A MARTINGALE SYSTEM THEOREM AND APPLICATIONS

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1. Introduction

Let $(W, \mathfrak{F}, P)$ be a probability space with points $\omega \in W$ and let $(y_n, \mathfrak{F}_n)$, $n = 1, 2, \cdots$, be an *integrable stochastic sequence:* y_n is a sequence of random variables, $\mathfrak{F}_n$ is a sequence of σ-algebras with $\mathfrak{F}_n \subset \mathfrak{F}_{n+1} \subset \mathfrak{F}$, y_n is measurable with respect to $\mathfrak{F}_n$, and $E(y_n)$ exists, $-\infty \leqq E(y_n) \leqq \infty$. A random variable $s = s(\omega)$ with positive integer values is a *sampling variable* if $\{s \leqq n\} \in \mathfrak{F}_n$ and $\{s < \infty\} = W$. (We denote by $\{\cdots\}$ the set of all ω satisfying the relation in braces, and understand equalities and inequalities to hold up to sets of P-measure 0.) We shall be concerned with the problem of finding, if it exists, a sampling variable s which maximizes $E(y_s)$.

To define a sampling variable s amounts to specifying a sequence of sets $B_n \in \mathfrak{F}_n$ such that

$$0 = B_0 \subset \cdots \subset B_n \subset B_{n+1} \subset \cdots; \bigcup_1^\infty B_n = W, \tag{1}$$

the sampling variable s being defined by

$$\{s \leqq n\} = B_n, \qquad \{s = n\} = B_n - B_{n-1}. \tag{2}$$

We shall be particularly interested in the case in which the sequence $(y_n, \mathfrak{F}_n)$ is such that the sequence of sets

$$B_n = \{E(y_{n+1}|\mathfrak{F}_n) \leqq y_n\} \tag{3}$$

satisfies (1). We shall call this the *monotone case*. In this case a sampling variable s is defined by

$$\{s \leqq n\} = \{E(y_{n+1}|\mathfrak{F}_n) \leqq y_n\}, \tag{4}$$

and s satisfies

$$E(y_{n+1}|\mathfrak{F}_n) \begin{cases} > y_n, & s > n, \\ \leqq y_n, & s \leqq n. \end{cases} \tag{5}$$

The relations (5) will be fundamental in what follows.

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In the monotone case we have for the sampling variable s defined by (4) the following characterization:

$$(6)\qquad s = \text{least positive integer } j \text{ such that } E(y_{j+1}|\mathfrak{F}_j) \leqq y_j.$$

Now even in the nonmonotone case we can always define a random variable s by (6), setting $s = \infty$ if there is no such j; let us call it the *conservative* random variable. The following statement is evident: the necessary and sufficient condition that there exists a sampling variable s satisfying (5) is that we are in the monotone case, and in this case s is the conservative random variable.

In section 3 we are going to show that *in the monotone case, under certain regularity assumptions, the conservative sampling variable s maximizes $E(y_s)$.*

2. An example

Before proceeding with the general theory we shall give a simple and instructive example of the monotone case in the form of a *sequential decision problem.*

Let $x, x_1, x_2, \cdots$ be a sequence of independent and identically distributed random variables with $E(x^+) < \infty$, where we denote $a^+ = \max(a, 0)$, $a^- = \max(-a, 0)$. We observe the sequence $x_1, x_2, \cdots$ sequentially and can stop with any $n \geqq 1$. If we stop with x_n we receive the reward $m_n = \max(x_1, \cdots, x_n)$, but the cost of taking the observations $x_1, \cdots, x_n$ is some strictly increasing function $g(n) \geqq 0$, so that our net gain in stopping with x_n is $y_n = m_n - g(n)$. The decision whether to stop with x_n or to take the next observation x_{n+1} must be a function of $x_1, \cdots, x_n$ alone. *Problem:* what stopping rule maximizes the expected value $E(y_s)$, where s is the random sample size defined by the stopping rule? We assume that the distribution function $F(u) = P\{x \leqq u\}$ is known. That $E(y_n)$ exists follows from the inequality

$$(7)\qquad y_n^+ \leqq x_1^+ + \cdots + x_n^+,$$

which implies that $E(y_n^+) < \infty$.

