SEQUENTIAL PROCEDURES FOR SELECTING THE BEST EXPONENTIAL POPULATION

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1. Introduction

A sequential procedure is considered for selecting that particular one of k exponential populations with the largest expected life (or the smallest failure rate). A. Birnbaum recently treated this problem [2], [3]. This paper extends his problem in two directions:

(i) The known guarantee period A_i, before which no items of the ith population fail, can be zero or of positive duration.

(ii) The number of populations k is assumed to be two or more.

In [3] it is assumed that $A_i = 0$ and $k = 2$. The case in which all populations have the same unknown guarantee period A is also treated. The effect of proceeding as if $A = 0$ when actually $A > 0$ is studied.

In life-testing experiments the problem is to utilize information provided by early failures to make decisions without waiting for all the units on test to fail. Frequently, it happens that even the expected lifetime of a single unit is longer than the experimenter is willing to wait before reaching a decision.

We shall assume in this paper that failure is a well defined and clearly recognizable phenomenon. We are interested in comparing the expected life (or the failure rate) of two or more populations. There are two classes of methods of reducing the time needed to reach a decision. One class consists of physical methods of accelerating the rate of failure (without introducing new causes of failure). The other class, with which this paper is concerned, consists of statistical methods.

One statistical method is to increase the initial number of units on test. Another is to replace each failure immediately by a new unit. A third is to use an appropriate *sequential* procedure to reach a decision. We shall consider some procedures which embody all three of the above features.

2. Definitions and assumptions

Suppose there are given k populations $\Pi_i (i = 1, 2, \cdots, k)$ such that the lifetimes of components taken from Π_i are distributed according to the delayed exponential density

$$f(x;\ A_i, \mu_i) = \frac{1}{\mu_i} e^{-(x-A_i)/\mu_i} \qquad \text{for } x \geqq A_i \tag{1}$$

(and $f = 0$ elsewhere), where the scale parameters $\mu_i > 0$ are assumed to be unknown, and also, since the location parameter A_i denotes a time delay, we shall assume that $A_i \geqq 0 (i = 1, 2, \cdots, k)$. Let the *ordered values* of the scale parameter μ be denoted by

$$\theta_1 \geqq \theta_2 \geqq \cdots \geqq \theta_k . \tag{2}$$

The goal of our proposed procedures is to select the population with the largest scale parameter, that is, with $\mu = \theta_1$. With regard to the location parameters A_i we shall consider two cases:

Case 1: The A_i are known. We shall be particularly interested in the special cases:

1*a*) The A_i have the common value $A > 0$.

1*b*) The A_i have the common value 0.

Case 2: The A_i have a common, unknown value $A \geqq 0$.

The special case in which the A_i have a *common* value A will be of particular importance throughout the paper. The case in which the A_i are different and unknown is not treated.

It will be assumed in this paper that

(1) Experimentation is started with the same number n of components from each of the k populations.

(2) All kn components are put on test at the same time (say at time $t_0 = 0$).

(3) Each failure from Π_i is immediately replaced by a new component from $\Pi_i (i = 1, 2, \cdots, k)$.

The above assumptions are made mostly for convenience of exposition and because they represent the method by which the test would most conveniently be carried out in practice.

It should be noted that, for any strictly monotonic function $\lambda = \lambda(\mu)$ of μ, the proposed procedures can also be used to select the population with the largest λ if $\lambda(\mu)$ is strictly increasing, or to select the population with the smallest λ if λ is strictly decreasing. For example, the expected component life for the ith population is $\mu_i + A_i$ and if there is a common A then the problem of selecting the population with the largest scale parameter is equivalent to the problem of selecting the population with the largest expected life. For another example, the equilibrium failure rate [defined below between (37) and (38)] for the ith population is given by

$$\lambda_i = \frac{n}{\mu_i + A_i} \qquad (i = 1, 2, \cdots, k) \tag{3}$$

and if there is a common A then the problem of selecting the population with the largest scale parameter is equivalent to the problem of selecting the population with the smallest λ-value.

