RANDOM OPERATOR EQUATIONS

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1. Introduction and summary

In recent years a number of papers have appeared in which random variables with values in spaces more general than the space of real or complex numbers were dealt with. These papers differ mainly in their approach to the generalization of notions known from classical probability theory. Thus measurability, integrability, convergence, and so on, are treated differently in different papers.

Many of these papers studied some problems connected with the theory of random operator equations. It should be mentioned that Czechoslovak probabilists have systematically attacked this part of probabilistic functional analysis since the year 1955, when Špaček [22] published his paper on random equations. Dealing with contraction mappings, Špaček proved the measurability of the random fixed point in the case of the space of real numbers and, under some further restrictive assumptions, the measurability of the fixed point for arbitrary complete metric spaces. All Špaček's restrictive assumptions were removed in [12], provided the complete metric space is question is separable. The proof of this result is based on the limit theorem for arbitrary metric spaces (see [13]) and on the substitution theorem for separable metric spaces (see [13]).

Under essentially the same restrictive assumptions, Špaček studied in [23] the measurability of invertible random transformations, being partially motivated by the necessary and sufficient condition for almost sure regularity of "function-space type" measurable transformations as developed in [21].

Further papers closely connected with random operator equations are concentrated mainly in the Transactions of the First and Second Prague Conferences. Of these [9], [13], [16], and [20] deal with general properties of generalized random variables; [7], [8], [10], [11], [14], and [17] deal with stochastic approximation methods and experience theory; and [15] deals with inverse and adjoint transformations. Some other papers published in the Transactions, though not directly related to random operator equations, have played an important part in the process of forming the ideas of experience theory and their connection with general stochastic approximation methods.

So much for the contributions of the Czechoslovak school of probabilistic functional analysis. Probabilists of other countries have, of course, contributed to a great extent to the development of this area of probability theory, and a number of references can be found in [9].

In the present paper some recent work concerning random operator equations is treated from a unified viewpoint; also some new results are stated. In sections 2 and 3 indispensable notions are introduced and three auxiliary theorems are recalled. Section 4 is devoted to the generalization of the well-known principle of contraction mappings. Linear bounded random transformations, which are not necessarily contraction mappings, are dealt with in section 5. Some results from experience theory are stated in section 6. General theorems on random integral equations are given in section 7, whereas section 8 is devoted to the discussion of some problems raised by Bharucha-Reid in [3].

2. Prerequisites

First of all we shall introduce some basic concepts indispensable for our further considerations. Throughout the whole paper, unless there is a statement to the contrary, we shall employ the following notation:

$(\Omega, \mathfrak{S}, \mu)$ denotes a probability space with a complete probability measure μ; that is, Ω is a nonempty set, $\mathfrak{S}$ a σ-algebra of subsets of the space Ω, and μ such a probability measure that $\mu(E_0) = 0$ implies $E \in \mathfrak{S}$ for every $E \subset E_0$.

X and Z are arbitrary separable Banach spaces and $\mathfrak{X}$ and $\mathfrak{Z}$ the σ-algebras of all Borel subsets of the spaces X and Z, respectively.

$E(X)$ denotes the algebra of endomorphisms on X; that is, the space of all linear bounded operators defined on the space X and taking values from the space X. Similarly, $\mathfrak{E}(X)$ is the σ-algebra of all Borel subsets of the space $E(X)$, provided the norm topology in $E(X)$ is assumed.

We now give some definitions.

DEFINITION 1. *A mapping V of the space Ω into the space Z is called a generalized random variable if $\{\omega: V(\omega) \in B\} \in \mathfrak{S}$ for all $B \in \mathfrak{Z}$. Two generalized random variables V and W are assumed to be equivalent if $V(\omega) = W(\omega)$ with probability one.*

DEFINITION 2. *A mapping T of the Cartesian product space $\Omega \times X$ into the space Z is called a random transformation if $T(\cdot, x)$ is for every $x \in X$ a generalized random variable.*

DEFINITION 3. *A mapping P of the Cartesian product space $\Omega \times D \times X$ into the space Z, where D is a direction, is called a generalized stochastic process if $P(\cdot, t, x)$ is a generalized random variable for every $t \in D$ and $x \in X$.*

All the notions of functional analysis, such as the inverse operator, the adjoint operator, and so on, are carried over in probabilistic functional analysis in an "almost sure" way; that is, for example, the mapping S is said to be the inverse of the random transformation T if $\mu\{\omega: T[\omega, S(\omega, x)] = x \text{ for every } x \in X\} = 1$.

Finally, we recall three theorems on generalized random variables (see [13]).

THEOREM 1. *If $V_1, V_2, \cdots$ is a sequence of generalized random variables with values in the space Z converging almost surely to the mapping V of the space Ω into the space Z, then V is a generalized random variable with values in the space Z.*

THEOREM 2. *A mapping V of the space Ω into the space Z is a generalized ran-*

dom variable if and only if for every linear bounded functional $f \in \Delta \subset Z^*$, where Δ is total on the whole space Z, the compound mapping $f(V)$ is a real-valued random variable.

THEOREM 3. *Let V be a generalized random variable with values in the space X; and T an almost surely continuous random transformation of the Cartesian product space $\Omega \times X$ into the space Z. Then the mapping W of the space Ω into the space Z defined for every $\omega \in \Omega$ by $W(\omega) = T[\omega, V(\omega)]$ is a generalized random variable with values in the space Z.*

REMARK 1. It should be noted that in general the spaces X and Z need not be separable Banach spaces; also, the σ-algebras $\mathfrak{X}$ and $\mathfrak{Z}$ need not be the σ-algebras of Borel subsets. Thus, in definitions 1 through 3, Z can be an arbitrary nonempty set and $\mathfrak{Z}$ an arbitrary σ-algebra of subsets of Z. Similarly, in definitions 2 and 3, X can be an arbitrary nonempty set without any particular σ-algebra $\mathfrak{X}$ specified.

REMARK 2. Theorem 1 holds even in the case Z is a (not necessarily separable and complete) metric space.

REMARK 3. Theorem 2 remains valid if the space Z is replaced by a separable normed linear space.

REMARK 4. Theorem 3 holds whenever X is a separable metric space and Z an arbitrary (not necessarily separable) metric space. Also, the Cartesian measurability of the random transformation T furnishes the measurability of W.