Let $\mathfrak{F}_n$ be the σ-algebra generated by $x_1, \cdots, x_n$. Then $(y_n, \mathfrak{F}_n)$ is an integrable stochastic sequence, and we have

$$(8)\qquad \begin{aligned} E(y_{n+1}|\mathfrak{F}_n) - y_n &= \int [m_{n+1} - m_n]\, dF(x_{n+1}) - [g(n+1) - g(n)] \\ &= \int (x - m_n)^+\, dF(x) - f(n), \end{aligned}$$

where we have set

$$(9)\qquad f(n) = g(n+1) - g(n) = \text{cost of taking the } (n+1)\text{st observation.}$$

Since we have assumed $g(n)$ to be strictly increasing, and $f(n) > 0$, it is easily seen that there exist unique constants α_n such that

$$(10)\qquad \int (x - \alpha_n)^+\, dF(x) = f(n), \qquad n \geqq 1.$$

By (8) and (10),

$$(11)\qquad E(y_{n+1}|\mathfrak{F}_n) \begin{cases} > y_n & \text{if} \quad m_n < \alpha_n, \\ \leqq y_n & \text{if} \quad m_n \geqq \alpha_n. \end{cases}$$

The conservative random variable s defined by (6) is therefore

$$(12)\qquad s = \text{least positive integer } j \text{ such that } m_j \geqq \alpha_j.$$

We are in the monotone case if and only if this s is a sampling variable and for every n

$$(13)\qquad \{E(y_{n+1}|\mathfrak{F}_n) \leqq y_n\} \subset \{E(y_{n+2}|\mathfrak{F}_{n+1}) \leqq y_{n+1}\},$$

that is, $m_n \geqq \alpha_n$ implies $m_{n+1} \geqq \alpha_{n+1}$, which will certainly be the case, since $m_n \leqq m_{n+1}$, if $\alpha_n \geqq \alpha_{n+1}$, that is, if $f(n)$ is *nondecreasing* and hence α_n is *nonincreasing. We shall henceforth assume this to hold.* We shall now show that in this case the conservative random variable s is in fact a sampling variable, that is, that $P\{s < \infty\} = 1$. We have

$$(14)\qquad \{s > n\} = \{m_n < \alpha_n\},$$

and hence

$$(15)\qquad P\{s < \infty\} = 1 - \lim_n P\{s > n\} = 1 - \lim_n P\{m_n < \alpha_n\}$$
$$\geqq 1 - \lim_n P\{m_n < \alpha_1\} = 1 - \lim_n P^n\{x < \alpha_1\} = 1,$$

since by hypothesis $f(1) > 0$ so that by (10), $P\{x < \alpha_1\} < 1$. In fact, for any $r \geqq 0$,

$$(16)\qquad E(s^r) = \sum_{n=1}^{\infty} n^r P\{s = n\} \leqq \sum_{n=1}^{\infty} n^r P\{s > n - 1\}$$
$$\leqq 1 + \sum_{n=2}^{\infty} n^r P\{m_{n-1} < \alpha_1\}$$
$$= 1 + \sum_{n=2}^{\infty} n^r P^{n-1}\{x < \alpha_1\} < \infty,$$

so that s has finite moments of all orders.