At any time t, measured from t_0, each population has a number of failures associated with it. Let the *ordered values* of these integers be denoted by $r_i = r_i(t)$ so that

$$r_1 \leqq r_2 \leqq \cdots \leqq r_k . \tag{4}$$

Equal values (or, more precisely, the populations associated with equal values) are to be ranked by a random device which gives equal probability to each ordering. Then each subscript refers back to one of the k populations. For any component we shall denote the A-value as the *guarantee period.* The total amount of aging beyond the guarantee period will be designated as the *Poisson life.* Let the total Poisson life from the population with r_i failures be denoted by $L_i (i = 1, 2, \cdots, k)$. By definition we have $L_i \geqq 0 (i = 1, 2, \cdots, k)$.

3. Specifications for case 1*b*

It will be convenient to consider first the case 1*b*) in which the common A-value is zero. *Before experimentation starts,* the experimenter is asked to specify two constants,

P_0 and a_0 such that $\frac{1}{k} < P_0 < 1$ and $a_0 > 1$, which determine the proposed rule. For simplicity, we shall assume that $P_0 > \frac{1}{2}$ for any k. The resulting rule then has the property that *it will correctly select the population with the largest scale parameter with probability at least P_0 whenever*

$$\frac{\theta_1}{\theta_2} \geqq a_0. \tag{5}$$

Another related property is that after experimentation is completed one can make, at the $100P_0$ per cent confidence level, the *confidence statement*

$$\theta_s \leqq \theta_1 \leqq a_0\theta_s, \tag{6}$$

where θ_s is the scale parameter of the selected population.

If we let $\epsilon_0 = (1 - P_0)/P_0$ then the proposed sequential rule can be stated as follows:

Rule R_{1b}.—"Continue experimentation until the inequality

$$a_0^{-(r_2-r_1)} + a_0^{-(r_3-r_1)} + \cdots + a_0^{-(r_k-r_1)} \leqq \epsilon_0 \tag{7}$$

is satisfied. Then stop and select the population with the smallest number r_1 of failures as the one having the largest scale parameter."

The problem of two or more populations having the same value r_1 at stopping time is omitted above since, assuming $P_0 > \frac{1}{2}$, we have $\epsilon_0 < 1$ and hence from (7) no two populations can have the same value r_1 at stopping time. For $k = 2$ the inequality (7) reduces to the simple form

$$r_2 - r_1 \geqq s \tag{8}$$

where s is the smallest integer not less than $\log(1/\epsilon_0)/\log a_0$.

It should be noted that the above procedure stops only at a failure time, never between failures, since (7) depends on t only through the quantities $r_i(t)$.

4. Numerical illustrations

Suppose the preassigned constants are $P_0 = .95$ and $a_0 = 19^{1/4} = 2.088$ so that $\epsilon_0 = 1/19$. Then for $k = 2$ the procedure is to stop when $r_2 - r_1 \geqq 4$. For $k = 3$ it is easy to check that the procedure reduces to the simple form: "Stop when $r_2 - r_1 \geqq 5$." For $k > 3$ either calculations can be carried out as experimentation progresses or a table of stopping values can be constructed before experimentation starts. For $k = 4$ and $k = 5$ such tables are shown below.

TABLE I

TO CARRY OUT THE SEQUENTIAL RULE FOR $P_0 = .95$, $a_0 = 19^{1/4}$

($k = 4$)

r_2-r_1	r_3-r_1	r_4-r_1	
5	5	9	
5	6	6	
6	6	6	*

* See note to Table II, p. 102.