3. Some fundamental notions

Let X and Z be two separable Banach spaces, T a mapping of the space X into the space Z, and z a fixed element from Z. If Σ denotes the set of those elements $x \in X$ for which the equality $T(x) = z$ holds, that is if $\Sigma = \{x: T(x) = z\}$, then any $x \in \Sigma$ will be called a solution of the operator equation

$$T(\xi) = z. \tag{1}$$

If the set Σ is empty, we say that the operator equation (1) does not possess a solution; if it is nonempty we say that (1) is solvable. In case Σ consists of exactly one point we say that (1) has a unique solution.

Now, in addition, let $(\Omega, \mathfrak{S}, \mu)$ be a probability space with a complete probability measure μ and let $\mathfrak{X}$ and $\mathfrak{Z}$ be the σ-algebras of all Borel subsets of the space X and Z, respectively. If T is a random transformation of the Cartesian product space $\Omega \times X$ into the space Z, then

$$T(\cdot, \xi) = z \tag{2}$$

is said to be a random operator equation.

However, (2) does not express the most general form of a random operator equation. That is, the right side of (2) need not be a fixed element from the space Z, but can be replaced by a generalized random variable with values in the space Z. Moreover, it should be remarked that the solution of a random operator equation does in general depend on the choice of $\omega \in \Omega$. Consequently, the most general form of a random operator equation may be more precisely written as

$$T[\cdot, \xi(\cdot)] = z(\cdot), \tag{3}$$

where, as mentioned above, T is a random transformation of the Cartesian product space $\Omega \times X$ into the space Z and z is a generalized random variable with values in the space Z.

Similarly, as in the deterministic case, with only the exception of neglecting a set of probability measure zero, the wide sense solutions of (3) are defined. That is, every mapping W of the space Ω into the space X which satisfies the equality $T[\omega, W(\omega)] = z(\omega)$ for every ω from a set Ω_0 of probability measure one is said to be a wide sense solution of the equation (3). However, following the spirit of our previous papers, it is quite natural to require the condition of measurability to be fulfilled so that we may speak about random solutions. Thus if the wide sense solution is also measurable it will be called the random solution, and we can state

DEFINITION 4. *Every generalized random variable x with values in the space X satisfying the condition $T[\omega, x(\omega)] = z(\omega)$ with probability one will be called the random solution of the random operator equation* (3).

Evidently there may exist wide sense solutions that are not random solutions. Moreover, if the random operator equation (3) has more than one solution for every ω from a set of positive probability measure then there may be, depending on the σ-algebra $\mathfrak{S}$ of course, many wide sense solutions that are not measurable. As a simple illustration of this fact let us give

EXAMPLE 1. Let $X = R$ be the space of all real numbers, E a nonmeasurable subset of the space Ω, and T a random transformation of the Cartesian product space $\Omega \times X$ into the space X, defined for every $\omega \in \Omega$ and $x \in X$ by $T(\omega, x) = x^2 - 1$. Then the mapping W of the space Ω into the space X, defined by $W(\omega) = 1$ for $\omega \in E$ and by $W(\omega) = -1$ otherwise, is a wide sense solution, but not a random solution, of the random operator equation

$$T[\cdot, \xi(\cdot)] = 0. \tag{4}$$

Roughly speaking, we are therefore interested mainly in the case when for every $\omega \in \Omega$ there exists a unique solution of the deterministic operator equation $T(\omega, \xi) = z(\omega)$. More precisely, we shall investigate most frequently the case when there exists a unique wide sense solution, provided we identify two wide sense solutions differing only on a set of probability measure zero. Nevertheless, even under this restriction, the unique wide sense solution need not be measurable, as is shown by

EXAMPLE 2. Let $\Omega = R$ be the space of all real numbers, $\mathfrak{S}$ the σ-algebra of all at most denumerable sets of real numbers and their complements, and μ a complete probability measure defined by $\mu(E) = 0$ if E is at most denumerable and by $\mu(E) = 1$ otherwise. Further, let $X = R$ be also the space of all real numbers with the σ-algebra $\mathfrak{X} = \mathfrak{R}$ of all Borel subsets of the space X, and T a random transformation of the Cartesian product space $\Omega \times X$ into the space X defined by $T(\omega, x) = 0$ for $\omega = x$ and by $T(\omega, x) = 1$ otherwise. Then the unique wide sense solution W of the random operator equation (4), given for every $\omega \in \Omega$ by $W(\omega) = \omega$, is not a random solution because for example the set $\{\omega: W(\omega) \leqq 0\}$ does not belong to $\mathfrak{S}$.

Many other questions concerning the relationship between wide sense solutions and random solutions of the same random operator equation arise, and the greater part of them have been as yet unsolved. Unfortunately, we too are unable to present some useful theory of random operator equations unless some further assumptions are imposed. In the present paper we shall discuss mainly three particular cases, namely if (a) the separable Banach space X equals the separable Banach space Z; or (b) the random transformation T is almost surely linear and bounded; or (c) both (a) and (b) occur.

REMARK 5. All the considerations of section 3 can be generalized to the case when $(X, \mathfrak{X})$ and $(Z, \mathfrak{Z})$ are arbitrary measurable spaces.

4. Principle of contraction mappings

Considering the case $X = Z$ it is easy to verify that the operator equation (1) is equivalent to the operator equation $\xi + g(\xi)[T(\xi) - z] = \xi$, where g is a nonvanishing real-valued function defined on the space X. Therefore, denoting by S the operator defined for every $x \in X$ by $S(x) = x + g(x)[T(x) - z]$, every solution x_0 of the operator equation (1) is at the same time a fixed point of the operator S; that is, a point with the property $S(x_0) = x_0$; and vice versa.

Consequently, it is not a surprising fact that in the theory of random operator equations the main role is played by probabilistic versions of the well-known principle of contraction mappings and its many modifications and generalizations. Under appropriate assumptions this principle may furnish the existence, uniqueness, and measurability of the random solution of a random operator equation. Therefore, from this point on we shall use the "fixed-point" terminology rather than the "solution" terminology, though the latter may seem to many probabilists more lucid.

The following theorem is a useful starting point for other theorems of this kind.

