ASYMMETRIC ORIENTED PERCOLATION ON A PLANE

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1. Introduction

The percolation process considered is over the lattice of integer points (x, y), the bonds (all of unit length) being parallel to the x- and y-axes and positively oriented. The "x" bonds have probability $1 - p$ and the "y" bonds probability $1 - p'$ of being dammed. Estimates are given for "critical" pairs of values p, p'.

The terminology used is that introduced by S. R. Broadbent and J. M. Hammersley in [1] and the medium here considered is a crystal in the sense defined there. Its atoms are the points (x, y), where x and y are integers, while its bonds are of two types, dammed with probability $q = 1 - p$, $q' = 1 - p'$ respectively, namely

(1)
$$(x, y) \rightarrow (x + 1, y) \qquad \text{with } q = 1 - p,$$
$$(x, y) \rightarrow (x, y + 1) \qquad \text{with } q' = 1 - p'.$$

Given a single wet atom as source, and the general principle that fluid flows from a wet atom along an undammed oriented bond to wet another atom, we now consider the

PROBLEM. *For what values of p and p' is it true that, with probability 1, only a finite number of atoms in all will be wet?*

We shall see that the answer to this question is unchanged if the source consists of any finite number of wet atoms. We shall further see that there exists a curve passing through the points $(0, 1)$ and $(1, 0)$ such that if the point (p, p') lies inside the curve [that is, on the same side as the origin $(0, 0)$] then, with probability 1, only a finite number of atoms will be wet, whereas, if (p, p') lies outside the curve, there is a nonzero probability that an infinity of atoms will be wet.

Hammersley [2], [3] has given lower and upper bounds for the value of p where the line $p = p'$ meets this curve. Estimates are here given for the values of ρ where the line $p = \rho \cos \theta$, $p' = \rho \sin \theta$ meets the curve, for all values of $\theta = 0, \pi/32, \cdots, \pi/2$. This is believed to be the first published example of a percolation process in which the bonds have not all the same probability of being dammed.

2. The Markov chain

There are many ways in which the process described can be considered as a Markov chain. For example, we may consider the successive states to be defined on successive diagonals, $x + y$ = constant, by the specification of all the wet atoms on that particular diagonal. In this light we see that the states are enumerable, there is a single absorbing state (namely, the zero state, consisting of no wet atoms) which is accessible from all other states, and all the nonzero states form a single aperiodic intercommunicating class. Hence, if $\mathbf{P}$ denotes the stochastic matrix which defines this Markov chain $\mathfrak{P}$ (its first row referring to the zero state), it follows that $\mathbf{II} = \lim_{n \to \infty} (\mathbf{P})^n$ exists, and that its elements π_{ij} satisfy the following conditions,

$$(2) \qquad \pi_{00} = 1, \qquad \pi_{ij} = 0 \qquad\qquad j \neq 0,$$

and

$$(3) \qquad \begin{array}{lll} \text{either} & \text{(3a) } \pi_{i0} = 1 & \text{all } i, \\ \text{or} & \text{(3b) } \pi_{i0} < 1 & i \neq 0. \end{array}$$

In the case (3a) there is unit probability that some diagonal, and every subsequent diagonal, has no wet atoms, so that the total number of wet atoms is finite, whereas in the case (3b), provided $i \neq 0$ (that is, that there is at least one wet atom as source) there is nonzero probability that every diagonal contains at least one wet atom, so that the total number of wet atoms is infinite.

Recalling (F. G. Foster [4]) the definition of a nondissipative Markov chain as one for which $\sum_j \pi_{ij} = 1$ for all i, we see that there is nonzero probability of an infinity of wet atoms, from any nonzero source, if and only if the Markov chain is not nondissipative. This proves the remark immediately following the statement of the problem in section 1 and also enables us to restate the

PROBLEM. *For what values of p and p' is the Markov chain nondissipative?*

A direct attack on this problem has proved intractable, so we turn our attention to the simpler problem provided by a class of pseudopercolation processes $\mathfrak{P}_r$ related to our main process $\mathfrak{P}$.