TABLE II

TO CARRY OUT THE SEQUENTIAL RULE FOR $P_0 = .95$, $a_0 = 19^{1/4}$

($k = 5$)

$r_2 - r_1$	$r_3 - r_1$	$r_4 - r_1$	$r_5 - r_1$	
5	5	9	10	
5	5	10	10	*
5	6	6	8	
5	6	7	7	
5	7	7	7	*
6	6	6	6	

* The rows marked with an * could be omitted without affecting the test since every integer in these rows is at least as great as the corresponding integer in the previous row. They are shown here to illustrate a systematic method which insures that all the necessary rows are included.

In the above form the proposed rule is to stop when, for at least one row (say row j) in the table, the observed row vector $(r_2 - r_1, r_3 - r_1, \cdots, r_k - r_1)$ is such that *each component* is at least as large as the corresponding component of row j.

Properties of the procedure. For $k = 2$ the above procedure is an example of a Wald sequential probability ratio test [5] as modified by Girshick [4]. A. Birnbaum [2], [3] has remarked that we can regard the successive failures as independent binomial observations with the probability of the next failure arising from the population with parameter θ_i constantly equal to

$$p_i = \frac{\frac{1}{\theta_i}}{\frac{1}{\theta_1} + \frac{1}{\theta_2}} \qquad (i = 1, 2). \tag{9}$$

The problem then is to select the population associated with the smaller probability. Using Wald's formulas one can easily compute the *OC* and *ASN* functions for the procedure. Both of these depend only on the true value of the ratio θ_1/θ_2 which we will denote by a. The *OC* function $P(a)$ gives us the probability of a correct selection and the *ASN* function $E(F; a)$ gives us the expected number of failures needed to complete the experiment. Since there is no excess over the boundary we have the following *exact* results:

$$P(a) = \frac{a^s}{1 + a^s} \tag{10}$$

$$P(1) = \tfrac{1}{2}, \quad P(a_0) \geqq P_0, \quad P(\infty) = 1. \tag{11}$$

For the *ASN* function, we compute

$$E(F; a) = s\left(\frac{a+1}{a-1}\right)\left(\frac{a^s - 1}{a^s + 1}\right) \quad \text{for } a > 1, \tag{12}$$

$$E(F; 1) = s^2. \tag{13}$$

The time intervals between failures are independent identically distributed chance variables that do not depend on the stopping rule. The time interval between failures for a single population is an exponential chance variable. Hence, the time interval between

failures is the minimum of two exponential chance variables which is again exponential. We then obtain, letting τ denote the interval between failures,

$$E\{\tau;\ a,\ \theta_2\} = \frac{1}{n}\frac{\theta_2 a}{1+a}. \tag{14}$$

Hence, letting T denote the *total* expected time for the experiment we have by multiplying (12) and (13) by (14).

$$\begin{aligned} E\{T;\ a,\ \theta_2\} &= \frac{a\theta_2 s\,(a^s-1)}{n\,(a^s+1)\,(a-1)} \qquad \text{for } a>1, \\ E\{T;\ 1,\ \theta_2\} &= \frac{\theta_2 s^2}{2n}. \end{aligned} \tag{15}$$

For the numerical illustration treated above with $k = 2$ we have

$$P(a) = \frac{a^4}{1+a^4}, \tag{16}$$

$$E(F;\ a) = 4\,\frac{a+1}{a-1}\,\frac{a^4-1}{a^4+1} = \frac{4\,(a+1)^2\,(a^2+1)}{a^4+1}, \tag{17}$$

$$E(F;\ 1) = 16.0\,, \qquad E(F;\ a_0) = 10.2\,, \qquad E(F;\ \infty) = 4\,, \tag{18}$$

$$E(T;\ 1,\ \theta_2) = \frac{8\theta_2}{n}, \qquad E(T;\ a_0,\ \theta_2) = \frac{6.9\theta_2}{n}, \qquad E(T;\ \infty,\ \theta_2) = \frac{4\theta_2}{n}. \tag{19}$$

For $k > 2$ the proposed procedure is an application of a general sequential rule for selecting the "best" of k populations treated in [1]. Proof that for $k > 2$ the probability statement associated with (5) holds, and bounds on the probability of a correct decision can be found there.