THEOREM 4. *Let T be an almost surely continuous random transformation of the Cartesian product space $\Omega \times X$ into the space X satisfying the condition*

$$(5) \qquad \mu\left(\bigcup_{m=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcap_{x \in X} \bigcap_{y \in X} \{\omega: \|T^n(\omega, x) - T^n(\omega, y)\| \leqq \left(1 - \frac{1}{m}\right) \|x - y\|\}\right) = 1,$$

where for every $\omega \in \Omega$, $x \in X$, and $n = 1, 2, \cdots$ we set $T^1(\omega, x) = T(\omega, x)$ and $T^{n+1}(\omega, x) = T[\omega, T^n(\omega, x)]$. Then there exists a generalized random variable ϕ with values in the space X satisfying the relation

$$(6) \qquad u\{\omega: T[\omega, \phi(\omega)] = \phi(\omega)\} = 1.$$

Moreover, if there exists another generalized random variable ψ with the property $T[\omega, \psi(\omega)] = \psi(\omega)$ with probability one, then $\psi(\omega) = \phi(\omega)$ with probability one.

PROOF. Let us denote by E the set of those ω from the set occurring in round brackets of (5) for which the mapping $T(\omega, \cdot)$ is continuous. Evidently, according to the assumption, $\mu(E) = 1$. Now let us define the mapping ϕ of the space Ω into the space X so that for every $\omega \in E$ the point $\phi(\omega)$ equals the unique fixed point of the mapping $T(\omega, \cdot)$ and for every $\omega \in \Omega - E$ we set $\phi(\omega) = \theta$, where θ is the null element of the Banach space X. Thus relation (6) holds. In order to prove the measurability of the mapping ϕ we make use of theorems 1 and 3. The remaining statement follows immediately from the uniqueness of the fixed point of the mapping $T(\omega, \cdot)$ for every $\omega \in E$.

The theorem just proved forms a generalization of the author's previous result, the stronger assumptions of which enable one to formulate condition (5) in a more lucid and seemingly less restrictive way.

THEOREM 5. *Let T be an almost surely continuous random transformation of the Cartesian product space $\Omega \times X$ into the space X and c a real-valued random variable so that*

$$\mu\{\omega: c(\omega) < 1\} = 1, \tag{7}$$

and

$$\mu\{\omega: ||T(\omega, x) - T(\omega, y)|| \leqq c(\omega)||x - y||\} = 1 \tag{8}$$

for every two elements x and y from X. Then there exists a generalized random variable ϕ with values in the space X for which (6) *holds.*

PROOF. Let us denote by E_1 the set occurring in (7), by $E_{x,y}$ the set occurring in (8), and by E_2 the set of those $\omega \in \Omega$ for which $T(\omega, \cdot)$ is continuous in x. Since the Banach space X is separable, we can replace the intersections in the expression $\bigcap_{x\in X} \bigcap_{y\in X} [E_{x,y} \cap E_1 \cap E_2]$ by intersections over a countable dense subset of the space X. Thus condition (5) is fulfilled with $n = 1$, which proves our theorem.

Roughly speaking, heretofore we were interested in the existence, uniqueness, and measurability of the solution of the random operator equation (3) for a given generalized random variable z with values in the space X. However, very often one has to consider the same questions for an arbitrary generalized random variable z with values in the space X. This is equivalent to the question of the existence and measurability of the inverse of a random transformation T.

The following result is an immediate consequence of theorem 5.

THEOREM 6. *Let c be a real-valued random variable and T an almost surely continuous random transformation of the Cartesian product space $\Omega \times X$ into the space X satisfying the condition* (8). *Then for every real number $\lambda \neq 0$ such that $c(\omega) < |\lambda|$ with probability one there exists a random transformation S that is the inverse of the random transformation $(T - \lambda I)$, I denoting the identity operator.*

PROOF. Evidently, since $\lambda \neq 0$, the random transformation $(T - \lambda I)$ is invertible whenever the random transformation $[(1/\lambda)T - I]$ is invertible, and vice versa. However, for every $z \in X$ the random transformation T_z defined for every $\omega \in \Omega$ and $x \in X$ by $T_z(\omega, x) = (1/\lambda)T(\omega, x) - z$ is almost surely a contraction mapping. Therefore by theorem 5 there exists a unique random fixed point x_z satisfying the relation $x_z(\omega) = (1/\lambda)T[\omega, x_z(\omega)] - z$ with probability one. Since the last statement is equivalent to the invertibility of the random transformation $[(1/\lambda)T - I]$, theorem 6 is proved.

It is well known that any linear bounded operator A satisfies the Lipschitz condition with constant $||A||$, which is at the same time the smallest constant with such a property. Therefore, making use of this fact and some of the classical results about linear bounded operators, we can state

THEOREM 7. *Let T be an almost surely linear bounded random transformation of the Cartesian product space $\Omega \times X$ into the space X. Then for every real number $\lambda \neq 0$ such that $\mu(\bigcup_{n=1}^{\infty} \{\omega: ||T^n(\omega, \cdot)|| < |\lambda|^n\}) = 1$ there exists a linear bounded random transformation S that is the inverse of the random transformation $(T - \lambda I)$. Thus we have $\mu(\bigcap_{x\in X} \{\omega: S(\omega, x) = (-1/\lambda) \sum_{n=0}^{\infty} \lambda^{-n}T^n(\omega, x)\}) = 1$, where the sum is meant uniformly.*

Random operator equations in general, and random integral equations of Fredholm type in particular, have been considered by Bharucha-Reid in [1] through [5]. Although most parts of these papers are carried out in separable Orlicz spaces, some theorems are stated for separable Banach spaces. In this section we mention theorems 2.1 through 2.3 from [3] only, a stronger version of which can be proved using theorem 7. We can, namely, state

THEOREM 8. *Let T be a random transformation of the Cartesian product space $\Omega \times X$ into the space X which is for every $\omega \in \Omega$ linear and bounded. Then for every real number $\lambda \neq 0$ the set $\Omega(\lambda) = \{\omega: ||T(\omega, \cdot)|| < |\lambda|\}$ belongs to the σ-algebra $\mathfrak{S}$, the random transformation $(T - \lambda I)$ is invertible for every $\omega \in \Omega(\lambda)$, the resolvent operator $R_{\lambda,T}$ exists for every $\omega \in \Omega(\lambda)$, and for these ω we have $R_{\lambda,T}(\omega, \cdot) = -\sum_{n=1}^{\infty} \lambda^{-n}T^{n-1}(\omega, \cdot)$. Finally, for every $\omega \in \Omega(\lambda)$ the solution $s(\omega)$ of the operator equation $T(\omega, \xi) - \lambda\xi = z(\omega)$ is for every generalized random variable z with values in the space X given by $s(\omega) = R_{\lambda,T}[\omega, z(\omega)]$, where the resolvent operator $R_{\lambda,T}$, and consequently also the solution s, is measurable with respect to the σ-algebra $\Omega(\lambda) \cap \mathfrak{S}$.*

PROOF. Theorem 8 follows immediately from our preceding theorems and from well-known classical results, because of the fact that for almost all $\omega \in \Omega$ we have $||T(\omega, \cdot)|| = \sup_{n\geqq 1} ||T(\omega, x_n)||$ where $x_1, x_2, \cdots$ is a countable dense set of the sphere $\{x: ||x|| \leqq 1\}$.