It is of interest to consider the special case $g(n) = cn$, $0 < c < \infty$. Here $f(n) = c$ and $\alpha_n = \alpha$, where α is defined by

$$(17)\qquad \int (x - \alpha)^+ \, dF(x) = c,$$

and s is the first $j \geqq 1$ for which $x_j \geqq \alpha$. Hence

$$P\{s = j\} = P\{x \geqq \alpha\}P^{j-1}\{x < \alpha\},$$

$$E(s) = \frac{1}{P\{x \geqq \alpha\}},$$

$$E(y_s) = \sum_{j=1}^{\infty} P\{s = j\}E(m_j - cj|s = j), \tag{18}$$

$$E(m_j|s = j) = E(x_j|x_1 < \alpha, \cdots, x_{j-1} < \alpha, x_j \geqq \alpha) = \frac{1}{P\{x \geqq \alpha\}} \int_{\{x \geqq \alpha\}} x\, dF(x),$$

so that

$$E(y_s) = \frac{1}{P\{x \geqq \alpha\}}\left[\int_{\{x \geqq \alpha\}} x\, dF(x) - c\right] \tag{19}$$

$$= \frac{1}{P\{x \geqq \alpha\}}\left[\int (x - \alpha)^+ dF(x) - c + \alpha P\{x \geqq \alpha\}\right] = \alpha,$$

an elegant relation.

3. General theorems

In the following three lemmas we assume that $(y_n, \mathfrak{F}_n)$ is any integrable stochastic sequence and that s and t are any sampling variables such that $E(y_s)$ and $E(y_t)$ exist.

LEMMA 1. *If for each* n,

$$E(y_s|\mathfrak{F}_n) \geqq y_n \qquad \text{if} \quad s > n, \tag{20}$$

and

$$E(y_t|\mathfrak{F}_n) \leqq y_n \qquad \text{if} \quad s = n,\ t > n, \tag{21}$$

then

$$E(y_s) \geqq E(y_t). \tag{22}$$

Conversely, if $E(y_s)$ *is finite and* (22) *holds for every* t, *then* (20) *and* (21) *hold for every* t.

PROOF.

$$E(y_s) = \sum_{n=1}^{\infty} \int_{\{s=n,\, t \leqq n\}} y_s\, dP + \sum_{n=1}^{\infty} \int_{\{s=n,\, t>n\}} y_n\, dP \tag{23}$$

$$= \sum_{n=1}^{\infty} \int_{\{s \geqq n,\, t=n\}} y_s\, dP + \sum_{n=1}^{\infty} \int_{\{s=n,\, t>n\}} y_n\, dP$$

$$\geqq \sum_{n=1}^{\infty} \int_{\{s \geqq n,\, t=n\}} y_n\, dP + \sum_{n=1}^{\infty} \int_{\{s=n,\, t>n\}} y_t\, dP$$

$$= E(y_t).$$

To prove the converse, for a fixed n let

$$V = \{s > n \quad \text{and} \quad E(y_s|\mathfrak{F}_n) < y_n\}; \tag{24}$$

then $V \in \mathfrak{F}_n$. Define

$$t' = \begin{cases} s, & \omega \notin V, \\ n, & \omega \in V. \end{cases} \tag{25}$$

Then t' is a sampling variable. Since $E(y_s)$ is finite, by (22) $E(y_n) < \infty$ and then $E(y_{t'})$ exists. But

$$\begin{aligned} E(y_{t'}) &= \int_{\{t'=s\}} y_{t'}\, dP + \int_V y_{t'}\, dP = \int_{\{t'=s\}} y_s\, dP + \int_V y_n\, dP \\ &\geqq \int_{\{t'=s\}} y_s\, dP + \int_V y_s\, dP = E(y_s). \end{aligned} \tag{26}$$

But by (22), $E(y_{t'}) \leqq E(y_s)$. Hence

$$\int_V y_n\, dP = \int_V y_s\, dP \tag{27}$$

and therefore $P(V) = 0$, which proves (20). To prove (21) let

$$V = \{s = n, t > n, \quad \text{and} \quad E(y_t|\mathfrak{F}_n) > y_n\}, \tag{28}$$

and define

$$t' = \begin{cases} s, & \omega \notin V, \\ t, & \omega \in V. \end{cases} \tag{29}$$