5. Specifications for case 1: Known location parameters

Before experimentation starts, the experimenter specifies three constants P_0, a_0, β_0, such that $\frac{1}{k} < P_0 < 1$, $a_0 > 1$ and $\beta_0 > 0$. To simplify the discussion we shall assume that $P_0 > \frac{1}{2}$ for any k. The resulting rule R_1 has the property that *it will select the population with the largest scale parameter with probability at least P_0 whenever we have both*

$$\frac{\theta_1}{\theta_2} \geqq a_0 \qquad \textit{and} \qquad \frac{1}{\theta_2} - \frac{1}{\theta_1} \geqq \beta_0. \tag{20}$$

The rule can now be stated in terms of a_0, β_0, $\epsilon_0 = (1 - P_0)/P_0$ and $A^* =$ maximum $A_i (i = 1, 2, \cdots, k)$.

Rule R_1. "Continue experimentation for at least A^* units of time until the inequality

$$\sum_{i=2}^{k} a_0^{-(r_i - r_1)}\, e^{-\beta_0 (L_1 - L_i)} \leqq \epsilon_0 \tag{21}$$

is satisfied. Then stop and select the population with the smallest number r_1 of failures as the one having the largest scale parameter. If, at stopping time, two or more populations have the same value r_1, then choose *one of these* by an independent experiment giving equal probability to each."

For $k = 2$ the inequality (21) reduces to

$$(r_2 - r_1) \log a_0 + (L_1 - L_2)\beta_0 \geqq \log (1/\epsilon_0) . \tag{22}$$

Remarks.

(1) If $A_i = A (i = 1, 2, \cdots, k)$ then the instruction to wait at least $A^* = A$ units of time is redundant since, with $\epsilon_0 < 1$, the inequality (21) cannot be satisfied before A.

(2) The procedure based on (21) may result in termination not only at failures but also between failures.

(3) The same rule can be used if different populations start at different times, if the number n of units from each population is different, or even if experimentation is carried out without replacement. The instruction would in general be to continue until each population had at least one unit aged past its guarantee period.

(4) The main disadvantage of this rule is that it requires more bookkeeping than a rule based only on the number of observed failures.

(5) The terms of the sum in (21) represent likelihood ratios. If at any time each term is less than unity then we shall regard the decision to select the population with r_1 failures as optimal. Since $\epsilon_0 < 1$ each term must be less than unity at termination.

Properties of rule R_1. The *OC* and *ASN* functions for R_1 will be approximated by comparing R_1 with another procedure R_1^*. We shall assume a common value A for the location parameters. Since

$$L_i \cong nT - r_i A \text{ for all } i , \tag{23}$$

then at stopping time

$$L_1 - L_i \cong (r_i - r_1) A . \tag{24}$$

Substituting this in (21) and letting

$$\gamma_0 = a_0 e^{\beta_0 A} \tag{25}$$

gives us a new rule which we denote by R_1^*.

Rule R_1^*. "Continue experimentation until the inequality

$$\sum_{i=2}^{k} \gamma_0^{-(r_i - r_1)} \leqq \epsilon_0 \tag{26}$$

is satisfied. Then stop and select the population with r_1 failures as the one with the largest scale parameter."

For rule R_1 the experimenter need only specify P_0 and the smallest value γ_0 of the *single* parameter

$$\gamma = \frac{\theta_1}{\theta_2} \exp\left[A\left(\frac{1}{\theta_2} - \frac{1}{\theta_1}\right)\right] \tag{27}$$

that he desires to detect with probability at least P_0.