REMARK 6. Theorems 4 and 5 remain valid if X is a complete separable metric space.

5. Linear bounded operators

Heretofore we have assumed case (a) or even (c). Now we shall state two theorems in which the spaces X and Z may be different separable Banach spaces, provided the random transformation T under consideration is almost surely linear and bounded. The proofs are omitted, because both the theorems are only slight modifications of theorems 1 and 2 in [15].

THEOREM 9. *The inverse of an almost surely linear bounded invertible random transformation T of the Cartesian product space $\Omega \times X$ into the space Z is a random transformation of the Cartesian product space $\Omega \times Z$ into the space X.*

THEOREM 10. *Let X and Z be two Banach spaces whose first adjoint spaces X^* and Z^* are separable; let $\mathfrak{X}$, $\mathfrak{Z}$, $\mathfrak{X}^*$, and $\mathfrak{Z}^*$ be the σ-algebras of all Borel subsets of the spaces X, Z, X^*, and Z^*, respectively. If T is an almost surely linear bounded mapping of the Cartesian product space $\Omega \times X$ onto the space Z, then the following two conditions are equivalent:* (i) *for almost all elements $\omega \in \Omega$ the mapping $T(\omega, \cdot)$ of the space X onto the space Z is invertible; and* (ii) *for almost all elements $\omega \in \Omega$ the range of the adjoint mapping $T^*(\omega, \cdot)$ is the whole space X^*. Further, if these conditions are satisfied then T^* is invertible and the inverse mapping $(T^*)^{-1}$ to the adjoint mapping T^* is almost surely equal to the adjoint mapping S^* of the inverse S to the mapping T. Moreover, if one of the mappings T, S, T^*, S^*, is a random transformation then all four mappings are random transformations.*

6. Experience theory

One of the most important problems in the theory of random operator equations is the question of the measurability of the solution, which has been dealt with in the preceding theorems. In this section we shall be concerned with another important question, namely, the relationship between the random solution of the random operator equation and the solution of the corresponding deterministic operator equation. More precisely, we shall discuss the case when the random operator equation (3) is such that the Bochner integrals $\int_\Omega z(\omega)\, d\mu(\omega) = z$ and $\int_\Omega T(\omega, x)\, d\mu(\omega) = S(x)$ exist for every $x \in X$. Let us assume that the solution of the deterministic operator equation $S(\xi) = z$ is equal to y. The question arises whether the expected value of the random solution of the random operator equation (3) exists, and if so, whether it is equal to the deterministic solution y.

It is not difficult to construct an example showing that there are cases in which the answer is affirmative. A most trivial one is that one when the probability measure μ is the Dirac measure; that is, when there exists an element $\omega_0 \in \Omega$ such that $\mu(\omega_0) = 1$. Another still trivial illustration is

EXAMPLE 3. Let T be a random transformation of the Cartesian product space $\Omega \times X$ into the space X defined for every $\omega \in \Omega$ and $x \in X$ by $T(\omega, x) = cx + V(\omega)$, where $c \neq 0$ is a real number and V is such a generalized random variable with values in the space X that the Bochner integral $\int_\Omega V(\omega)\, d\mu(\omega)$ equals θ, where θ is the null element of the space X. Then the expected value of the unique random solution of the random operator equation $T[\cdot, \xi(\cdot)] = \theta$ is equal to the solution of the operator equation $S(\xi) = \theta$ where, of course, $S(x) = cx$ for every $x \in X$.

It is also not difficult to give examples where the answer to the above stated question is negative.

However, there are many cases in which we are interested in the solution of the deterministic operator equation corresponding to the random operator equation, rather than in the expected value or even in the probability distribution of the random solution of the random operator equation under consideration. For example, the case of determining *LD* 50 is one of them.

For the sake of brevity and definiteness, let us call the deterministic operator equation associated with the random operator equation by means of "expected-value" correspondence simply the regression operator equation. Thus, an important problem of the theory of random operator equations is that of reaching the solution of the regression operator equation, and it is to this problem that the remainder of this section is devoted.

Since a detailed case history of this and similar problems is given by Driml and the author in [7], we will not go into details here, but will simply state four useful theorems only. Nevertheless, it should be noted that this branch of probability theory, often called experience theory, is very closely connected with stochastic approximations methods as developed by Robbins and Monro, and later by other authors.

The following theorem is due to Driml and the author and can be considered one of the basic theorems of experience theory.

THEOREM 11. *Let T be a generalized stochastic process mapping the Cartesian product space $\Omega \times [0, \infty) \times X$ into the space X and almost surely continuous with respect to both the arguments $t \geqq 0$ and $x \in X$ simultaneously. Let there exist an element $\hat{x} \in X$, a real-valued random variable c, and let the following assumptions together with* (7) *be satisfied*

$$\mu\{\omega: \lim_{t\to\infty} \|t^{-1} \int_0 T(\omega, s, \hat{x})\, ds - \hat{x}\| = 0\} = 1; \tag{9}$$

$$\mu\{\omega: \|T(\omega, t, x) - T(\omega, t, y)\| \leqq c(\omega)\|x - y\|\} = 1 \tag{10}$$

for every $t \geqq 0$ and every two elements x and y from X. Further, let $x_t(\cdot)$ be the solution of the random operator equation

$$\begin{aligned} \xi_0(\cdot) &= T[\cdot, 0, \xi_0(\cdot)]; \\ \xi_t(\cdot) &= t^{-1} \int_0^t T[\cdot, s, \xi_s(\cdot)]\, ds \qquad \textit{for} \quad t > 0. \end{aligned} \tag{11}$$

