A MATHEMATICAL FORMULATION OF VARIATIONAL PROCESSES OF ADAPTIVE TYPE

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1. Summary

The questions we shall discuss in what follows belong to two fields which formerly were quite disjoint, the classical theory of probability and the classical calculus of variations. That there is now considerable overlap is due to the rise in scientific interest in the field of control processes. Although it is only within the last few years that the theory of feedback control has penetrated the academic curriculum and become a respectable member of the mathematical community, the conventional formulation is already far outmoded. In order to treat current and future problems of any significance, it is absolutely essential to introduce stochastic elements. These, however, enter in entirely novel ways, not in the fairly well understood fashion of conventional stochastic processes, but in connection with "learning processes" (compare [2]), or, as we shall henceforth say, *adaptive processes*.

In what follows we show how the functional equation technique of dynamic programming can be used to treat adaptive control processes, and how continuous processes can be defined in terms of the discrete versions.

2. Introduction

In order to prepare a suitable background for the introduction of the new features, let us review the elementary ideas of feedback control processes. We are, of course, here interested only in the mathematical presentation of these concepts, and shall ignore any of the difficulties of engineering or statistical application.

One version of the feedback control problem is that of maximizing a functional of the form

$$(2.1) \qquad J(y) = \int_0^T F(x, y) \, dt$$

over all functions $y(t)$, where x and y are connected by means of a differential equation

$$(2.2) \qquad \frac{dx}{dt} = G(x, y), \qquad x(0) = c,$$

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and y may be subject to further constraints which, in general, will depend upon $x(t)$.

Although problems of this genre appear to belong in a very natural way to the calculus of variations, and thus to be susceptible to classical techniques, they are often more advantageously treated by means of the theory of dynamic programming [1], [2]. It turns out to be convenient from many points of view, conceptual, analytic, and computational, to consider a discrete version of the foregoing problem.

Let us agree to maximize the function

$$(2.3) \qquad J_N(y) = \sum_{k=0}^{N} F(x_k, y_k),$$

where

$$(2.4) \qquad x_{k+1} = x_k + G(x_k, y_k), \qquad\qquad x_0 = c,$$

over the set of y: $y_1, y_2, \cdots, y_N$, with, as above, possibly some constraints present. Not only are problems posed in this form much more amenable to the application of digital computers, but, what is often forgotten, they frequently represent more realistic descriptions of the original physical process.

So far, everything has been very deterministic. Let us now introduce stochastic elements. In place of the transformation of (2.2), let us suppose that x_{k+1} is obtained by means of a stochastic transformation

$$(2.5) \qquad x_{k+1} = x_k + G(x_k, y_k, r_k), \qquad\qquad x_0 = c.$$

In place of the original maximization problem, let us consider the problem of maximizing the expected value of the function

$$(2.6) \qquad J_N(y) = \sum_{k=0}^{N} F(x_k, y_k, r_k).$$

At the moment, we take the r_k to represent a sequence of independent random variables, and the y_k are to be chosen in feedback fashion. By this we mean that y_k is chosen with knowledge of $x_0, x_1, \cdots, x_{k-1}, y_0, y_1, \cdots, y_{k-1}, r_0, r_1, \cdots, r_{k-1}$, but not of r_k, nor of any of the following x, y, or r.

In [3] we discussed in some detail the use of the functional equation techniques of dynamic programming to treat optimization problems of this nature. Our emphasis there was upon the use of discrete processes to lay a foundation for the rigorous formulation of continuous processes.

In this paper we wish to discuss corresponding problems arising in cases in which the distribution functions for the r_k are only partially known. The problems we discuss here represent only a small part of the cornucopia of questions which the theory of feedback control thrusts upon us. In a series of papers with R. Kalaba [4]–[8], we have laid a foundation for the study of such questions.

3. Multistage decision processes

To treat the optimization problems described in the preceding paragraph, as well as those of more complex nature, we use the concept of a multistage

decision process. Let p be a point in a space S and $T(p, q)$ a set of transformations, defined for all $p \in S$ and $q \in S_1$, a second space, with the property that $T(p, q) \in S$ for all $p \in S$ and $q \in S_1$.