3. The processes $\mathfrak{P}_r$

For each positive integer r, the process $\mathfrak{P}_r$ is the same as $\mathfrak{P}$ with the modification that additional sources (wet atoms) are introduced in the course of the process itself in the following manner; namely, that every rth diagonal (and these alone) have the property that all atoms on the diagonal between any two wet atoms are themselves wet. Thus any state $j = 0, 1, 2, \cdots$ of the Markov chain $\mathfrak{P}_r$ is given by specifying merely the total number j of (necessarily contiguous) wet atoms on the diagonal in question, while the transition probabilities are obtained from those of the rth power of $\mathbf{P}$ by identifying states of $\mathfrak{P}$ with the same total span ($=j$) and deleting the rows corresponding to states of $\mathfrak{P}$ in which the wet atoms are not all contiguous.

Before proceeding to study the "critical curve" (see section 1) for the process $\mathfrak{P}_r$ we may mention its relevance to the problem of finding the critical curve for $\mathfrak{P}$. Since any particular realization of $\mathfrak{P}_r$ certainly contains at least as many wet atoms as the corresponding realization of the process $\mathfrak{P}$, the critical curve for $\mathfrak{P}_r$ certainly lies inside or on the critical curve for $\mathfrak{P}_r$, so that we may obtain sequences of lower bounds for the critical probabilities for $\mathfrak{P}$. On the other hand, by letting $r \to \infty$, we obtain processes $\mathfrak{P}_r$ as nearly as we please coinciding with $\mathfrak{P}$, so that our sequences will tend to the required critical probabilities for $\mathfrak{P}$. A rigorous proof of this observation has been provided by Hammersley and is given in section 10.

Now it is easily seen that the elements $p_{ij}^{(r)}$ of the stochastic matrix $\mathbf{P}^{(r)}$ defining the Markov chain $\mathfrak{P}_r$ satisfy the conditions of the corollary to theorem 3 of [5], pages 832–833, in particular we may take $1 - \alpha = [\min(p, p')]^r$. Notice that the subscripts in the first condition (48) should be $i_0 j$, not $i j_0$.

Hence the following limit exists

$$(4) \qquad E = E^{(r)}(p, p') = \lim_{i \to \infty} \sum_{j=0}^{\infty} (j - i) p_{ij}^{(r)}$$

and the process $\mathfrak{P}_r$ is nondissipative if $E < 0$ and not if $E > 0$.

Since the general considerations of section 2 apply as well to $\mathfrak{P}_r$ as to $\mathfrak{P}$, it follows that the critical curve for the process $\mathfrak{P}_r$ is the locus of points (p, p') such that

$$(5) \qquad E^{(r)}(p, p') = 0.$$

4. The end-effects of $\mathfrak{P}_r$

For any given diagonal $x + y = br$ of the subset corresponding to states of $\mathfrak{P}_r$, let W denote the aggregate of wet atoms on this diagonal and define

$$(6) \qquad \xi = \min \{x : (x, y) \in W\}, \qquad \eta = \min \{y : (x, y) \in W\},$$

with $\xi = \eta = \infty$ if W is null. Also let ξ^* and η^* denote the corresponding quantities on the next diagonal (for $\mathfrak{P}_r$), namely $x + y = (b + 1)r$, and let

$$(7) \qquad h_k^{(r)}(p, p'; i) = P\{\xi^* - \xi \geqq k | i\},$$

where i is the state of the former diagonal $x + y = br$. Finally let

$$(8) \qquad h_k = h_k^{(r)}(p, p') = \lim_{i \to \infty} h_k^{(r)}(p, p'; i).$$

Then the following propositions are obvious.

$$(9) \qquad h_k^{(r)}(p, p'; i) \text{ is independent of } i \text{ if } i \geqq k.$$

$$(10) \qquad \lim_{i \to \infty} E\{\xi^* - \xi | \xi^* \neq \infty\} = \sum_{k=1}^{\infty} h_k^{(r)}(p, p') = \sigma^{(r)}(p, p'), \text{ say.}$$

$$(11) \qquad E^{(r)}(p, p') = E\{j - i\} = r - \sigma^{(r)}(p, p') - \sigma^{(r)}(p', p).$$

We now turn to the evaluation of $h_k^{(r)}(p, p')$ and hence of $\sigma^{(r)}(p, p')$.