Then x_t is for every $t \geqq 0$ a generalized random variable and we have

$$\mu\{\omega: x_t(\omega) \textit{ is continuous in } t\} = 1 \tag{12}$$

and

$$\mu\{\omega: \lim \|x_t(\omega) - \hat{x}\| = 0\} = 1. \tag{13}$$

PROOF. Let us denote by C_∞ the space of all continuous mappings f of the space $[0, \infty)$ into the space X such that the relation $\lim_{t\to\infty} \|f(t) - \hat{x}\| = 0$ or, as we shall write hereafter, such that $f(t) \to \hat{x}$ holds. Introducing the distance function ρ for every pair of elements f and g from C_∞ by $\rho(f, g) = \sup_{t \geqq 0} ||f(t) - g(t)||$ the space C_∞ becomes a separable metric space whose σ-algebra of all Borel subsets is the σ-algebra generated by the class of sets $\{f : f(t) \in B\}$, where t runs over $[0, \infty)$ and B runs over $\mathfrak{X}$. Further, let us denote by S the operator of the Cartesian product space $\Omega \times C_\infty$ defined by $[S(\omega, f)](0) = T[\omega, 0, f(0)]$ and by $[S(\omega, f)](t) = t^{-1} \int_0^t T[\omega, s, f(s)]\, ds$ for every $f \in C_\infty$, $t > 0$, and every $\omega \in E$, where E equals the intersection over all $t \geqq 0$, $x \in X$, and $y \in X$ of the sets occurring in (7), (9), (10), and the set of those $\omega \in \Omega$ for which $T(\omega, t, x)$ is continuous in both t and x simultaneously. For every $\omega \in \Omega - E$, $f \in C_\infty$, and $t \in [0, \infty)$ let us put $[S(\omega, f)](t) = \hat{x}$. First of all let us prove that the mapping S maps the Cartesian product space $\Omega \times C_\infty$ into the space C_∞. Choose arbitrarily $\omega \in E$, $f \in C_\infty$, and $\delta > 0$. Then there exists a real number t_0 such that for every $t \geqq t_0$ we have $||f(t) - \hat{x}|| \leqq \delta/3$ and simultaneously $||t^{-1} \int_0^t T(\omega, s, \hat{x})\, ds - \hat{x}|| \leqq \delta/3$. Then for every $t \geqq t_0\rho(f, h)3/\delta$, where h denotes the function for which $h(t) = \hat{x}$ for every $t \in [0, \infty)$, we can write $||[S(\omega, f)](t) - \hat{x}|| \leqq t^{-1} \int_0^t ||T(\omega, s, f(s)) - T(\omega, s, \hat{x})||\, ds + ||[S(\omega, h)](t) - \hat{x}|| \leqq c(\omega)(t_0/t)\rho(f, h) + c(\omega)(1 - t_0/t)(\delta/3) + \delta/3 \leqq \delta$. Since the $\Omega - E$ part is trivial we have proved that S maps the Cartesian product space $\Omega \times C_\infty$ into the space C_∞. Now, let us prove that S is a contraction mapping. Thus, let us take arbitrarily $f \in C_\infty$, $g \in C_\infty$, and $\omega \in E$. (For $\omega \in \Omega - E$ we get a singular case.) Then using (10) we have $\rho[S(\omega, f), S(\omega, g)] \leqq c(\omega)\rho(f, g)$. The mapping S being a random contraction transformation of the Cartesian product space $\Omega \times C_\infty$ into the complete separable metric space C_∞, we can apply theorem 5, which asserts that there exists a generalized random variable ϕ with values in the space C_∞ so that $S[\omega, \phi(\omega)] = \phi(\omega)$ holds with probability one. However, we have $E \in \mathfrak{S}$, $\mu(E) = 1$; hence if we define x_t for every $\omega \in \Omega$ and every $t \in [0, \infty)$ by $x_t(\omega) = [\phi(\omega)](t)$ we immediately obtain all the assertions of theorem 11.

A generalization of the preceding theorem for almost surely linear bounded random transformations is

THEOREM 12. *Let T be a generalized stochastic process mapping the Cartesian product space $\Omega \times [0, \infty) \times X$ into the space X and almost surely continuous with respect to both the arguments $t \in [0, \infty)$ and $x \in X$ simultaneously. Denote by $T_{t_1, t_2, \cdots, t_n}$ the mapping of the Cartesian product space $\Omega \times X$ into the space X defined for every $\omega \in \Omega$, $x \in X$, and $j = 1, 2, \cdots$ by $T_{t_1}(\omega, x) = T(\omega, t_1, x)$ and $T_{t_1, \cdots, t_{j+1}}(\omega, x) = T[\omega, t_{j+1}, T_{t_1, \cdots, t_j}(\omega, x)]$. Let there exist an element $\hat{x} \in X$ such that condition* (9) *and the following conditions are satisfied.* (i) *$T(\omega, t, \alpha x + \beta y) = \alpha T(\omega, t, x) + \beta T(\omega, t, y)$ holds with probability one for every $t \in [0, \infty)$, $x \in X$, $y \in X$, $\alpha \in R$, and $\beta \in R$ separately;* (ii) *$\sup_{t \geqq 0} ||T(\omega, t, \cdot)|| < \infty$ with probability one; and* (iii) *$\mu(\bigcup_{m=1}^\infty \bigcup_{n=1}^\infty \bigcap_{x \in X} \bigcap_{y \in X} \{\omega : ||T_{t_1, \cdots, t_n}(\omega, x - y)|| \leqq (1 - 1/m)||x - y||\}) = 1$. Further, let $x_t(\cdot)$ be the solution of the random oper-*

ator equation (11). *Then x_t is for every $t \in [0, \infty)$ a generalized random variable, the solution is unique in the "almost sure" sense, and* (12) *and* (13) *hold.*

PROOF. The proof of theorem 12 follows essentially that of theorem 11, the greatest difference being in making use of the property that the Bochner integral $\int_\Omega A[f(\omega)]\, d\mu(\omega)$ equals $A\left[\int_\Omega f(\omega)\, d\mu(\omega)\right]$ for every linear bounded operator A.

Theorems 11 and 12 and many other theorems of the same type can claim to be generalizations of the fact that under certain appropriate assumptions the "decision process" converges to the sought fixed point of the regression transformation with probability one. However, in practical situations the statistician is unlikely to know whether all the necessary conditions are fulfilled. In particular the contraction property would in many cases be very difficult to verify. Nevertheless, some justification of the statistician's decision about the fixed point of the regression transformation is contained in

THEOREM 13. *Let T be a generalized stochastic process mapping the Cartesian product space $\Omega \times [0, \infty) \times X$ into the space X and let $\hat{x}$ be a fixed element of the space X. Let $x_t(\cdot)$ be the solution of the random operator equation* (11) *and let the following relations together with* (13) *hold.* (i) $\|T[\omega, t, x_t(\omega)] - T(\omega, t, \hat{x})\| \to 0$ *with probability one;* (ii) $\int_0^t \|T[\omega, s, x_s(\omega)]\|\, ds < \infty$ *with probability one for every $t \in [0, \infty)$. Then* (9) *holds.*

PROOF. According to relation (12) it will be sufficient to prove that $t^{-1} \int_0 T(\omega, s, \hat{x})\, ds - x_t(\omega) \to \theta$ with probability one, or that $t^{-1} \int_0^t T(\omega, s, \hat{x})\, ds - t^{-1} \int_0^t T[\omega, s, x_s(\omega)]\, ds \to \theta$ with probability one. However, the last relation follows immediately from our assumptions, which proves theorem 13.