Starting with a point p_1, a choice of q_1 is made, leading to a new point $p_2 = T(p_1, q_1)$. Repeating the process, a choice of q_2 leads to a third point $p_3 = T(p_2, q_2)$, and so on. The set of q, that is, $[q_1, q_2, \cdots, q_N]$, is called a *policy*, and the process itself is called a multistage decision process.

Let us now suppose that the q_i are to be chosen so as to maximize a preassigned criterion function

$$(3.1) \qquad F(p_1, p_2, \cdots, p_N; q_1, q_2, \cdots, q_N).$$

A policy which maximizes is called an *optimal policy*.

Since the problem of determining optimal policies in this generality is much too difficult, let us restrict ourselves to the case where F is separable,

$$(3.2) \qquad F_N = R(p_1, q_1) + R(p_2, q_2) + \cdots + R(p_N, q_N).$$

Fortunately, in many significant applications, F can be taken to have this form.

The case where only the term $R(p_N, q_N)$ appears is called *terminal control* in engineering circles. If the number of stages, N, is itself a function of the sequence of states and decisions, we speak of an *implicit* variational problem.

The basic problem is that of determining optimal policies and the value of the maximum of F.

4. Functional equation approach

For a variety of reasons which we shall not enter into, conventional methods of calculus are seldom operative by themselves. Let us introduce the sequence of functions $\{f_N(p_1)\}$, defined by the relation

$$(4.1) \qquad f_N(p_1) = \max_{\{q\}} R(p_1, q_1) + R(p_2, q_2) + \cdots + R(p_N, q_N),$$

for $N = 1, 2, \cdots$, and $p_1 \in S$.

An application of the principle of optimality ([1], p. 83) or, in this case, some simple manipulation, yields the basic recurrence relation

$$(4.2) \qquad f_N(p_1) = \max_{q_1} R(p_1, q_1) + f_{N-1}[T(p_1, q_1)],$$

for $N = 2, 3, \cdots$, with

$$(4.3) \qquad f_1(p_1) = \max_{q_1} R(p_1, q_1).$$

These equations yield two sequences, the sequence of maxima, $\{f_N(p_1)\}$, and the sequence of policy functions, $\{q_N(p_1)\}$. The function $q_N(p_1)$ is the choice of q_1 which is made when the system is in state p_1 and there are N stages remaining.

5. Discussion

The usual approach to the foregoing maximization problem attempts to determine the set $[q_1, q_2, \cdots, q_N]$ at one time, using variational techniques. In

place of this, we determine q_1 in terms of p_1 and N, then q_2 in terms of p_2 and $N - 1$, and so on. This is *feedback control*. We determine the "control vector" q_i in terms of the current state of the system, p_i, and the duration of the process, $N - i$.

For deterministic processes, the two approaches are equivalent. For stochastic processes, they diverge rapidly. We shall pursue the "feedback" approach since it is both easier to follow and much the more important.

6. Stochastic multistage decision processes

Let us now suppose that a choice of q_1 in state p_1 yields a state $p_2 = T(p_1, q_1, r_1)$, where r_1 is a random vector with a given distribution function $dG(r_1)$. As above, we assume that $T(p_1, q_1, r_1) \in S$ for $p_1 \in S$, $q_1 \in S'$ and r_1 chosen from $dG(r_1)$.

In place of the maximization problem in section 3, we consider the problem of maximizing the expected value of

$$(6.1) \qquad F_N = R(p_1, q_1, r_1) + R(p_2, q_2, r_2) + \cdots + R(p_N, q_N, r_N),$$

over all feedback policies $[q_1, q_2, \cdots, q_N]$. By this we mean that q_1 is chosen with a knowledge of p_1. After q_1 is determined, r_1 is obtained from $dG(r_1)$, giving rise to p_2. Then q_2 is selected, with knowledge of p_2, r_2 is obtained from $dG(r_2)$, yielding p_3, and so on.