We shall compute the *OC* and *ASN* functions for R_1^* and use these as an approximation to the *OC* and *ASN* functions for R_1. For any time values $\theta_1 \geqq \theta_2$ we have only to replace a_0 by γ_0 specified for rule R_1^* or computed from (25) for rule R_1 and a by γ com-

puted from (27) in the formulas derived for the rule R_{1b}. We thus obtain (omitting P_0 in the notation for the rules)

$$(28)\quad P\{\gamma; R_1(a_0, \beta_0)\} \cong P\{\gamma; R_1^*(\gamma_0)\} = \frac{\gamma^{s^*}}{1+\gamma^{s^*}}$$

$$(29)\quad E\{F; R_1(a_0, \beta_0)\} \cong E\{F; R_1^*(\gamma_0)\} \cong \begin{cases} s^*\left(\frac{\gamma+1}{\gamma-1}\right)\left(\frac{\gamma^{s^*}-1}{\gamma^{s^*}+1}\right) & \text{for } \gamma > 1 \\ s^{*2} & \text{for } \gamma = 1 \end{cases}$$

where s^* is the smallest integer not less than

$$(30)\quad \frac{\log(1/\epsilon_0)}{\log \gamma_0} = \frac{\log(1/\epsilon_0)}{\log a_0 + \beta_0 A}.$$

6. Alternative procedures

It is often desirable to make a test simpler even if a certain amount of efficiency has to be sacrificed. The question arises whether the simpler rule R_{1b} which depends only on the numbers of observed failures can also be used when $A_i = A > 0 (i = 1, 2, \cdots, k)$ without seriously invalidating the probability requirement. To examine this we shall define two other rules R_1' and R_1'' which have a higher probability of a correct selection than R_1 and then we shall compare R_1'' with R_{1b}.

Rule R_1'. "Continue experimentation until the k inequalities

$$(31)\quad \sum_{i=2}^{k} a_0^{-(r_i - r_1)} e^{-\beta_0(L_1 - L_i)} \leqq \epsilon_0,$$

$$(32)\quad L_1 \geqq L_i \qquad (i = 2, 3, \cdots, k)$$

are simultaneously satisfied. Then choose the population with r_1 failures."

Since R_1 and R_1' both take the optimum terminal decision and R_1' requires at least as much experimentation as R_1 then the probability of a correct selection is greater for R_1' than it is for R_1, regardless of the true situation.

Rule R_1''. "Continue experimentation until the k inequalities

$$(33)\quad \sum_{i=2}^{k} a_0^{-(r_i - r_1)} \leqq \epsilon_0$$

$$(34)\quad L_1 \geqq L_i \qquad (i = 2, 3, \cdots, k)$$

are simultaneously satisfied. Then choose the population with r_1 failures."

Since $L_1 \geqq L_i$ at stopping time the terms in (33) are at least as large as the corresponding terms in (31). Hence the rule R_1'' always requires more experimentation than the rule R_1'. It follows that the probability of a correct selection is greater for the rule R_1'' than for R_1'.

To go from R_1'' to R_{1b} it is only necessary to delete the inequalities (34) but it cannot be shown that these inequalities are redundant. However, it is reasonable to suspect that the effect of introducing these inequalities in R_1' is of the same order of magnitude as the effect of deleting them from R_1''. This argument is nonrigorous; it should be clear, however, from the above that the rule R_{1b} can be used without seriously affecting the probability requirement.

We can obtain *OC* and *ASN* functions for R_{1b} when $A > 0$ and $k = 2$ by comparing R_{1b} with R_1^*. Since these two rules have the same form we can obtain the *OC* and *ASN*

functions for R_{1b} by replacing γ_0 in (28) and (29) by a_0. The resulting formulas are functions of γ which is computed from (27). If we disregard the error introduced by the approximation (24) then we can write (again omitting P_0)

$$(35)\quad P\{\gamma; R_{1b}(a_0)\} = P\{\gamma; R_1^*(a_0)\} \geqq P\{\gamma; R_1^*(\gamma_0)\} = P\{\gamma; R_1(a_0, \beta_0)\}$$