Some other theorems dealing with experience theory problems can be found in [6], [7], [8], [10], [11], [14], and [17].

Now, in order to make theorem 13 more lucid, we shall state an immediate corollary of it. For this purpose we introduce two definitions.

DEFINITION 5. *We say that T is a weakly stationary generalized stochastic process of the Cartesian product space $\Omega \times [0, \infty) \times X$ into the space X if for every integer n, every n-tuple of nonnegative numbers $t_1, \cdots, t_n$, every n-tuple of sets $B_1, \cdots, B_n$ from the σ-algebra $\mathfrak{X}$, every positive number s, and every $x \in X$ we have* $\mu[\bigcap_{k=1}^n \{\omega: T(\omega, t_k, x) \in B_k\}] = \mu[\bigcap_{k=1}^n \{\omega: T(\omega, t_k + s, x) \in B_k\}]$.

DEFINITION 6. *We say that T is a weakly ergodic generalized stochastic process of the Cartesian product space $\Omega \times [0, \infty) \times X$ into the space X if for every linear bounded functional f from a total set $\Delta \subset X^*$ and every $x \in X$ we have* $t^{-1} \int_0^t f[T(\omega, s, x)]\, ds \to \int_\Omega f[T(\omega, 0, x)]\, d\mu(\omega)$ *with probability one.*

Using these definitions we can state

THEOREM 14. *Let T be a weakly stationary and weakly ergodic generalized stochastic process of the Cartesian product space $\Omega \times [0, \infty) \times X$ into the space X so that it is almost surely continuous in both the arguments t and x simultaneously.*

Suppose that $\int_\Omega ||T(\omega, 0, x)|| \, d\mu(\omega) < \infty$ *for every* $x \in X$. *Denote by* $x_t(\cdot)$ *the solution of the random operator equation* (11). *Then* (13) *implies* $S(\hat{x}) = \hat{x}$, *where for every* $x \in X$ *we put* $S(x) = \int_\Omega T(\omega, 0, x) \, d\mu(\omega)$.

Now, as the assumption of both weak stationarity and weak ergodicity as well as the assumption of almost sure continuity are quite natural in many practical cases, theorem 14 ensures that having constructed the decision process x_t by solving the random operator equation (11) this process does converge, if it converges at all, to the fixed point of the regression transformation S with probability one, whether or not the relations (7) and (10) hold.

REMARK 7. Other theorems in this section can be reformulated using the notions of weak stationarity and weak ergodicity.

REMARK 8. Analogues of theorems 11 through 14 for sequences of random transformations can be easily formulated.

7. Random integral equations

In the remainder of this paper we shall deal with a special type of random operator equations, namely, with random integral equations. Two particular cases will be treated: the case of random integral equations in the space of continuous functions and the case of random integral equations in Orlicz spaces.

Let us denote by C the space of all continuous functions defined on the closed interval $[0, d]$, $0 < d$. Introducing the norm $||x|| = \max_{0 \leqq u \leqq d} |x(u)|$ the space C becomes a separable Banach space.

First of all we shall recall a well-known result about integral equations in the space C.

THEOREM 15. *If* $k(u, v)$ *is bounded for every* $u \in [0, d]$ *and every* $v \in [0, d]$, *and if all discontinuity points of* k *are located on a finite number of curves* $v = \phi_i(u)$, *for* $i = 1, \cdots, n$, *where the functions* ϕ_i *are continuous, then the formula* $y(u) = \int_0^d k(u, v)x(v) \, dv$ *defines a compact linear operator on the space* C *into itself.*

Thus, let us denote by K the space of all functions k defined and bounded on the set $[0, d] \times [0, d]$ all of whose discontinuity points are located on a finite number of curves $v = \phi_i(u)$ and such that for every $u \in [0, d]$, $v \in [0, d]$, and for every sequence of real numbers $d \geqq \delta_1 > \delta_2 > \cdots > \delta_n \to 0$, we have $k(u, 0) = \lim_{n\to\infty} k(u, \delta_n)$ and $k(u, v) = \lim_{n\to\infty} k(u, v - \delta_n)$, provided $\delta_1 \leqq v$ in the latter case. Introducing the norm by $||k|| = \sup |k(u, v)|$, where sup is taken over $u \in [0, d]$ and $v \in [0, d]$, the space K becomes a separable normed linear space.

We can now state a result concerned with the relationship between the measurability of the random integral operator and the measurability of its kernel.

THEOREM 16. *Let* k *be a mapping of the space* Ω *into the space* K *and let the mapping* T *of the Cartesian product space* $\Omega \times C$ *into the space* C *be defined for every* $\omega \in \Omega$ *and every* $x \in C$ *by*

(14) $$T(\omega, x) = \int_0^d k(\omega, \cdot, v)x(v)\, dv.$$

Then the mapping T is for every $\omega \in \Omega$ a compact linear transformation of the space C into itself. Moreover the following four statements are equivalent.

(i) *T is a random transformation;*

(ii) *$\{\omega: k(\omega, u, v) < r\} \in \mathfrak{S}$ for every $u \in [0, d]$, $v \in [0, d]$, and every $r \in R$;*

(iii) *$\{\omega: k(\omega, \cdot, \cdot) \in B\} \in \mathfrak{S}$ for every $B \in \mathfrak{K}$;*

(iv) *$\{\omega: T(\omega, \cdot) \in B\} \in \mathfrak{S}$ for every $B \in \mathfrak{E}(C)$.*

PROOF. The compactness follows immediately from the classical result. To prove the equivalence of the four statements let us observe that both the set $\{g_{u,v}: u \in [0, d], v \in [0, d]\}$ and the set $\{h_{u,x}: u \in [0, d], x \in C\}$ are total sets of linear bounded functionals on the space K, provided we define for every $u \in [0, d]$, $v \in [0, d]$, $x \in C$, and $k \in K$

