FRACTIONAL REPLICATION IN INDUSTRIAL RESEARCH

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1. Introduction

It is interesting to compare the current state of industrial experimentation with the situation in experimental agriculture twenty years ago. The distribution of statistical knowledge among workers in the two fields seems to be about the same. The two groups would agree that their problems are of great complexity; that many factors are operative; that no detailed analytic theory of process operation is available; and that in addition to the effects of the "known" factors, large amounts of unpredictable variation are present.

Industrial experimentation, at least in the process industries, differs strikingly from agricultural work in the speed with which experimental results can be obtained. Experimental units are often successive runs on the same piece of equipment, but even quite large designs rarely take more than a few weeks to complete. A further difference, I believe, lies in the number of dependent variables that must be measured or calculated. The important properties of a type of rayon, or of a new kind of steel, or of an improved cake-mix, cannot be summarized by a single number. More usually, from five to ten dependent variables are required. Finally, the usefulness of blocks of homogeneous experimental material appears to be less in industrial work, but this appearance may be due to our not knowing how to specify blocks.

Industrial experimentation may be viewed as a continuum extending all the way from slight modifications of existing operation to the development of entirely new processes. Balanced designs of the type to be discussed are easier to use near the "existing process" end of this scale, but experience is rapidly being gained with the development of new processes.

Factors whose first-order effects are large, say of the order of 3 or 4 standard deviations, are usually discovered by making a few runs. The symbol σ_R will be used for the standard deviation of runs made under "fixed" conditions. If we must estimate a first-order effect of the order of σ_R with say a 95% confidence interval of half-width $\sigma_R/2$, then

$$(1) \qquad 1.96\sigma_R\sqrt{\frac{4}{N_R}} = \tfrac{1}{2}\sigma_R$$

from which N_R, the number of statistically independent runs required, is roughly 64. This number is conservative for two reasons. Chemists and plant engineers are likely to underestimate their σ_R, often by a factor of one-half. In the second place, since a considerable number of effects are usually measured in the same experiment, the usual 95% level of confidence is often not high enough. To reach the 99% level (per effect), roughly $(2.58/1.96)^2$ or 1.7 times as many runs must be made as at the 95% level.

At least 64 runs are required, then, half at each of two levels of a particular factor, in order to estimate main effects of magnitude σ_R to ± 50 per cent. There will be no loss in precision, but a great gain in generality, if a 2^6 factorial is substituted. Indeed, there is little risk in planning to study 8 two-level factors (and all their two-factor interactions) in a "quarter replicate of a 2^8," abbreviated here 2^{8-2}. This scheme measures each main effect along with (confounded with) three higher order (four- or five-factor) interactions. It also measures each two-factor interaction, abbreviated here 2 f.i., along with three higher order (three- or four-factor) interactions. Such a design will be called a "two-factor interaction clear" design, or 2 f.i.c. for short.

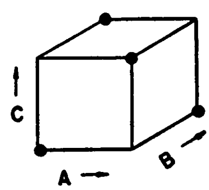
Larger designs are necessary, and are frequently run, if more factors must be studied (with all 2 f.i.c.), or if the error standard deviation has been underestimated. If p factors are to be studied, then $p(p + 1)/2$ degrees of freedom must be available to estimate all main effects and all 2 f.i. It will be shown below that for values of p from 7 to 16, roughly two to three times that number of runs must be made.

2. Fractional replication of factorial designs, 2^{p-q}, with all two-factor interaction clear

Fractional replication was introduced by Finney [1], [2], discussed and extended by Kempthorne [3], [4]. A wide range of useful cases was tabulated by Brownlee, Kelly, and Loraine [5]. More recently a very complete atlas of designs up to 2^{12-5} has been prepared by Clatworthy, Connor, and Zelen [6]. An extended discussion suitable for

TABLE I

REPRESENTATIONS OF THE ONE-HALF REPLICATE OF THE 2^3 FACTORIAL

Representation	1			2			3			4	5	6
Factors	*a*	*b*	*c*	*a*	*b*	*c*	*a*	*b*	*c*			
Run Number												
1	1	1	0	1	1	−1	+	+	−	*ab*		"Even with *ABC*"
2	1	0	1	1	−1	1	+	−	+	*ac*		
3	0	1	1	−1	1	1	−	+	+	*bc*		
4	0	0	0	−1	−1	−1	−	−	−	(1)		

engineers is given in [14]. Most of what follows is only adaptation from the work of these writers.