thus providing another nonrigorous "proof" that R_{1b} satisfies the required probability condition. In fact, we can now estimate the increase in the probability of a correct selection as well as the increase in the expected number of failures due to using R_{1b} instead of R_1. Defining s and s^* as in (10) and (30), respectively, we have

$$(36)\quad P(\gamma; R_{1b}) - P(\gamma; R_1) \cong \frac{\gamma^s}{1+\gamma^s} - \frac{\gamma^{s^*}}{1+\gamma^{s^*}},$$

$$(37)\quad E(F; R_{1b}) - E(F; R_1) \cong \frac{\gamma+1}{\gamma-1}\left[\frac{s(\gamma^s-1)}{\gamma^s+1} - \frac{s^*(\gamma^{s^*}-1)}{\gamma^{s^*}+1}\right].$$

7. Equilibrium approach

Consider first the case of n items on test all n and their replacements from a *single* delayed exponential population with parameters (μ, A). Let T_j denote the length of the time interval between the jth and $j+1$st failures $(j = 0, 1, 2, \cdots)$ with the convention that the 0th failure denotes the starting time. As time increases to infinity the expected number of failures per unit time clearly approaches $n/(\mu + A)$, which is called the equilibrium failure rate. The inverse of this is the expected time between failures "at equilibrium," say $E(T_\infty)$. The question as to how the quantities $E(T_i)$ approach $E(T_\infty)$ which interests us now is also of considerable interest in its own right. It is easy to see that

$$(38)\quad E(T_0) = A + \frac{\mu}{n} \geqq \frac{\mu + A}{n} = E(T_\infty).$$

In fact, since all units are new at starting time then

$$(39)\quad E(T_i) \leqq E(T_0) \qquad (i = 1, 2, \cdots).$$

If we compare the replacement procedure with $A > 0$ with the special case $A = 0$ we obtain

$$(40)\quad E(T_i) \geqq \frac{\mu}{n} \qquad (i = 1, 2, \cdots)$$

and if we compare it with the nonreplacement case we obtain

$$(41)\quad E(T_i) \leqq \frac{\mu}{n-i} \qquad (i = 1, 2, \cdots, n-1).$$

These comparisons show that the difference of the two members in (40) is small when A/μ is small and the difference in (41) is small (for $i < n$) when A/μ is large. Straightforward but tedious computation gives the exact results

$$(42)\quad E(T_1) = \frac{\mu}{n-1}\left[1 - \frac{1}{n}\, e^{-(n-1)A/\mu}\right] \qquad \text{for } n > 1,$$

$$(43)\quad E(T_2) = \frac{\mu}{n-2}\left[1 - \frac{(n+2)(n-1)}{n^2}\, e^{-(n-2)A/\mu} + \frac{n-2}{n-1}\, e^{-(n-1)A/\mu} - \frac{(n-2)}{n^2(n-1)}\, e^{-2(n-1)A/\mu}\right] \qquad \text{for } n > 2.$$

It can be shown that the polynomial formed by subtracting (42) from (43) and setting $y = \exp(-A/\mu)$ has for $n > 2$ a double root at $y = 1$. Then, using Descartes's rule of signs, it follows that for $n > 2$ and $0 < y \leqq 1$

$$E(T_1) \leqq E(T_2), \tag{44}$$

equality holding only for $A = 0$.

For $n = 2$ we obtain

$$E(T_2) = A + \mu[-\tfrac{1}{4} + e^{-A/\mu} - \tfrac{1}{4}e^{-2A/\mu}] \tag{45}$$

and comparing this with (42) for $n = 2$ we obtain again the inequality (44), equality holding only for $A = 0$. For $n = 1$ the result is trivial since equality holds. On the basis of these results one is tempted to conjecture that for any n

$$E(T_1) \leqq E(T_2) \leqq \cdots \leqq E(T_{n-1}) \leqq E(T_0), \tag{46}$$

but this has not been shown.