Introducing the sequence of functions

$$(6.2) \qquad f_N(p_1) = \max_{\{q\}} \underset{\{r\}}{E} (F_N),$$

$N = 1, 2, \cdots$, we see that

$$(6.3) \qquad f_1(p_1) = \max_{q_1} \int R(p_1, q_1, r_1) \, dG(r_1),$$

and, as above,

$$(6.4) \qquad f_N(p_1) = \max_{q_1} \int \{R(p_1, q_1, r_1) + f_{N-1}[T(p_1, q_1, r_1)]\} \, dG(r_1)$$

for $N = 2, 3, \cdots$.

We see then that stochastic processes of this type can be treated in very much the same fashion as the deterministic processes discussed earlier.

7. Prediction and information theory

Let us note in passing that these techniques can be used to provide new approaches to prediction and information theory, and extensions of the previous results. For prediction theory, see Kalman [9], and Bellman [10]; for information theory, see Bellman and Kalaba [7], [11], and Marschak [12].

8. Adaptive processes

We now wish to consider processes in which not enough is known to use a formulation of the type given above. There are many ways of treating processes

of this type, and it is never clear as to which is the proper way of doing this. Nor is it clear that this adjective "proper" has any meaning in this context.

It must be recognized, however ruefully or regretfully, that no definitive theory of uncertainty can ever exist. The theories that are used will depend upon the applications that are made and the personal philosophy of the user.

We are thinking of processes in which

(a) cause and effect may not be known;

(b) the state of the system at any time may not be known;

(c) the range of decisions may not be known;

(d) the utility functions, for example, $R(p, q)$, may not be known;

(e) the duration of the process may not be known;

(f) it may not be known whether deterministic or stochastic influences are paramount, or whether the process is a one-person or multi-person process.

These are not problems which conventional mathematical techniques are designed to treat. We propose to show how they can be precisely formulated and treated analytically by means of the foregoing mathematical apparatus, the functional equation approach of dynamic programming. For some other approaches which appear promising, see Robbins [13], Box [14].

9. Information pattern

In treating processes involving uncertainty, our hope is that the multistage nature of the situation will enable us to reduce the level of uncertainty stage by stage. This idea leads to some interesting ideas concerning asymptotic behavior which we shall discuss below.

It is not to be expected that in all cases a simplification will ensue as the process continues. There is little difficulty in displaying processes which complicate to an extraordinary degree as additional information is obtained.

Without worrying about such matters, let us formulate an important type of adaptive process. We follow the brief sketch given in [8]. Let the state of the system S be specified, as usual, by a point p in phase space, and by an *information pattern* P. This information pattern represents the information about the process that we retain in order to determine some of the properties of the decision process which are initially unknown. In our case, let us assume that only the distribution function for r is unknown. The simplest information pattern that one can think of in this case is the entire previous history of the process. Generally, one can do much better than this and substantially compress the vast amount of data.

The state of the system is now specified by a point in an extended phase space, $[p, P]$. A choice of a decision vector q results in a transformation of p into $T_1(p, P; q, r)$, and P into $T_2(p, P; q, r)$. Here r is a random vector variable, specified by an a priori probability distribution $dG(p, P; q, r)$, itself a part of the information pattern P.

Let us suppose, for the sake of simplicity, that the new state p_1 is known

after the decision q_1 has been made. Let the a priori single stage return be $\phi[T_1(p, P; q, r), T_2(p, P; q, r)]$. Then, introducing the function

$$(9.1) \qquad f_N(p, P) = \min E[\phi(p_N, P_N)],$$

where, as in the preceding cases, the minimum is taken over feedback control policies, we have the functional equations

$$(9.2) \qquad f_N(p, P) = \min_q \int f_{N-1}[T_1(p, P; q, r), T_2(p, P; q, r)] \, dG(p, P; q, r),$$

for $N = 2, 3, \cdots$, with

$$(9.3) \qquad f_1(p, P) = \min_q \int \phi[T_1(p, P; q, r), T_2(p, P; q, r)] \, dG(p, P; q, r).$$

These relations can be used to establish the existence of optimal policies and to study further properties of the multistage process. In particular, as we shall discuss below, they can be used as a basis for the construction of a theory of continuous processes.