Then

$$\begin{aligned} E(y_{t'}) &= \int_{\{t'=s\}} y_{t'}\, dP + \int_V y_{t'}\, dP = \int_{\{t'=s\}} y_s\, dP + \int_V y_t\, dP \\ &\geqq \int_{\{t'=s\}} y_s\, dP + \int_V y_n\, dP = \int_{\{t'=s\}} y_s\, dP + \int_V y_s\, dP = E(y_s), \end{aligned} \tag{30}$$

and again $P(V) = 0$, which proves (21).

LEMMA 2. *If for each n,*

$$E(y_{n+1}|\mathfrak{F}_n) \geqq y_n, \qquad s > n, \tag{31}$$

and if

$$\liminf_n \int_{\{s>n\}} y_n^+\, dP = 0, \tag{32}$$

then for each n,

$$E(y_s|\mathfrak{F}_n) \geqq y_n, \qquad s \geqq n. \tag{33}$$

PROOF. (compare [2], p. 310). Let $V \in \mathfrak{F}_n$ and $U = V\{s \geqq n\}$. Then

$$(34)\qquad \begin{aligned}\int_U y_n\,dP &= \int_{V\{s=n\}} y_n\,dP + \int_{V\{s>n\}} y_n\,dP\\ &\leqq \int_{V\{s=n\}} y_n\,dP + \int_{V\{s>n\}} y_{n+1}\,dP\\ &= \int_{V\{n\leqq s\leqq n+1\}} y_s\,dP + \int_{V\{s>n+1\}} y_{n+1}\,dP\\ &\leqq \cdots \leqq \int_{V\{n\leqq s\leqq n+r\}} y_s\,dP + \int_{V\{s>n+r\}} y_{n+r}\,dP\\ &\leqq \int_{V\{n\leqq s\leqq n+r\}} y_s\,dP + \int_{\{s>n+r\}} y_{n+r}^+\,dP.\end{aligned}$$

Therefore

$$(35)\qquad \int_U y_n\,dP \leqq \int_{V\{s\geqq n\}} y_s\,dP + \liminf_n \int_{\{s>n\}} y_n^+\,dP = \int_U y_s\,dP,$$

which is equivalent to (33).

LEMMA 3. *If for each n,*

$$(36)\qquad E(y_{n+1}|\mathfrak{F}_n) \leqq y_n, \qquad s \leqq n,$$

and if

$$(37)\qquad \liminf_n \int_{\{t>n\}} y_n^-\,dP = 0,$$

then

$$(38)\qquad E(y_t|\mathfrak{F}_n) \leqq y_n, \qquad s = n, \quad t \geqq n.$$

PROOF. Let $V \in \mathfrak{F}_n$ and $U = V\{s = n, t \geqq n\}$. Then

$$(39)\qquad \begin{aligned}\int_U y_n\,dP &= \int_{V\{s=n,\,t=n\}} y_n\,dP + \int_{V\{s=n,\,t>n\}} y_n\,dP\\ &\geqq \int_{V\{s=n,\,t=n\}} y_n\,dP + \int_{V\{s=n,\,t>n\}} y_{n+1}\,dP\\ &= \int_{V\{s=n,\,n\leqq t\leqq n+1\}} y_t\,dP + \int_{V\{s=n,\,t>n+1\}} y_{n+1}\,dP\\ &\geqq \cdots \geqq \int_{V\{s=n,\,n\leqq t\leqq n+r\}} y_t\,dP + \int_{V\{s=n,\,t>n+r\}} y_{n+r}\,dP\\ &\geqq \int_{V\{s=n,\,n\leqq t\leqq n+r\}} y_t\,dP - \int_{\{t>n+r\}} y_{n+r}^-\,dP.\end{aligned}$$

Therefore

(40) $$\int_U y_n\, dP \geqq \int_{V\{s=n,\, t\geqq n\}} y_t\, dP - \liminf_n \int_{\{t>n\}} y_n^-\, dP = \int_U y_t\, dP,$$

which is equivalent to (38).