Six equivalent methods of representation of fractional replicates are in use. They are shown for the 2^{3-1} in table I. The symbols *ab*, *ac*, *bc*, and (1) are also used to indicate the responses to runs made under the respective conditions. It is obvious that $[ab + ac - bc - (1)]/2$ measures the average effect of changing the factor A from its "low" to its "high" level. It is less obvious that the $B \times C$ interaction, that is, the *difference* in the effects of B at high and at low C, is measured by the same number. The effect of B at high C may be taken as $(bc - ac)$, and the effect of B at low C, as $[ab - (1)]$. Half the difference between these two effects may be written $[-ab - ac + bc + (1)]/2$ and this is the negative of the quantity used to measure A. Thus the contrast given measures $A - BC$. The effects A and $-BC$ are said to be confounded; each effect is called the *alias* of the other.

Similarly for the four runs given in table I, B and $-AC$ are confounded, as are C and

$-AB$. Finally, the average of the four runs is confounded with $-ABC$. (Use of the complementary set of four runs, *a*, *b*, *c*, and *abc*, gives $+BC$ as the alias of A, $+AC$ as the alias of B, and so on.)

The use of any other set of four runs will confound main effects with the grand mean, or, avoiding this, will confound main effects with each other.

All these relations can be summarized quite compactly. The symbols for all effects and interactions, in the present example, A, B, C, AB, AC, BC, and ABC, together with the identity I constitute a group, if we define multiplication in the usual way, and add the "cancellation rule" $A^2 = B^2 = C^2 = I$. The alias of any effect, for example, A, is now found by multiplying the effect by $-ABC$. The product is $-BC$. Similarly, the alias of $-ABC$ is I, and so the mean of the eight members of the full 2^3 is estimated by the four runs specified, confounded with $-ABC$. The entire system of aliases can be seen in summary from the so-called *defining contrast*, $I = -ABC$. Multiplication of both sides of this "equation" by any set of letters in the effect group, produces a pair of effects that are those measured together. The two terms I and $-ABC$ are called the *alias subgroup*.

The alias subgroup is also used to specify the run conditions. It is only necessary to write down all the combinations of lower-case letters that have an *even* number of letters, or none, in common with every member of the alias subgroup.

In similar fashion, one alias subgroup for the 2^{5-1} could be written

$$I = -ABCDE\,, \tag{2}$$

meaning that AB is measured along with $-CDE$, etc. The 16 run conditions are easy to write down: *abcd*, *abce*, *abde*, $\cdots$, *de*, (1).

The confounding patterns for the quarter replicates for 5, 6, 7, 8, 9 factors can be visualized by the scheme of table II. Each line is to be read three times; the first time

TABLE II

CONDENSED ALIAS SUBGROUPS OF THE QUARTER REPLICATES, 2^{p-2}

(p = number of factors; N_R = number of runs)

p	N_R	Alias Subgroup
5	8	$AB\underline{C}DE$*
6	16	$AB\underline{CD}EF$
7	32	$ABC\underline{DE}FG$
8	64	$ABC\underline{DE}FGH$**
9	128	$ABC\underline{DEF}GHJ$

* Read: $I = ABC = CDE = ABDE$.
** Smallest 2 f.i.c. design for 8 factors.

through the underlined letters, then starting with the underlined letters, and, finally, excluding them.

Thus for eight factors, $p = 8$, the line $ABC\underline{DE}FGH$ is read

$$I = ABCDE = DEFGH = ABCFGH\,.$$

Any one of the latter three members can be viewed as the product of the other two, which are then called the *generators* of the alias subgroup. It can be seen from this table that the 2^{8-2}, requiring 64 runs, is the smallest quarter replicate with all 2 f.i.c.

Any one of four sets of run conditions can be used. The first set is *even* with respect to the two generators. Its first few members are, then, *abcdfgh, abcdf, abcdg, abcdh, abcefgh.* Another quarter replicate is even with respect to the first generator, but odd with respect to the second; thus *abcdfg, abcdfh, abcdgh,* are members of this fraction. The first set measures $A - BCDE - ADEFGH + BCFGH$, etc. The second set measures $A + BCDE - ADEFGH - BCFGH$, and so on.

Writing out in alphabetical order the full alias subgroups for several of the published fractional replicate designs reveals, if it was not already obvious, that the order of appearance of each letter in the successive members of each subgroup follows the regular patterns familiar in specifying treatment combinations in the 2^p series.