10. Sequential machines, coin weighing, and search processes

The further study of information patterns inevitably leads to a consideration of sequential machines and search processes in general. As an illustration of the way in which the information can become complicated in an extraordinary fashion as a process continues, consider the well-known puzzle of locating a defective coin in a batch of N coins, given an equal arm balance, and its extensions.

The initial information is that a batch of N coins contains one defective coin. After a weighing, involving the comparison of two groups of k coins chosen from this original set of N coins, we know that the defective coin is in one of these two sets of k coins, or in the remaining batch of untested $N - 2k$ coins. Thus the form of the information remains constant with each succeeding test, or stage of the process, and simplifies to the point where we can eliminate the original uncertainty.

Consider what happens, however, when we start with the knowledge that there are two defective coins. Comparing two sets of k coins each, we are led to the following possibilities:

(a) If the scale balances, there is either one defective coin in each of the k-sets, or none, which means that there are two defectives in the remaining $N - 2k$ coins.

(b) If the scale unbalances, there is either one defective coin or two defective coins in one of the k-sets, and either one or none left in the remaining $N - 2k$ coins.

It is easy to see that as the testing process continues, the information pattern increases in size and in complexity. If we allow perfectly general testing policies

which admit the mixing of different batches, it appears to be hopeless to attempt to keep track of the process.

Problems of this nature are of great practical importance and extremely difficult to handle by means of analytic techniques. For some preliminary work on the foregoing problem by means of functional equation techniques, see Bellman [15], Bellman and Gluss [16], Cairns [17]. For a discussion of problems of related nature, see M. Sobel and P. A. Groll [18], R. Dorfman [19], and P. Ungar [20].

11. Continuous adaptive processes

A simple and conceptually important way to found a theory of continuous processes of any type is by a passage to the limit in a theory of discrete processes. In some situations it is not difficult to construct a theory of the continuous process directly. In these cases it is essential to establish the equivalence of the two approaches. Many theorems of this type exist in connection with the study of differential and difference equations, in the field of partial differential and difference equations, and in the theory of probability.

In some fields, only recently developed, the continuous theories do not exist and seem quite difficult to formulate. For these, a passage from the discrete to the continuous seems to be the easiest and safest approach.

One advantage of using the passage to the limit approach lies in the fact that we can in many cases establish the existence of a limiting continuous process under conditions which are far weaker than those necessary to impose in order to guarantee the existence of a continuous process constructed directly.

In order to illustrate these comments which at first may not seem reasonable, let us consider a problem in the calculus of variations.

Suppose that we wish to determine the minimum of the functional

$$(11.1) \qquad J(u) = \int_0^T g(u, u') \, dt$$

over all functions $u(t)$ satisfying the initial condition $u(0) = c$. This problem is far more complex than it may seem at first glance. In the first place, to assure that an actual minimum, rather than an infimum, exists, strong conditions must be imposed upon the function $g(u, u')$. Secondly, the standard variational technique, which leads to the Euler equation, possesses many drawbacks; see [2] for a detailed discussion.

We have then a situation in which it is not easy to establish the existence of a solution, and not easy to obtain the solution once the existence has been established.

Consider, however, a discrete version of the foregoing problem. Suppose that we wish to minimize

$$(11.2) \qquad J_N = \sum_{k=0}^{N} g(u_k, v_k)\Delta, \qquad\qquad u_0 = c,$$

where $u_{k+1} = u_k + v_k\Delta$. Very mild conditions upon the function g will enable us to assert the existence of an attained minimum. Furthermore, if we allow u and v to assume only a finite set of values, all we ask is that the function $g(u, v)$ be defined for the allowable values of u and v.

The recurrence relation

$$(11.3) \qquad f_N(c) = \min_v g(c, v)\Delta + f_{N-1}(c + v\Delta)$$

yields a constructive way of obtaining the desired minimum value.

The question naturally arises as to the relation between the discrete and continuous versions of this multistage decision process. We suspect that as $\Delta \to 0$ the discrete process will converge to the continuous process, if we impose sufficient regularity conditions upon $g(u, u')$.

This is indeed the case. It is quite easy to show that the conditions usually imposed upon $g(u, u')$ to guarantee the existence of a solution are strong enough to yield the desired limiting behavior. See the proof by Fleming in [1].