For the case of two populations with the same n, with scale parameters $\theta_1 \geqq \theta_2$ and a common location parameter A, we have similarly

$$E(T_0) = A + \frac{\theta_1\theta_2}{n(\theta_1+\theta_2)} \geqq \frac{(\theta_1+A)(\theta_2+A)}{n(\theta_1+\theta_2+2A)} = E(T_\infty), \tag{47}$$

$$E(T_i) \leqq E(T_0) \qquad (i = 1, 2, \cdots), \tag{48}$$

$$\frac{\theta_1\theta_2}{n(\theta_1+\theta_2)} \leqq E(T_i) \leqq \frac{\theta_1\theta_2}{n\theta_2+(n-i)\theta_1} \qquad (i = 1, 2, \cdots, n), \tag{49}$$

$$\frac{\theta_1\theta_2}{n(\theta_1+\theta_2)} \leqq E(T_i) \leqq \frac{\theta_1}{2n-i} \qquad (i = n+1, n+2, \cdots, 2n-1), \tag{50}$$

where the first inequality in (49) and (50) holds for all i. The extension to the case of k populations is straightforward.

Duration of the experiment. For the sequential rule R_1 with $k = 2$ we can now write down upper and lower bounds as well as approximations to the expected duration $E(T)$ of the experiment. From (47), (48) and (49)

$$A + \frac{\theta_1\theta_2 E(F;\gamma)}{n(\theta_1+\theta_2)} \leqq E(T) \leqq E(T_0)E(F;\gamma). \tag{51}$$

As an estimate of $E(T)$ we would use

$$E(T) \cong E(T_\infty)E(F;\gamma). \tag{52}$$

The upper bound in (51) can be improved by bounding $E(T_i)$ in the equation $E(T) = E\left[\sum_{i=0}^{F-1} E(T_i)\right]$ by the minimum of $E(T_0)$ and the right members of (49) or (50), or by use of the appropriate exact expression (42), (43) or (45).

8. Case 2: Common unknown location parameter

In this case there are several reasonable sequential procedures that can be used. The most conservative procedure is to use R_{1b}, that is, to act under the assumption that $A = 0$. Then by the discussion in case 1 the probability requirement is satisfied for all $A \geqq 0$. The *OC* and *ASN* functions, which are now functions of A also, were already

obtained above. Of course, we need not consider values of A greater than the smallest observed lifetime of all units tested to failure.

9. Comparison of sequential and nonsequential procedures

We shall now compare the sequential procedure R_{1b} and a nonsequential, nonreplacement procedure R_3 which will now be defined.

Rule R_3. "Continue without replacement until each population has produced r failures. Then stop and accept the last population that produced r failures."

It is immediately evident that we need only wait until $k-1$ populations each have r failures since the population to be selected is then determined. Let R_3' denote the rule R_3 so altered.

For $k=2$ the probability of a correct decision using R_3' is

$$P(a; R_3') = \int_0^\infty \int_0^{x_r} f(y_r, \theta_2)\, f(x_r, \theta_1)\, dy_r dx_r \tag{53}$$

where

$$f(x, \mu) = \frac{r}{\mu} C_r^n (1 - e^{-x/\mu})^{r-1} e^{-x/\mu} e^{-(n-r)x/\mu}. \tag{54}$$

This reduces to

$$P(a; R_3') = 1 - (rC_r^n)^2 \sum_{j=1}^{r} \frac{(-1)^{j-1}}{n-r+j} C_{j-1}^{r-1} \{B[r, n-r+1+a(n-r+j)]\}^{-1} \tag{55}$$

where $B(x, y)$ is the (complete) beta function. Although equation (53) is derived under the assumption that $A=0$, it is clear that the result (55) holds for any $A \geqq 0$.