5. The processes $\mathfrak{P}_1$ and $\mathfrak{P}_2$

In the case of $\mathfrak{P}_1$ we easily calculate

$$(12) \qquad h_k^{(1)}(p, p'; i) = q'(qq')^{k-1} \qquad\qquad i \geqq k \geqq 1,$$

where $q = 1 - p$, $q' = 1 - p'$.

Hence, from (6) and (12), the critical curve for $\mathfrak{P}_1$ is

$$(13) \qquad\qquad p + p' - \frac{1}{2}pp' = 1.$$

Similarly for the process $\mathfrak{P}_2$ we find

$$(14) \qquad \begin{aligned} h_1^{(2)}(p, p'; i) &= 1 - p'^2 & i \geqq 1, \\ h_k^{(2)}(p, p'; i) &= [1 - p'(1 - qq')]^2[1 - (1 - qq')^2]^{k-2} & i \geqq k \geqq 2. \end{aligned}$$

Hence, $\qquad 1 - \sigma^{(2)}(p, p') = [2p'(1 - qq') - 1](1 - qq')^{-2}$,

and the critical curve for $\mathfrak{P}_2$ is given by

$$(15) \qquad\qquad (p + p')(p + p' - pp') = 1.$$

In particular, for the symmetric case $p = p'$, (13) and (15) provide lower bounds for the critical probability p_d for $\mathfrak{P}$, namely,

$$(16) \qquad\qquad p_d \geqq 2 - \sqrt{2} = 0.58578 \cdots$$

$$p_d \geqq (t|t^3 - 2t^2 + \frac{1}{2} = 0) = 0.59697 \cdots$$

The second lower bound here was quoted by Hammersley in [6], together with the improved lower bound provided by $\mathfrak{P}_3$.

6. The end-effect as a Markov chain

Suppose that the condition (wet or dry) of all the atoms $\{(x, -x) : x \geqq 0\}$ is known. Consider the chain formed by the consecutive vertical segments

$$(17) \qquad\qquad S_c = \{(c, s - c) : 1 \leqq s \leqq r\} \qquad c = 0, 1, 2, \cdots.$$

Any such segment may be in any one of 2^r distinct states (that is, specifications of its wet and dry atoms) which may be determined uniquely, and ordered $0, 1, \cdots, 2^r - 1$, by specifying the integer

$$(18) \qquad\qquad i = i(S_c) = \sum_{s=1}^{r} 2^{r-s}v(s),$$

where

$$(19) \qquad v(s) = v_c(s) = \frac{1}{0} \text{ if the atom } (c, s - c) \text{ is } \frac{\text{dry}}{\text{wet}}.$$

Clearly, if the atom $(0, 0)$ is wet, we have

$$(20) \qquad P\{i(S_0) = i_0\} = \begin{cases} p'^r & \text{if } i_0 = 0, \\ (p')^{r-m}q' & \text{if } i_0 = 2^m - 1, \qquad m = 1, \cdots, r, \\ 0 & \text{otherwise.} \end{cases}$$

The stochastic matrix of transition probabilities from states of S_c to states of S_{c+1} depends (apart from p and p') only on the (known) conditions of the two atoms $(c, -c)$ and $(c + 1, -c - 1)$. Let us denote the four stochastic matrices in question by $\mathbf{A}_{\alpha\beta}^{(r)}$, with $\alpha, \beta = 0, 1$, where

(21)
$$\alpha = \begin{cases} 1 \\ 0 \end{cases} \quad \text{if the atom } (c, -c) \text{ is } \begin{array}{l} \text{dry,} \\ \text{wet,} \end{array}$$

$$\beta = \begin{cases} 1 \\ 0 \end{cases} \quad \text{if } (c + 1, -c - 1) \text{ is } \begin{array}{l} \text{dry,} \\ \text{wet.} \end{array}$$