Thus for the quarter replicate just mentioned, the full alias subgroup might be written:

								A	*B*	*C*	*D*	*E*	*F*	*G*	*H*
A	*B*	*C*	*D*	*E*				1	1	1	1	1	0	0	0
A	*B*	*C*			*F*	*G*	*H*	1	1	1	0	0	1	1	1
			D	*E*	*F*	*G*	*H*	0	0	0	1	1	1	1	1
I								0	0	0	0	0	0	0	0

The numbers in the right panel are written out first in rows, but are then viewed in columns. The three types of columns that appear, which represent here the presence of a factor by a $\underline{1}$ and its absence by a $\underline{0}$, are the three that appear in the first representation in table I. Each column allocated to a new factor (and letter) adds members in only two positions. Each row is a symbol for a member of an alias subgroup. The columns headed S_p (p goes from 2 to 9) in table III show the number of letters in each member

TABLE III

GENERATION OF THE QUARTER REPLICATES, 2^{p-2},
FOR $p = 6, 7, 8, 9$

A	*B*	*C*	*D*	*E*	*F*	*G*	*H**	*J*	S_2	S_3	S_4	S_5	S_6	S_7	S_8^*	S_9	**
1	1	0	1	1	0	1	1	0	2	2	3	4	4	5	6	6	*ABDEGH*
1	0	1	1	0	1	1	0	1	1	2	3	3	4	5	5	6	*ACDFGJ*
0	1	1	0	1	1	0	1	1	1	2	2	3	4	4	5	6	*BCEFHJ*
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	*I*

* Smallest 2 f.i.c. design for $p = 8$. ** Alias subgroup for 2^{9-2}.

of the alias subgroup, after p columns have been accumulated. The first column in which all entries, except, of course, the last, are 5 or greater is S_8, and so the 2^{8-2} is the smallest 2 f.i.c. design for eight factors.

At the right of table III the alias subgroup for the 2^{9-2} is returned to the literal symbols. The smaller 2^{p-2} can all be read from the same column by dropping letters in reverse order, starting with J. The treatment combinations can be derived from the generators: (1) *abc, ad, be, cf, dg, eh, fj,* using only the first p of the eight given.

In the alias subgroup, repeated use of the same columns is permitted. This repetition merely states symmetry of the alias relations for two or more letters. Repetition of this kind in the treatment specification part of the design would enforce full confounding of the letters repeated.

Table IV shows the analogous generation of the 2^{p-3}, the "eighth replicates." The 2^{10-3} in 128 runs is the smallest 2 f.i.c. ten-factor design.

In table V the same procedure is used to derive the 2^{p-4} series. To facilitate smooth accumulation of letters, the matrix of zeros and ones is written in symmetrical form and the zeros have been left blank. The successive sums are written below the matrix. The 2^{11-4} in 128 runs has all 2 f.i.c. This design will then usually replace the 2^{10-3} since one more factor together with all its interactions can be studied, with no increase in the size of the experiment.

TABLE IV

GENERATION OF THE EIGHTH REPLICATES, 2^{p-3}, FOR $p = 7, 8, 9, 10, 11$

A	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*J*	*K**	*L*	S_7	S_8	S_9	S_{10}^*	S_{11}	**
1	1	1	0	0	0	1	1	1	1	1	4	5	6	7	8	*ABCGHJKL*
1	1	0	0	1	1	0	1	1	0	0	4	5	6	6	6	*ABEFHJ*
1	0	1	1	0	1	0	1	0	1	0	4	5	5	6	6	*ACDFHK*
1	0	0	1	1	0	1	1	0	0	1	4	5	5	5	6	*ADEGHL*
0	1	1	1	1	0	0	0	1	1	0	4	4	5	6	6	*BCDEJK*
0	1	0	1	0	1	1	0	1	0	1	4	4	5	5	6	*BDFGJL*
0	0	1	0	1	1	1	0	0	1	1	4	4	4	5	6	*CEFGKL*
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	*I*

* Smallest 2 f.i.c. design for $p = 10$. ** Alias subgroup for 2^{11-3}.