(15) $$g_{u,v}(k) = k(u, v)$$

and

(16) $$h_{u,x}(k) = \int_0^d k(u, v)x(v)\, dv.$$

Further, the set $\{g_u: u \in [0, d]\}$, where we put for every $u \in [0, d]$ and $x \in C$

(17) $$g_u(x) = x(u),$$

is a total set of linear bounded functionals on the space C. Finally, the set $\{g_{u,x}: u \in [0, d], x \in C\}$ is a total set of linear bounded functionals on the space $F = \{T: T(x) = \int_0^d k(\cdot, v)x(v)\, dv, k \in K, x \in C\}$, provided we set for every $u \in [0, d]$ and every $x \in C$

(18) $$g_{u,x}(T) = g_u[T(x)].$$

Now, the space F is a normed linear space and since for every $T \in F$ we have

(19) $$\|T\| = \sup \|T(x)\| = \sup \left\|\int_0^d k(\cdot, v)x(v)\, dv\right\|$$
$$= \sup \max \left|\int_0^d k(u, v)x(v)\, dv\right| \leqq \sup \max d\|k\|\,\|x\| = d\|k\|,$$

where sup is taken over $\|x\| = 1$ and max over $u \in [0, d]$, the separability of the normed linear space K implies the separability of the space F. Thus, all the spaces C, K, and F being separable normed linear spaces, the equivalence of our four statements follows from theorem 2; from the equalities (15) through (18); from the relation $\mathfrak{F} = F \cap \mathfrak{E}(C)$; and from the fact that $h_{u,x}(k) = g_{u,x}(T)$ holds for every $u \in [0, d]$, $x \in C$, and every $T \in F$ and $k \in K$ joint by relation (14).

The structure of the resolvent set of a linear bounded random transformation is described by

THEOREM 17. *Let T be a linear bounded random transformation of the Cartesian product space $\Omega \times X$ into the space X so that $\{\omega: T(\omega, \cdot) \in B\} \in \mathfrak{S}$ for every $B \in \mathfrak{E}(X)$. Let us denote by $\rho(T)$ the set of those pairs $(\omega, \lambda) \in \Omega \times R$ for which* *the linear random transformation* $(T - \lambda I)$ *has a linear bounded inverse. Then for every* $\lambda \in R$ *we have* $\{\omega: (\omega, \lambda) \in \rho(T)\} \in \mathfrak{S}$.

PROOF. According to the well-known theorem on resolvent sets in Banach algebras the set $B_\lambda = \{L: L \in E(X), \lambda \in \rho(L)\}$ is for every $\lambda \in R$ open, and, therefore, an element of the σ-algebra $\mathfrak{E}(X)$. Hence, theorem 17 follows from the equality $\{\omega: (\omega, \lambda) \in \rho(T)\} = \{\omega: T(\omega, \cdot) \in B_\lambda\}$ which holds for every $\lambda \in R$.

A sufficient condition for the invertibility of the linear random transformation $(T - \lambda I)$ is given in

THEOREM 18. *Let all the assumptions of theorem* 16 *be fulfilled. Let in addition the real number* λ *satisfy the inequality* $d\|k(\omega, \cdot, \cdot)\| < |\lambda|$ *with probability one. Then the linear random transformation* $(T - \lambda I)$ *is invertible, that is,* $\mu\{\omega: (\omega, \lambda) \in \rho(T)\} = 1$.

PROOF. The desired conclusion follows immediately from (19) and from theorem 7 for $n = 1$.

The possibility of reaching the solution of the regression integral equation using only one realization of the stationary and ergodic kernel is furnished by

THEOREM 19. *Let* k *be a mapping of the Cartesian product space* $\Omega \times [0, \infty)$ *into the space* K *so that* $\{\omega: k(\omega, t, u, v) < r\} \in \mathfrak{S}$ *for every* $u \in [0, d]$, $v \in [0, d]$, *and* $r \in R$; *so that* k *is stationary and ergodic; and so that* $k(\omega, t, \cdot, \cdot)$ *is continuous in* t *with probability one. Further, let* z *be a stationary and ergodic random transformation of the Cartesian product space* $\Omega \times [0, \infty)$ *into the space* X *and let* $\lambda \neq 0$ *be any real number satisfying*

$$\mu\{\omega: d\|k(\omega, 0, \cdot, \cdot)\| < |\lambda|\} = 1. \tag{20}$$

Denote by $x_t(\cdot)$ *the unique random solution of the random operator equation*

$$\int_0^d k(\cdot, 0, \cdot, v)[\xi_0(\cdot)](v)\, dv - \lambda\xi_0(\cdot) = z(\cdot, 0) \tag{21}$$

$$t^{-1} \int_0^t \int_0^d k(\cdot, s, \cdot, v)[\xi_s(\cdot)](v)\, dv\, ds - \lambda\xi_t(\cdot) = t^{-1} \int_0^t z(\cdot, s)\, ds \quad \textit{for every } t > 0.$$

Then there exists a unique solution $\hat{x} \in C$ *of the operator equation*

$$\int_0^d q(\cdot, v)\xi(v)\, dv - \lambda\xi = z_0 \tag{22}$$

for which (13) *holds, if* $\int_\Omega k(\omega, 0, \cdot, \cdot)\, d\mu(\omega) = q(\cdot, \cdot)$ *and* $\int_\Omega z(\omega, 0)\, d\mu(\omega) = z_0$.

PROOF. The existence of a unique solution follows from theorem 18 applied to the kernel q, and (13) is a consequence of theorem 11.

The next theorem forms an analogue of theorem 18 for Volterra kernels.

THEOREM 20. *Let all the assumptions of theorem* 16 *be fulfilled. Let in addition the kernel* k *satisfy the condition* $\mu\{\omega: k(\omega, u, v) = 0\} = 1$ *for every* $0 \leqq u < v \leqq d$. *Then for every real number* $\lambda \neq 0$ *the linear random transformation* $(T - \lambda I)$ *is invertible.*

PROOF. From our assumptions we get after several lines of computation the result that for almost every $\omega \in \Omega$ and every $n = 1, 2, \cdots$ we have $\|T^n(\omega, \cdot)\| \leqq$

$|\lambda d|^n \|k(\omega, \cdot, \cdot)\|^n/n!$ which enables us to make use of theorem 7, which yields the desired result.

Finally, we shall prove a theorem of experience type using the random solution of the random operator equation (21) for random integral equations of Volterra type.