The more interesting problem is to determine conditions upon $g(u, u')$ which will guarantee that the limit of the discrete process exists as $\Delta \to 0$. We can then define a continuous process, not directly by way of (11.1), but in this fashion.

It turns out that this program can be carried out. In [21] it was shown that using only the recurrence relation of (11.3) and imposing upon $g(u, u')$ conditions which are far weaker than those required in the classical theory, the existence of a limit for $f_N(c)$ as $\Delta \to 0$ can be established. It follows that we have a concept of a continuous variational process which generalizes that of the classical version.

Perhaps the most important aspect of this approach is that it enables us to introduce the idea of a continuous process in situations in which no classical theory exists. In [3] we discussed this for stochastic variational processes. It is clear that we can in a similar fashion build upon the formulation of discrete adaptive processes we have given in the preceding pages to formulate a theory of continuous variational processes of adaptive type. Similarly, we can construct a theory of continuous multistage games, of ordinary stochastic or adaptive type. See [1] for the formulation of the discrete multistage version.

The problems of convergence of the return function and of the optimal policies are quite complex. They require a blend of classical analysis and probability theory which has not heretofore existed.

12. Reduction of dimensionality and sufficient statistics

The functional equations we derived to treat adaptive control processes involve, in many cases, functions of functions. Although these functions can be used to establish the existence of optimal policies, they are not well suited to analytic investigation nor to computational work.

In order to obtain analytic and numerical results, it is essential that we reduce these functions of functions to ordinary functions. In many cases, we can perform this reduction by using the concept of sufficient statistics. This idea enables us

to reduce the information pattern from a set of functions to an ordinary vector.

As an example of this, consider a process in which a certain random variable r assumes only the two values 0 and 1 with unknown probabilities of respectively $1 - p$ and p. After the process has continued for M stages, we have acquired an information pattern $[0, 0, 1, 1, 1, \cdots, 0, 1]$ consisting of the values assumed by r over the preceding M trials.

In place of this set of values which increases in size as the process continues, we can often use merely the number of 1 and the number of 0 which have been tabulated. In these cases, the order of occurrence is of no importance.

In place of a function $f_N(c; S)$, where $S = [0, 0, 1, 1, 1, \cdots, 0, 1]$, we will now have a function $f_N(c, m, n)$, where m is the number of 1 and n is the number of 0. See Bellman [22], Bellman and Kalaba [4], Freimer [23], [24], for applications of this idea. Clearly, this technique can be used in many ways.

One technique which has not been investigated as yet is that of "asymptotic sufficient statistics." Perhaps the best example of this is the central limit theorem. If the random variables x_i are drawn from an unknown distribution, and if it is desired to determine the distribution of $z_N = \sum_{i=1}^{N} x_i$, we know that for large N it is sufficient to tabulate merely the two sums $\sum_{i=1}^{N} x_i$, $\sum_{i=1}^{N} x_i^2$.

If, as in many cases, we are interested only in steady-state policies, which is to say asymptotic policies, results of this type will enable us to reduce the dimension of the problem greatly. Other techniques for reduction of dimensionality will be found in Bellman [25], Beckwith [26], Bellman and Kalaba [27], Bellman and Dreyfus [28].

13. Linear equations and quadratic criteria, I

In view of the analytic complexity of the general problem, and with application of the method of successive approximations in mind, it is worth while to consider processes governed by linear equations and quadratic criteria.

Let us consider first the scalar case. Write

$$(13.1) \qquad u_{n+1} = au_n + v_n + r_n, \qquad\qquad u_0 = c,$$

and suppose that the v_i are to be chosen so as to minimize the expected value of the quadratic form

$$(13.2) \qquad J_N = \sum_{n=0}^{N} (u_n^2 + \lambda v_n^2).$$

Consider first the stochastic case where the r_i are independent random variables with known distributions, which for simplicity of notation we shall take to be same.