We can now state the main result of the present paper.

THEOREM 1. *Let $(y_n, \mathfrak{F}_n)$ be an integrable stochastic sequence in the monotone case and let s be the conservative sampling variable*

(41) $$s = \text{least positive integer } j \text{ such that } E(y_{j+1}|\mathfrak{F}_j) \leqq y_j.$$

Suppose that $E(y_s)$ exists and that

(42) $$\liminf_n \int_{\{s>n\}} y_n^+\, dP = 0.$$

If t is any sampling variable such that $E(y_t)$ exists and

(43) $$\liminf_n \int_{\{t>n\}} y_n^-\, dP = 0,$$

then

(44) $$E(y_s) \geqq E(y_t).$$

PROOF. From lemmas 1, 2, and 3 and relations (5).

We shall now establish a lemma (see [2], p. 303) which provides sufficient conditions for (42) and (43).

LEMMA 4. *Let $(y_n, \mathfrak{F}_n)$ be a stochastic sequence such that $E(y_n^+) < \infty$ for each $n \geqq 1$, and let s be any sampling variable. If there exists a nonnegative random variable u such that*

(45) $$E(su) < \infty,$$

and if

(46) $$E[(y_{n+1} - y_n)^+|\mathfrak{F}_n] \leqq u, \qquad s > n,$$

then

(47) $$E(y_s^+) < \infty, \qquad \lim_n \int_{\{s>n\}} y_n^+\, dP = 0.$$

PROOF. Define

(48) $$z_1 = y_1^+, \quad z_{n+1} = (y_{n+1} - y_n)^+ \text{ for } n \geqq 1, \quad w_n = z_1 + \cdots + z_n.$$

Then

(49) $$y_n^+ \leqq w_n$$

(and hence $y_n^+ \leqq w_s$ if $s \geqq n$), and by (46)

(50) $$E(z_{n+1}|\mathfrak{F}_n) \leqq u, \qquad s > n.$$

Hence

$$
\begin{aligned}
(51)\qquad E(y_s^+) &\leqq E(w_s) = \sum_{n=1}^{\infty} \int_{\{s=n\}} w_n \, dP = \sum_{n=1}^{\infty} \sum_{j=1}^{n} \int_{\{s=n\}} z_j \, dP \\
&= \sum_{j=1}^{\infty} \sum_{n=j}^{\infty} \int_{\{s=n\}} z_j \, dP = \sum_{j=1}^{\infty} \int_{\{s>j-1\}} z_j \, dP \\
&= E(y_1^+) + \sum_{j=2}^{\infty} \int_{\{s>j-1\}} E(z_j|\mathfrak{F}_{j-1}) \, dP \\
&\leqq E(y_1^+) + \sum_{j=2}^{\infty} \int_{\{s>j-1\}} u \, dP = E(y_1^+) + \sum_{j=2}^{\infty} \sum_{n=j}^{\infty} \int_{\{s=n\}} u \, dP \\
&= E(y_1^+) + \sum_{n=2}^{\infty} \int_{\{s=n\}} (n-1)u \, dP = E(y_1^+) + E(su) - E(u) \\
&\leqq E(y_1^+) + E(su) < \infty,
\end{aligned}
$$

and hence from (49)

$$
(52)\qquad \lim_{n} \int_{\{s>n\}} y_n^+ \, dP \leqq \lim_{n} \int_{\{s>n\}} w_s \, dP = 0.
$$

REMARK. Lemma 4 remains valid if we replace a^+ by a^- or by $|a|$ throughout.