Then each of the matrices $\mathbf{A}_{\alpha\beta}^{(r)}$ is square of order 2^r by 2^r, and we easily verify that they are defined recursively by the equations

(22)
$$\mathbf{A}_{00}^{(r+1)} = \begin{bmatrix} (1 - qq')\mathbf{A}_{00}^{(r)} & qq'\mathbf{A}_{01}^{(r)} \\ (1 - qq')\mathbf{A}_{10}^{(r)} & qq'\mathbf{A}_{11}^{(r)} \end{bmatrix}, \quad \mathbf{A}_{01}^{(r+1)} = \begin{bmatrix} p\mathbf{A}_{00}^{(r)} & q\mathbf{A}_{01}^{(r)} \\ p\mathbf{A}_{10}^{(r)} & q\mathbf{A}_{11}^{(r)} \end{bmatrix},$$

$$\mathbf{A}_{10}^{(r+1)} = \begin{bmatrix} p'\mathbf{A}_{00}^{(r)} & q'\mathbf{A}_{01}^{(r)} \\ p'\mathbf{A}_{10}^{(r)} & q'\mathbf{A}_{11}^{(r)} \end{bmatrix}, \quad \mathbf{A}_{11}^{(r+1)} = \begin{bmatrix} \mathbf{O}^{(r)} & \mathbf{A}_{01}^{(r)} \\ \mathbf{O}^{(r)} & \mathbf{A}_{11}^{(r)} \end{bmatrix},$$

with

(23)
$$\mathbf{A}_{00}^{(0)} = \mathbf{A}_{01}^{(0)} = \mathbf{A}_{10}^{(0)} = \mathbf{A}_{11}^{(0)} = [1].$$

7. The operations ϑ and φ

We may regard (22) as an operation which transforms a set of four matrices $\mathbf{A}_{\alpha\beta}^{(r)}$ into a set of four matrices $\mathbf{A}_{\alpha\beta}^{(r+1)}$. Let us denote this operation by ϑ. Then, if $a_{00}, a_{01}, a_{10}, a_{11}$ are any four numbers, regarded as square matrices of order 2^0, it follows that $\vartheta^r(a_{00}, a_{01}, a_{10}, a_{11})$ denotes a set of four square matrices of order 2^r. These four matrices will be identified individually as

(24) $$\vartheta_{\alpha\beta}^r(a_{00}, a_{01}, a_{10}, a_{11}), \qquad \alpha, \beta = 0, 1.$$

Similarly, the operation φ, defined by

$$\mathbf{u}_{00}^{(r+1)} = \begin{bmatrix} (1 - qq')\mathbf{u}_{00}^{(r)} + qq'\mathbf{u}_{01}^{(r)} \\ (1 - qq')\mathbf{u}_{10}^{(r)} + qq'\mathbf{u}_{11}^{(r)} \end{bmatrix}, \quad \mathbf{u}_{01}^{(r+1)} = \begin{bmatrix} p\mathbf{u}_{00}^{(r)} + q\mathbf{u}_{01}^{(r)} \\ p\mathbf{u}_{10}^{(r)} + q\mathbf{u}_{11}^{(r)} \end{bmatrix},$$

$$\mathbf{u}_{10}^{(r+1)} = \begin{bmatrix} p'\mathbf{u}_{00}^{(r)} + q'\mathbf{u}_{01}^{(r)} \\ p'\mathbf{u}_{10}^{(r)} + q'\mathbf{u}_{11}^{(r)} \end{bmatrix}, \quad \mathbf{u}_{11}^{(r+1)} = \begin{bmatrix} \mathbf{u}_{01}^{(r)} \\ \mathbf{u}_{11}^{(r)} \end{bmatrix},$$

transforms a set of four column vectors into another such set of four with twice the dimension, and an r-fold iteration of this transformation on four scalars $a_{\alpha\beta}$ (regarded as vectors of dimension 1) yields four vectors of dimension 2^r which will be denoted individually by $\varphi_{\alpha\beta}^r (a_{00}, a_{01}, a_{10}, a_{11})$ with $\alpha, \beta = 0, 1$.