THEOREM 21. *Let all the assumptions of theorem* 19, *with the exception of* (20), *be fulfilled. Let in addition the kernel k satisfy the condition $\mu\{\omega : k(\omega, t, u, v) = 0\} = 1$ for every $t \in [0, \infty)$ and every $0 \leqq u < v \leqq d$. Then there exists a unique solution $\hat{x} \in X$ of the operator equation* (22) *for which* (13) *holds.*

PROOF. The existence of a unique solution follows from theorem 20 applied to the kernel q, and (13) is furnished by theorem 12.

8. Random integral equations in Orlicz spaces

In this section some problems raised by Bharucha-Reid in [3] for random integral equations in separable Orlicz spaces are discussed. For the sake of brevity, the reader is referred to [19] and [24], where all the notions used here are defined.

Roughly speaking, given an n-dimensional interval U and a continuous function q, Bharucha-Reid considered in [3] random integral equations of the form

$$\int_{I_\omega} q(u, v)x(v)\, dv - \lambda x(u) = y(u), \tag{23}$$

where I_ω is for every $\omega \in \Omega$ an n-dimensional interval such that $I_\omega \subset U$. Although some results in [3] are given also for more general functions q, we will discuss mainly the case of continuous kernels.

However, it is known (see [19]) that, if U is a bounded closed subset of an n-dimensional Euclidean space, the Orlicz space of functions with domain U is isomorphic and isometric with the Orlicz space of functions with domain $[0, d]$, where d equals the Lebesgue measure of the set U, provided in both cases the same convex function is in action. Therefore we confine ourselves to the discussion of the one-dimensional case assuming $U = [0, d]$ and denoting the Orlicz space under consideration by M.

When dealing with measurability questions it is advantageous to rewrite the random integral equation (23) by introducing a random kernel k defined by $k(\omega, u, v) = q(u, v)$ for every $\omega \in \Omega$, $u \in [0, d]$, and every $v \in I_\omega$; and by $k(\omega, u, v) = 0$ for every $\omega \in \Omega$, $u \in [0, d]$, and every $v \in [0, d] - I_\omega$. Evidently, if q is a continuous deterministic kernel then the corresponding random kernel k is a mapping of the space Ω into the space K, and since every separable Orlicz space is at the same time a separable Banach space, theorem 16 gives a general answer to the measurability problem in words of the auxiliary random kernel k. Using this theorem we can immediately state necessary and sufficient conditions for the measurability of the integral transformation

$$T(\omega, x) = \int_{I_\omega} q(\cdot, v)x(v)\,dv \tag{24}$$

in terms of the random set I_ω. Denoting by Q the set of all continuous kernels q we have

THEOREM 22. *The mapping T of the Cartesian product space $\Omega \times M$ into the space M defined by* (24) *is for every $q \in Q$ a random transformation if and only if $\{\omega : v \in I_\omega\} \in \mathfrak{S}$ for every $v \in [0, d]$.*

PROOF. Theorem 22 is a special case of theorem 16 in which condition (ii) is specified by setting $k(\omega, u, v) = \chi_{I_\omega}(v)q(u, v)$ for every $\omega \in \Omega$, $u \in [0, d]$, and $v \in [0, d]$, where χ_A denotes the indicator of the set A.

However, nothing in general can be said about I_ω itself when a particular kernel $q \in Q$ is considered. As an illustration assume the case $q(u, v) = 0$ for every $u \in [0, d]$ and every $v \in [0, d]$. The integral transformation T defined by (24) equals in this case the null transformation and is consequently measurable regardless of the nature of the random set I_ω. Nevertheless, we can state

THEOREM 23. *Let $q \in Q$. Then the mapping T of the Cartesian product space $\Omega \times M$ into the space M defined by* (24) *is a random transformation if and only if $\chi_{I_\omega}(v)q(u, v)$ is a real-valued random variable for every $u \in [0, d]$ and every $v \in [0, d]$.*

PROOF. Theorem 23 follows immediately from theorem 16.

Let us now suppose that for every $\omega \in \Omega$ the random set I_ω is equal to the interval $[0, \beta(\omega)]$. Then we can state

THEOREM 24. *Under the assumption that $I_\omega = [0, \beta(\omega)]$ for every $\omega \in \Omega$ the mapping T of the Cartesian product space $\Omega \times M$ into the space M defined by* (24) *is for every $q \in Q$ a random transformation if and only if β is a real-valued random variable.*

PROOF. Theorem 24 is a consequence of theorem 22.

However, for the case $I_\omega = [\alpha(\omega), \beta(\omega)]$ the situation is no longer unambiguous. We have

THEOREM 25. *If $I_\omega = [\alpha(\omega), \beta(\omega)]$ for every $\omega \in \Omega$ and $\alpha < \beta$ with probability one, then the mapping T of the Cartesian product space $\Omega \times M$ into the space M defined by* (24) *is for every $q \in Q$ a random transformation if and only if α and β are two real-valued random variables.*

PROOF. Theorem 25 follows from theorem 22 and the relation $\{\omega : \beta(\omega) \geqq r_0\} = \bigcup \{\omega : \alpha(\omega) \leqq r \leqq \beta(\omega)\}$ which holds for every rational number r_0, provided the union is taken over all rational numbers $r \geqq r_0$.

On the other hand, if there exists a set $E \in \mathfrak{S}$ for which $\mu(E) > 0$ and $E \subset \{\omega : \alpha(\omega) = \beta(\omega)\}$ then, depending on the structure of the σ-algebra $\mathfrak{S}$ of course, α and β may not be random variables. This fact can be illustrated by

EXAMPLE 4. Let $(\Omega, \mathfrak{S}, \mu)$ be the same as in example 2 and let $I_\omega = [\alpha(\omega), \beta(\omega)]$ for every $\omega \in \Omega$, where $\alpha(\omega) = \beta(\omega) = \omega - [\omega/d]\,d$ for every $\omega \in \Omega$, assuming $[\omega/d]$ is defined by $(\omega/d) - 1 < [\omega/d] \leqq (\omega/d)$. Then $\chi_{I_\omega}(v)$ is for every $v \in [0, d]$ a real-valued random variable, but $\{\omega: 2\alpha(\omega) < d\} = \{\omega: 2\beta(\omega) < d\}$ does not belong to the σ-algebra $\mathfrak{S}$.

Hence we see that it is possible to avoid the condition required in [3] of the measurability of the solution of the random integral equation (23) from the discussion of the measurability of the integral operator T defined by (24). Moreover, the measurability of the solution for every generalized random variable y, the right side of equation (23), follows from our theorem 9, provided the linear random transformation $(T - \lambda I)$ is invertible.

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