4. Application to the sequential decision problem of section 2

Recalling the problem of section 2, let $x, x_1, x_2, \cdots$ be independent and identically distributed random variables with $E(x^+) < \infty$, $\mathfrak{F}_n$ the σ-algebra generated by $x_1, \cdots, x_n$, $g(n) \geqq 0$, $f(n) = g(n+1) - g(n) > 0$ and nondecreasing, $m_n = \max(x_1, \cdots, x_n)$, and $y_n = m_n - g(n)$. The constants α_n are defined by

$$
(53)\qquad E[(x - \alpha_n)^+] = f(n)
$$

and are nonincreasing; we are in the monotone case, and the conservative sampling variable s is the first $j \geqq 1$ such that $m_j \geqq \alpha_j$; thus

$$
(54)\qquad \{s > n\} = \{m_n < \alpha_n\}.
$$

We have shown in section 2 that

$$
(55)\qquad P\{s < \infty\} = 1, \qquad E(s^r) < \infty \quad \textit{for} \quad r \geqq 0.
$$

We wish to apply theorem 1. As concerns s it will suffice to show that $E(y_s^+) < \infty$ and that

$$
(56)\qquad \lim_{n} \int_{\{s>n\}} y_n^+ \, dP = 0,
$$

which we shall do by using lemma 4. Let

$$
(57)\qquad Y_n = m_n^+ - g(n).
$$

Then

$$Y_n^+ = y_n^+, \quad E(Y_n^+) = E(y_n^+) \leqq E(x_1^+ + \cdots + x_n^+) = nE(x^+) < \infty, \tag{58}$$

and

$$\begin{aligned} E[(Y_{n+1} - Y_n)^+|\mathfrak{F}_n] &= E\{[m_{n+1}^+ - m_n^+ - f(n)]^+|\mathfrak{F}_n\} \\ &\leqq E[(m_{n+1}^+ - m_n^+)|\mathfrak{F}_n] \leqq E(x_{n+1}^+|\mathfrak{F}_n) \\ &= E(x^+) < \infty. \end{aligned} \tag{59}$$

Hence by lemma 4, setting $u = E(x^+)$,

$$E(y_s^+) = E(Y_s^+) < \infty \tag{60}$$

and

$$\lim_n \int_{\{s>n\}} y_n^+ \, dP = \lim_n \int_{\{s>n\}} Y_n^+ \, dP = 0, \tag{61}$$

which were to be proved.

To establish the conditions on t of theorem 1 we assume that $Ex^- < \infty$; then since $y_n^- \leqq x_1^- + g(n)$ it follows that $E(y_n^-) < \infty$. Define a random variable u by setting

$$u(\omega) = f(n) \quad \text{if} \quad t(\omega) = n. \tag{62}$$

Since

$$(y_{n+1} - y_n)^- \leqq f(n) \tag{63}$$

and $f(n)$ is nondecreasing, it follows that

$$E[(y_{n+1} - y_n)^-|\mathfrak{F}_n] \leqq u \quad \text{if} \quad t \geqq n. \tag{64}$$

We now assume that $f(n) \leqq h(n)$, where $h(n)$ is a polynomial of degree $r \geqq 0$, and that $E(t^{r+1}) < \infty$. Then

$$E(tu) = \sum_{n=1}^{\infty} \int_{\{t=n\}} nf(n) \, dP \leqq \sum_{n=1}^{\infty} nh(n)P\{t = n\}. \tag{65}$$

Since

$$E(t^{r+1}) = \sum_{n=1}^{\infty} n^{r+1}P\{t = n\} < \infty, \tag{66}$$

it follows that $E(tu) < \infty$. Then by the remark following lemma 4,

$$E(y_t^-) < \infty \quad \text{and} \quad \lim_n \int_{\{t>n\}} y_n^- \, dP = 0, \tag{67}$$

and all the conditions of theorem 1 are established. Thus we have proved

THEOREM 2. *Suppose that $E|x| < \infty$ and that in addition to the conditions on $g(n)$ in the first paragraph of this section we have $f(n) \leqq h(n)$, where $h(n)$ is a polynomial of degree $r \geqq 0$. If t is any sampling variable for which $E(t^{r+1}) < \infty$ then $-\infty < E(y_t) \leqq E(y_s) < \infty$, where s is the conservative sampling variable de*fined by (54).