Thus, for example, the sum of all the columns of the 2^r by 2^r matrix $\vartheta_{\alpha\beta}^r(a, b, c, d)$ is the vector $\varphi_{\alpha\beta}^r(a, b, c, d)$.

8. The evaluation of $h_k^{(r)}(p, p')$ and $\sigma^{(r)}(p, p')$

We may suppose that we have an infinite source of wet atoms $\{(x, -x) : x \geqq 0\}$. Then h_k is the probability that the first k segments S_c, with

$c = 0, 1, \cdots, k - 1$, are all in states whose labels $i(S_c)$ are odd numbers. Hence, in order to calculate h_k, we may confine attention exclusively to the 2nd, 4th, $\cdots$, states, whose labels i are $1, 3, \cdots, 2^r - 1$. The appropriate one-step transition-probability matrix (which is not, of course, a stochastic matrix) consists of half the rows and columns of $\mathbf{A}_{00}^{(r)}$ and is

$$(26) \qquad \mathbf{B} = \mathbf{B}^{(r)}(p, p') = \vartheta_{00}^{r-1}(qq', q, q', 1).$$

Then $h_k^{(r)}(p, p')$ is the sum of the absolute probabilities of all such states in the kth segment S_{k-1}, given the transition matrix $\mathbf{B}$, and the initial probabilities (20). That is to say,

$$(27) \qquad h_k = h_k^{(r)}(p, p') = \mathbf{y}(\mathbf{B})^{k-1}\mathbf{u},$$

where $\mathbf{y}$ is a row vector and $\mathbf{u}$ a column vector whose elements are defined respectively by

$$(28) \qquad \begin{aligned} u_i &= 1 \text{ for } i = 0, 1, \cdots, 2^{r-1} - 1, \\ y_i &= \begin{cases} p'^{r-1-m}q' & \text{if } i = 2^m - 1, \\ 0 & \text{otherwise.} \end{cases} \qquad m = 0, 1, \cdots, r - 1, \end{aligned}$$

Hence, from (11)

$$(29) \qquad \sigma^{(r)}(p, p') = \mathbf{y}(\mathbf{I} - \mathbf{B})^{-1}\mathbf{u},$$

where $\mathbf{B}$, $\mathbf{y}$, and $\mathbf{u}$ are defined in (26) and (28).

We now notice that, for $r \geqq 2$, each even-numbered row of $\mathbf{B}$ is q times its successor (so that $\mathbf{B}$ is singular). Hence, if $\mathbf{T}$ denotes the square matrix of order 2^{r-1} whose elements t_{ij} are defined by

$$(30) \qquad t_{ij} = \begin{cases} q & \text{if } i = 2j, \\ 1 & \text{if } 2j - i = -1 \text{ or } 2^{r-1}, \\ 0 & \text{otherwise,} \end{cases}$$

then, on replacing $\mathbf{B}$ by $\mathbf{T}^{-1}\mathbf{B}\mathbf{T}$ and transforming $\mathbf{y}$ and $\mathbf{u}$ appropriately, we find that σ is given by (29) with

$$(31) \qquad u_i = \begin{cases} 1 & \text{if } 0 \leqq i \leqq 2^{r-2} - 1, \\ p & \text{if } 2^{r-2} \leqq i \leqq 2^{r-1} - 1. \end{cases}$$

$$(32) \qquad y_i = \begin{cases} (1 + p'q)p'^{r-2}q' & \text{if } i = 0, \\ p'^{r-2-m}q' & \text{if } i = 2^m - 1, \\ p'^{r-1}q' & \text{if } i = 2^{r-2}, \\ 0 & \text{otherwise.} \end{cases} \qquad m = 1, 2, \cdots, r - 2,$$

$$(33) \qquad \mathbf{B} = \begin{bmatrix} \mathbf{C} & \mathbf{D} \\ \mathbf{O} & \mathbf{O} \end{bmatrix},$$

where

$$(34) \qquad \begin{aligned} \mathbf{C} &= \vartheta_{00}^{r-2}(qq'(2 - qq'), q(1 + pq'), q'(1 + p'q), 1), \\ \mathbf{D} &= \vartheta_{00}^{r-2}(q'(1 - qq'), pq', p'q', 0), \end{aligned}$$

and obviously

$$(35) \qquad (\mathbf{I} - \mathbf{B})^{-1} = \begin{bmatrix} (\mathbf{I} - \mathbf{C})^{-1} & \mathbf{O} \\ \mathbf{O} & \mathbf{I} \end{bmatrix} \times \begin{bmatrix} \mathbf{I} & \mathbf{D} \\ \mathbf{O} & \mathbf{I} \end{bmatrix}.$$

