\, dx \tag{42}$$

and condition (35) takes the form

$$\sum_1^\infty \left[\int_0^1 |f(x) - [f]_k(x)|^2\, dx\right]^{1/2} < \infty. \tag{43}$$

The translation S acts on Ω in accordance with the formula $S_\tau x = 2^\tau x \pmod 1$.

Theorem 1 yields the following

COROLLARY 1. *Condition* (43) *implies*

$$\sigma^2 = \int_0^1 f^2(x)\, dx + 2 \sum_1^\infty \int_0^1 f(x) f(2^k x)\, dx < \infty \tag{44}$$

and if also $\sigma \neq 0$ *then*

$$P\left\{\frac{1}{\sigma\sqrt{n}}\sum_1^n f(2^k x) < z\right\} \to \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-u^2/2}\,du. \tag{45}$$

Condition (43) will be satisfied if, for example, any one of the following three conditions is satisfied.

(A) $f(x)$ is a function of bounded variation,

(B) $\int_0^1 |f(x) - f(x+h)|^2\,dx \leqq C|\log h|^{2+\epsilon}$, where C is some constant and $\epsilon > 0$,

(C) the Fourier coefficients $a_n = \int_0^1 e^{2\pi n x} f(x)\,dx$ decrease so fast that $|a_n| \leqq Cn^{1/2}(\log n)^{(3+\epsilon)/2}$.

Another result of Ibragimov relates to continued fractions. As before, let $x \in [0, 1]$ and let

$$x^{\cdot} = \cfrac{1}{a_1(x) + \cfrac{1}{a_2(x) + \cdots}} \tag{46}$$

be its expansion as a simple continued fraction. We shall write $x = [a_1(x), a_2(x), \cdots]$. Let us suppose that $Sx = [a_2(x), a_3(x), \cdots]$.

We shall study the measure μ on the Lebesgue sets, defined by the formula

$$\mu(A) = \frac{1}{\log 2}\int_A \frac{dx}{1+x}. \tag{47}$$

It transpires that the sequence $\xi(t) = a_t(x)$ is a process, with probability measure μ, which is stationary in the narrow sense, satisfies condition (2), and moreover is such that the corresponding function $\beta(\tau)$ decreases very rapidly, in fact

$$\beta(\tau) \leqq Ce^{-\lambda\sqrt{\tau}} \tag{48}$$

where C is a certain constant and $\lambda > 0$.

Theorem 5 has the following

COROLLARY 2. *Let the function $f(x)$ be such that*

$$\int_0^1 f(x)\mu(dx) = 0, \qquad \int_0^1 f^2(x)\,dx < \infty, \tag{49}$$

and

$$|f(x+h) - f(x)| \leqq C|\log^{-1-\epsilon} h|. \tag{50}$$

Then

$$\sigma^2 = \int_0^1 f^2(x)\mu(dx) + 2\sum_1^\infty \int_0^1 f(x)f(S^k x)\mu(dx) < \infty \tag{51}$$

and if moreover $\sigma \neq 0$, then

$$\mu\left[x: \frac{1}{\sigma\sqrt{n}}\sum_1^n f(S^k x) < z\right] \to \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-u^2/2}\,du. \tag{52}$$

6. Asymptotic expansions and large deviations

To the case of weakly dependent quantities we may transfer the results of [18] to [20], which make the central limit theorem more precise.

V. A. Statulevichus [17] studied sequences of random quantities $\xi_n(t)$ satisfying for every n the strong mixing condition (1) in which the functions $\alpha_n(\tau)$ have the form

$$\alpha_n(\tau) = (\tau\alpha_n)^{-a}, \qquad a > 0,\ \alpha_n > 0. \tag{53}$$

Let the random quantities $\xi_n(t)$ be uniformly bounded,

$$|\xi_n(t)| \leqq C \tag{54}$$

and let

$$\sigma^2\left[\sum_k^l \xi_n(t)\right] \geqq c\alpha_n(l-k). \tag{55}$$

THEOREM 6. *If the variables $\xi_n(t)$ satisfy conditions* (53) *and* (55) *and, as $n \to \infty$*

$$\alpha_n n^{1/2} \to \infty, \tag{56}$$

then, for $|u| \leqq k(\log n)^{1/2}$, when k is an integer satisfying the condition $3 \leqq k \leqq (a-3)^{1/2}$, the characteristic function $f_n(u)$ of the random variable

$$\eta_n = \frac{\sum_1^n \xi_n(t) - E\left[\sum_1^n \xi_n(t)\right]}{\sigma\left[\sum_1^n \xi_n(t)\right]} \tag{57}$$

admits the expansion

$$f_n(u) = e^{-u^2/2}\left[1 + \sum_{j=1}^{k-3} P_{nj}(iu)\frac{1}{r_n^j}\right] + O\left(\frac{|u|^k + |u|^{3k}}{r_n^{k-2}}\right)\exp\left[-\frac{u^2}{2}(1+\epsilon_n)\right] \tag{58}$$

where

$$r_n = \frac{\alpha_n^2}{n}\sigma^3\left[\sum_1^n \xi_n(t)\right] \tag{59}$$

$$C_1\alpha_n n \leqq \sigma^2\left|\sum_1^n \xi_n(t)\right| \leqq C_2\frac{n}{\alpha_n} \tag{60}$$

and the coefficients of the polynomials $P_{nj}(iu)$ are uniformly bounded in n, the constants implied by the $O(\cdot)$ notation depend on k and are uniformly bounded in n, and $\epsilon_n \to 0$ when $n \to \infty$.

Outside of the interval $|u| \leqq k(\log n)^{1/2}$ we have $|f_n(u)| \leqq \exp\{-cu^2\}$, where c is a positive constant.

Further, if the exponents α_n in condition (53) *are bounded away from zero, so that*

$$\alpha_n \geqq \alpha > 0, \tag{61}$$

and if, in addition, the constant $a \geqq 12$, *then, for* $1 \leqq x \leqq o(\sqrt{n})$, *the probability of large deviations*

$$P\{\eta_n > x\} = 1 - F_n(x), \tag{62}$$

say, satisfies the following Cramér-Petrov relation

$$\frac{1 - F_n(x)}{\int_x^\infty e^{-u^2/2}\, du} = \exp\left[\frac{x^3}{\sqrt{n}} \lambda_n\left(\frac{x}{\sqrt{n}}\right)\right]\left[1 + O\left(\frac{x}{\sqrt{n}}\right)\right] \tag{63}$$

where the power series for $\lambda_n(y)$ *converges for small* y *uniformly in* n.

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