If $g(n) = nc$ then $f(n) = c$ and we can take $r = 0$. Hence

COROLLARY 1. *If $E|x| < \infty$ and $y_n = m_n - cn$, $0 < c < \infty$, then if t is any sampling variable for which $E(t) < \infty$, $E(y_t) \leqq E(y_s) = \alpha$ [see (19)], where α is defined by $E(x - \alpha)^+ = c$ and s = the first $j \geqq 1$ such that $x_j \geqq \alpha$. Thus s is optimal in the class of all sampling variables with finite expectations.*

To replace the condition $E(t^{r+1}) < \infty$ in theorem 2 and corollary 1 by conditions on y_t we require the following theorem which is of interest in itself. We omit the proof.

THEOREM 3. *Let $F(u)$ be a distribution function. Define $G(u) = \prod_{n=1}^{\infty} F(u + n)$. Then $G(u)$ is a distribution function if and only if*

$$\int_0^{\infty} u\, dF(u) < \infty, \tag{68}$$

and for any integer $b \geqq 1$,

$$\int_0^{\infty} u^b\, dG(u) < \infty \tag{69}$$

if and only if

$$\int_0^{\infty} u^{b+1}\, dF(u) < \infty. \tag{70}$$

COROLLARY 2. *If $y_n = m_n - cn$, $0 < c < \infty$, and b is any integer $\geqq 1$, then*

$$E(\sup_{n \geqq 1} y_n^+)^b < \infty \tag{71}$$

if and only if

$$E(x^+)^{b+1} < \infty. \tag{72}$$

PROOF. We can assume $c = 1$. Define

$$G(u) = P\{\sup_{n \geqq 1} y_n^+ \leqq u\}. \tag{73}$$

Then for $u \geqq 0$,

$$\begin{aligned} G(u) &= P\{x_1 \leqq u + 1, x_2 \leqq u + 2, \cdots, x_n \leqq u + n, \cdots\} \\ &= \prod_{n=1}^{\infty} F(u + n). \end{aligned} \tag{74}$$

By theorem 3,

$$E(\sup_{n \geqq 1} y_n^+)^b = \int_0^{\infty} u^b\, dG(u) < \infty \tag{75}$$

if and only if

$$\int_0^{\infty} u^{b+1}\, dF(u) = E(x^+)^{b+1} < \infty. \tag{76}$$

THEOREM 4. *Assume $E|x| < \infty$, $E(x^+)^2 < \infty$. If $y_n = m_n - g(n)$ where $g(n)$ is a polynomial of degree $r \geqq 1$ such that*

$$g(1) > 0, \tag{77}$$

$g(n+1) - g(n)$ is positive and nondecreasing, then for any sampling variable t,

$$E(y_t) \leqq E(y_s), \tag{78}$$

where s is the conservative sampling variable defined by (54).

PROOF. By theorem 2, if $E(t^r) < \infty$ then (78) holds. Hence we can assume that $E(t^r) = \infty$. Now

$$\begin{aligned} & g(1) > 0, && f(1) = g(2) - g(1) > 0, \\ & g(2) \geqq g(1) + f(1), && g(3) - g(2) \geqq f(1), \\ & g(3) \geqq g(1) + 2f(1), && \\ & \cdots\cdots && \\ & g(n) \geqq g(1) + (n-1)f(1). && \end{aligned} \tag{79}$$

Let

$$a = \frac{1}{2} \min [g(1), f(1)] > 0. \tag{80}$$

Then by (79),

$$g(n) \geqq an \quad \text{for} \quad n \geqq 1. \tag{81}$$

Let

$$\tilde{y}_n = m_n - \frac{a}{2} n. \tag{82}$$

By corollary 2, $E(\tilde{y}_t^+) < \infty$. Then since

$$y_t = \tilde{y}_t + \frac{a}{2} t - g(t) \leqq \tilde{y}_t^+ - \frac{1}{2} g(t) \tag{83}$$

we have

$$E(y_t) \leqq E(\tilde{y}_t^+) - \frac{1}{2} E[g(t)] = -\infty, \tag{84}$$

so that (78) holds in this case too.