Taking out separately the contribution to σ arising from y_k, where $k = 2^{r-2}$, we find that

$$(36) \qquad \sigma^{(r)}(p, p') = p'^{r-1}q'p + q'\mathbf{z}(\mathbf{I} - \mathbf{C})^{-1}\mathbf{v},$$

where $q = 1 - p$, $q' = 1 - p'$, $\mathbf{C}$ is given by (34),

$$(37) \qquad z_i = \begin{cases} p'^{r-2}(1 + p'q) & \text{if } i = 0, \\ p'^{r-2-m} & \text{if } i = 2^m - 1, \qquad m = 1, 2, \cdots, r - 2, \\ 0 & \text{for all other values of } i, \qquad 0 \leqq i \leqq 2^{r-2} - 1, \end{cases}$$

and $\mathbf{v}$, the sum of the columns of $\mathbf{I} + p\mathbf{D}$ is defined by

$$(38) \qquad \mathbf{v} = \varphi_{00}^{r-2}[1 + pq'(1 - qq'), 1 + p^2q', 1 + pp'q', 1].$$

9. The critical curves for the processes $\mathfrak{P}_r$, $1 \leqq r \leqq 9$

The solutions for the cases $r = 1, 2$ are given by (13), (15). For other values of r we let $p = \rho \cos \theta$, $p' = \rho \sin \theta$, where $\theta = k\pi/32$ and $k = 1, \cdots, 8$. Then $\rho(r, k)$ denotes the critical value of ρ, for which $E^{(r)}(\rho \cos \theta, \rho \sin \theta) = 0$. This quantity is tabulated in table I, with accuracy within 10^{-7} for $1 \leqq r \leqq 7$, 3.10^{-5}

TABLE I

VALUES OF $\rho(r, k)$

r	$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 5$	$k = 6$	$k = 7$	$k = 8$
1	.9554740	.9191702	.8899865	.8670910	.8498639	.8378578	.8307705	.8284271
2	.9564690	.9224798	.8961947	.8762765	.8617366	.8518512	.8461185	.8442406
3	.9573331	.9250638	.9006607	.8824817	.8693897	.8605815	.8555107	.8538558
4	.9580929	.9271526	.9040640	.8870145	.8748159	.8666500	.8619658	.8604401
5	.9587677	.9288843	.9067607	.8904986	.8789034	.8711645	.8667360	.8652954
6	.9593722	.9303478	.9089599	.8932760	.8821163	.8746839	.8704384	.8690590
7	.9599177	.9316042	.9107938	.8955524	.8847233	.8775227	.8734161	.8720830
8	.96041	.93270	.91235	.89746	.88689	.87987	.87588	.87458
9	.9602	.9340	.9126	.8990	.8887	.8818	.8775	.8767
∞	.9668	.9440	.9240	.9110	.9015	.8951	.8919	.8905

for $r = 8$, and 10^{-3} for $r = 9$. In fact, when $r = 9$, the accuracy is probably 10^{-4} except that the case $k = 7$ is somewhat doubtful and the cases $k = 1, 2, 3$ are rather difficult to compute. Included in table I is the set of values of $\rho(\infty, k)$ obtained by the method of extrapolation explained in section 11.

10. Proof of convergence to the required limit

This proof is due to J. M. Hammersley. Consider the countable sample space $\Omega = \{\omega\}$ consisting of all the different specifications of dammed and undammed

bonds. Then fixing p and p' induces a fixed probability measure $\mu = \mu(p, p')$ on the subsets of Ω.