The expected time necessary to come to a decision for the rule R_3' is for $\theta_1 = \theta_2 = \mu$, say, and $k=2$

$$E(T) = \int_0^\infty 2x f(x, \mu)[1 - F(x, \mu)]\, dx, \tag{56}$$

where $f(x, \mu)$ is the density given by (54) and $F(x, \mu)$ is its c.d.f. This reduces to

$$E(T) = 2\mu (rC_r^n)^2 \sum_{i=1}^{r} \sum_{j=1}^{r} C_{j-1}^{r-1} C_{i-1}^{r-1} \frac{(-1)^{i+j}}{(n-r+j)[2(n-r)+i+j]^2}. \tag{57}$$

For $A > 0$ we need only add A to the right member of (57).

For $a = a_0$ the expected time using rule R_3' is obtained from an expression similar to (57) derived for the case $\theta_1 \neq \theta_2$. For $a = a_0$ the expected time using rule R_{1b} is obtained from (47) and expressions similar to (42) and (43) derived for the case $\theta_1 \neq \theta_2$.

For $n=10$, $r=3$, $a_0=2$, $k=2$ and any $A \geqq 0$ the probability attained is .789 when the true $a = a_0$. If we set $P_0 = .789$ and $a_0 = 2$ in the sequential rule R_{1b}, the rule takes the form

$$\text{Stop when } r_2 - r_1 \geqq 2. \tag{58}$$

For $a = a_0$ the sequential rule R_{1b} attains a higher probability which is easily shown to be .800 for $A=0$ and is greater than .800 for $A > 0$. The following table gives the ratio

of the expected times of the experiment for the two rules. The lower entry in each cell is the ratio of the probabilities of a correct selection for the two rules.

TABLE III

THE RATIOS $\frac{E(T; R_{1b})}{E(T; R_3')}$ AND $\frac{P(R_{1b})}{P(R_3')}$

($n = 10$; $a_0 = 2$, $P_0 = .789$)

	$A=0$	$A/\theta_2=1$
$\theta_1=\theta_2$ $a=1=\frac{a_0}{2}$	.867 1.000	.989 1.000
$\theta_1=2\theta_2$ $a=2=a_0$	.809 1.013	.942 1.160
$\theta_1=\infty$ $a=\infty$	.595 1.000	.906 1.000

The sequential rule R_{1b} gives an expected time at least as small and a probability at least as large as the nonsequential rule R_3' for each of the true situations considered and appears to be better for all true situations.

By comparing $R_{1b}(n)$ with $R_3'(n, r)$, averaged over $r=3$ and $r=4$ so that the probability is exactly .800 for $a = a_0$, we obtain an efficiency measure which varies continuously with P_0. This efficiency factor can be broken up into a product of two factors, one showing the efficiency gained by the replacement feature alone, and the other showing the efficiency gained by the sequential feature alone. The results are given below for $a_0 = 2$, $P_0 = .800$, $A = 0$, four values of n and three values of a.

TABLE IV

PER CENT REDUCTION IN AVERAGE EXPERIMENT TIME

a	n	Per Cent Reduction Due to Replacement Feature Alone	Per Cent Reduction Due to Sequential Feature Alone	Per Cent Reduction Due to Replacement and Sequential Features
1	4	29.5	12.7	38.5
	10	13.7	12.7	24.6*
	20	6.3	12.7	18.2
	∞	0.0	12.7	12.7
2	4	30.1	17.8	42.5
	10	13.9	17.8	29.2
	20	6.6	17.8	23.2
	∞	0.0	17.8	17.8
	4	31.5	38.8	58.1
	10	13.6	38.8	47.2*
	20	6.3	38.8	42.7
	∞	0.0	38.8	38.8

* Slightly inconsistent with columns 1 and 2 because of rounding.

Thus the advantage of a replacement technique is greatest for small values of n and is essentially independent of a while the advantage of a sequential procedure is greatest for large a and is independent of n.

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