REMARK. If in the case $g(n) = cn$ we define $\bar{y}_n = x_n - cn$, then

$$\bar{y}_n \leqq y_n, \qquad \bar{y}_s = y_s. \tag{85}$$

Hence for any sampling variable t,

$$E(\bar{y}_t) \leqq E(y_t) \leqq E(y_s) = E(\bar{y}_s), \tag{86}$$

so that s is also optimal for the stochastic sequence $(\bar{y}_n, \mathfrak{F}_n)$.

5. A result of Snell

As an application of lemmas 1 and 2, we are going to obtain Snell's result on sequential game theory [3].

LEMMA 5 (*Snell*). *Let $(y_n, \mathfrak{F}_n)$ be a stochastic sequence satisfying $y_n \geqq u$ for* *each n with $E|u| < \infty$. Then there exists a semimartingale $(x_n, \mathfrak{F}_n)$ such that for every sampling variable t and every n,*

$$E(x_t|\mathfrak{F}_n| \geqq x_n \quad \text{if} \quad t \geqq n, \qquad x_n \geqq E(u|\mathfrak{F}_n), \tag{87}$$

$$x_n = \min [y_n, E(x_{n+1}|\mathfrak{F}_n)], \tag{88}$$

and

$$\liminf x_n = \liminf y_n. \tag{89}$$

We will assume the validity of this lemma, and prove the following theorem by applying lemmas 1 and 2.

THEOREM 5 (*Snell*). *Let $(y_n, \mathfrak{F}_n)$ and $(x_n, \mathfrak{F}_n)$ satisfy the conditions of lemma 5. For $\epsilon \geqq 0$ define $s = j$ to be the first $j \geqq 1$ such that $x_j \geqq y_j - \epsilon$. If $\epsilon > 0$, then*

$$E(y_s) \leqq E(y_t) + \epsilon \tag{90}$$

for every sampling variable t. If $\epsilon = 0$ and if $P\{s < \infty\} = 1$, then (90) still holds.

PROOF. It is obvious that in both cases s is a sampling variable. We need to verify that $P\{s < \infty\} = 1$. If $\epsilon > 0$, by (89) this is true.

Since $(x_n, \mathfrak{F}_n)$ is a semimartingale,

$$E(x_{n+1}|\mathfrak{F}_n) \geqq x_n. \tag{91}$$

By (88) and the definition of s,

$$E(x_{n+1}|\mathfrak{F}_n) = x_n \quad \text{for} \quad s > n. \tag{92}$$

Since $-x_n \leqq E(-u|\mathfrak{F}_n)$ and $E|u| < \infty$, by lemma 2 and (92), we have

$$E(x_s|\mathfrak{F}_n) \leqq x_n \quad \text{for} \quad s > n. \tag{93}$$

By (87), (93), and lemma 1, we obtain $E(x_s) \leqq E(x_t)$, and therefore, by definition of s,

$$E(y_s) \leqq E(x_s) + \epsilon \leqq E(x_t) + \epsilon \leqq E(y_t) + \epsilon. \tag{94}$$

Thus the proof is complete.

J. MacQueen and R. G. Miller, Jr., in a recent paper [1], treat the problem of section 2 by completely different methods. Reference should also be made to a paper by C. Derman and J. Sacks [4], in which the formulation and results are very similar to those of the present paper.

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