Define the following subsets of Ω

$$S = \{\omega : \mathfrak{P} \text{ wets infinitely many atoms}\},$$
$$S_n = \{\omega : \mathfrak{P} \text{ wets some atom on } x + y = n\},$$
$$S^r = \{\omega : \mathfrak{P}_r \text{ wets infinitely many atoms}\},$$
$$S_n^r = \{\omega : \mathfrak{P}_r \text{ wets some atom on } x + y = n\}.$$

We have to prove that $\mu(S) = 0$ if and only if $\lim_{n\to\infty}\mu(S^n) = 0$, and we shall in fact prove the stronger result (45).

The following relations are evident:

$$(39) \qquad\qquad S = \bigcap_{n=1}^{\infty} S_n \subseteq S_{n+1} \subseteq S_n,$$

$$(40) \qquad\qquad S \subseteq S^r \subseteq S_r^r = S_r.$$

From (39) and the standard theory of measure functions

$$(41) \qquad\qquad \lim_{n\to\infty} \mu(S_n) = \mu(S).$$

Hence for prescribed $\epsilon > 0$, there exists $N(\epsilon)$ such that

$$(42) \qquad\qquad \mu(S_n) < \mu(S) + \epsilon, \qquad\qquad n > N(\epsilon).$$

By (40) we have

$$(43) \qquad\qquad \mu(S) \leqq \mu(S^r) \leqq \mu(S_r).$$

From (42) and (43) it follows that

$$(44) \qquad\qquad \mu(S) \leqq \mu(S^n) < \mu(S) + \epsilon, \qquad\qquad n > N(\epsilon),$$

and hence

$$(45) \qquad\qquad \lim_{n\to\infty} \mu(S^n) = \mu(S),$$

because ϵ is arbitrary.

11. The critical curve for the process $\mathfrak{P}$

We fix $k (= 1, 2, \cdots, 8)$. Then the sequence $\{\rho(r, k) : r = 1, 2, \cdots\}$ is a monotonic increasing sequence whose limit $\rho(\infty, k)$ provides a point on the required critical curve. It transpires that a formula which converges rapidly and which correctly gives the first nine terms of the sequence, to the accuracy to which they are known, is

$$(46) \qquad\qquad \rho(r, k) = \rho(\infty, k) - A\lambda^r - B\mu^r - C\nu^r,$$

where $A, B, C, \lambda, \mu, \nu$ are suitably chosen positive constants depending on k. Their values, together with those of p and p', are tabulated in table II. It will be noticed that the symmetric case $(p = p')$ is given by $k = 8$ and the table can be continued to $k = 16$ $(\theta = \pi/2)$ by interchanging p and p'. In the sym-

TABLE II

	$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 5$	$k = 6$	$k = 7$	$k = 8$
ρ_∞	.9668	.9440	.9240	.9110	.9015	.8951	.8919	.8905
λ	.9345	.9280	.8930	.8886	.8820	.8774	.8790	.8760
μ	.7445	.7348	.6341	.6152	.5914	.5780	.5817	.5744
ν	.2501	.4187	.2884	.2679	.2465	.2395	.2457	.2412
A	.0101	.0178	.0250	.0300	.0342	.0371	.0385	.0393
B	.0013	.0065	.0084	.0125	.0154	.0173	.0190	.0191
C	.0000	.0005	.0007	.0014	.0021	.0028	.0036	.0037
p	.9622	.9259	.8842	.8417	.7950	.7442	.6895	.6297
p'	.0948	.1842	.2682	.3486	.4249	.4973	.5658	.6297

metric case, Hammersley [3] has shown rigorously that $p = p' < 0.849585\cdots$. It must be emphasized that the extrapolation here is purely conjectural—all that has been rigorously provided is a few terms of a monotone sequence which is known to converge to the required limit.

The computation was carried out on Mercury in the Oxford University Computing Laboratory, and I here express my gratitude to the staff for their assistance. It will be seen from (22) that the calculation was particularly well adapted to the use of binary digits.

My thanks are also due to Mr. Hammersley, with whom I have had many valuable discussions.

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