Writing

$$(13.3) \qquad f_N(c) = \min_{v} E(J_N).$$

it is easy to see that

(13.4) $f_0(c) = c^2,$

and

(13.5) $f_N(c) = \min\limits_{v_0} [c^2 + \lambda v_0^2 + \underset{r_0}{E}\{f_{N-1}(ac + v_0 + r_0)\}]$

$= \min\limits_{v_0} [c^2 + \lambda v_0^2 + \int f_{N-1}(ac + v_0 + r) \, dG(r)].$

It is easy to show inductively that $f_N(c)$ is a quadratic in c, that is,

(13.6) $f_N(c) = u_N + v_N c + w_N c^2,$ $N = 0, 1, 2, \cdots,$

where u_N, v_N, and w_N are independent of c. Using this representation for the functions $f_N(c)$, we readily obtain representations for u_N, v_N, and w_N in terms of u_{N-1}, v_{N-1}, and w_{N-1}; see Bellman [29], Kramer [30], Beckwith [26], Freimer [23], Adorno [31].

These results can now be used for computational purposes and to study the asymptotic behavior of return functions and optimal policies as $N \to \infty$.

14. Linear equations and quadratic criteria, II

Let us now consider an adaptive version. Suppose that $\{r_n\}$ is a sequence of random variables with probability p of assuming the value 1 and $1 - p$ of 0. It is clear that we can use the idea of sufficient statistics. Let

(14.1) $f_N(c, m, n) = \min\limits_{v} \underset{r}{E}(J_N),$

where m of the 1-values and n of the 0-values have been observed for the r_i over the past $m + n$ stages.

Let $dG(p)$ be an initial a priori distribution for p and suppose that it is agreed to use the following transformations:

$$dG(p) \to \frac{p \, dG(p)}{\displaystyle\int_0^1 p \, dG(p)}, \qquad\qquad r_0 = 1,$$

(14.2)

$$dG(p) \to \frac{(1 - p) \, dG(p)}{\displaystyle\int_0^1 (1 - p) \, dG(p)}, \qquad\qquad r_0 = 0.$$

The result is that after $m + n$ trials with m of the 1 and n of the 0, we have as the new a priori distribution

(14.3) $dG_{m,n}(p) = \dfrac{p^m (1 - p)^n \, dG(p)}{\displaystyle\int_0^1 p^m (1 - p)^n \, dG(p)}.$

We use as an estimate for p for the next stage the value

(14.4) $p_{m,n} = \displaystyle\int_0^1 p \, dG_{m,n}(p) = \dfrac{\displaystyle\int_0^1 p^{m+1} (1 - p)^n \, dG(p)}{\displaystyle\int_0^1 p^m (1 - p)^n \, dG(p)}.$

Hence, the functional equation for $f_N(c, m, n)$ is

$$(14.5) \quad f_N(c, m, n) = \min_{v_0} \left[c^2 + \lambda v_0^2 + p_{m,n} f_{N-1}(ac + v_0 + 1) \right.$$
$$\left. + (1 - p_{m,n}) f_{N-1}(ac + v_0) \right].$$

As above, we can use the structural relation of (13.6) to simplify this relation.

15. Discussion

The problem of determining the asymptotic behavior of $f_N(c, m, n)$ as $N \to \infty$ has many complex features. It is to be expected that in one sense or another, $f_N(c, m, n) \sim f_N(c, p_0)$ as $N \to \infty$, where p_0 is the actual value of p. A particular version of this problem has been attacked by Adorno [31].

16. Open problems

In the foregoing sections we have indicated how a theory of adaptive control processes can be constructed. Associated with this approach, there are any number of analytic problems which we have explicitly or tacitly raised. Some of these are:

(a) Is the set of transformations in (14.2) the "best" way to modify a priori information?

(b) Is the estimate of (14.4) the "best" estimate for p?

(c) Asymptotically, does it make much difference what transformations we employ from stage to stage, and what a priori information we assume?

The analytic difficulties in this field are great, but the conceptual difficulties are greater. It seems reasonable to believe that there never will be definitive theories in this area, nor is it clear that the work "optimal" has an absolute meaning. We can summarize the situation simply by saying that all of the philosophical paradoxes of statistics and game theory are present, with their cousins and their sisters and their aunts.

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