TABLE V

GENERATION OF THE SIXTEENTH REPLICATES, 2^{p-4}, FOR $p < 13$

	A	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*J*	*K*	*L**	*M*				**
A	1		1		1		1		1		1		1		1	*ACEGJL* generator
B		1		1		1		1	1		1		1		1	*BDFHJL* generator
C	1		1			1		1		1		1	1		1	*ACFHKM* generator
D		1		1	1		1			1		1	1		1	*BDEGKM*
E	1			1		1	1		1			1		1	1	*ADFGJM* generator
F		1	1		1			1	1			1		1	1	*BCEHJM*
G	1			1	1			1		1	1			1	1	*ADEHKL*
H		1	1			1	1			1	1			1	1	*BCFGKL*
J	1	1			1	1			1	1			1	1		*ABEFJK*
K			1	1			1	1	1	1			1	1		*CDGHJK*
*L**	1	1					1	1			1	1	1	1		*ABGHLM*
M			1	1	1	1					1	1	1	1		*CDEFLM*
	1	1	1	1					1	1	1	1				*ABCDJKLM*
					1	1	1	1	1	1	1	1				*EFGHJKLM*
	1	1	1	1	1	1	1	1								*ABCDEFGH*

SUCCESSIVE SUMS

2	1	1	1	1	1	1	1	1	2	0	2	0	2	0	2	0
3	2	1	2	1	1	2	1	2	2	1	2	1	3	0	3	0
4	2	2	2	2	2	2	2	2	2	2	2	2	4	0	4	0
5	3	2	2	3	2	3	3	2	3	2	2	3	4	1	5	0
6	3	3	3	3	3	3	3	3	4	2	2	4	4	2	6	0
7	4	3	3	4	4	3	3	4	4	3	3	4	4	3	7	0
8	4	4	4	4	4	4	4	4	4	4	4	4	4	4	8	0
9	5	5	4	4	5	5	4	4	5	5	4	4	5	5	8	0
10	5	5	5	5	5	5	5	5	6	6	4	4	6	6	8	0
11*	6	6	5	5	5	5	6	6	6	6	5	5	7	7	8	0
12	6	6	6	6	6	6	6	6	6	6	6	6	8	8	8	0

* Smallest 2 f.i.c. design for $p = 11$. ** Alias subgroup for 2^{12-4}.

Because of their bulk, the corresponding tables for the 1/32 and 1/64 replicates are not reproduced. The 2^{13-5} and the 2^{14-6} are the resulting smallest 2 f.i.c. designs.

Table VI gives sets of generators of alias subgroups up to 2^{16-7}. The last column has been adapted from Brownlee, Kelly, and Loraine [5]. In each case letters may be dropped from the end of each generator until five-factor interactions appear. The generators diminished in this way will give the smallest 2 f.i.c. design for that number of factors.

TABLE VI

GENERATORS OF ALIAS SUBGROUPS FOR FRACTIONAL REPLICATES 2^{p-q}

(p = number of factors; q = degree of fractionation)

p	6–9	7–11	8–12	11–14	14	15–16
q	2	3	4	5	6	7
	ABDEGH	*ABCGHL*	*ABGHLM*	*ABCDMO*	*ABJNO*	*ABCDOQ*
	ACDFGJ	*ACDFHK*	*ACFHKM*	*ADFGHLO*	*ACEGJ*	*ABEHPQ*
		ADEGJL	*ADFGJM*	*BDEGHKO*	*ADFGM*	*ACKLNQ*
			BDEGKM	*EFGHMNO*	*BDFHJ*	*ADEGN*
				HJKLMO	*CDJLN*	*ADJKP*
					CDKMO	*BDJMNQ*
						CDFGPQ

TABLE VII

SMALLEST FRACTIONAL REPLICATES IN 2^{p-q} SERIES WITH ALL TWO-FACTOR INTERACTIONS CONFOUNDED WITH HIGHER ORDERS

p = number of factors
q = degree of fractionation
f = fraction f, 2^{-q}
N_R = number of runs
E = degree of freedom efficiency*
n_4 = maximum number of four-level factors

p	q	f	N_R	E	n_4
5	1	1/2	16	1.00	0
6	1	1/2	32	0.68	1
7	1	1/2	64	0.44	2
8	2	1/4	64	0.56	1
9	2	1/4	128	0.35	3
10	3	1/8	128	0.43	2
11	4	1/16	128	0.52	1
12	4	1/16	256	0.31	4
13	5	1/32	256	0.36	4
14	6	1/64	256	0.41	
15	7	1/128	256	0.47	

* $E = p(p+1)/2(N_R - 1)$.

(The 2^{13-5} given has 30 2 f.i. confounded with 3 f.i. The alias subgroup given by Brownlee, Kelly, and Loraine for the same situation has 44 such 2 f.i. All other designs for $p < 16$ given by these writers are reproduced by the present method.)

Some elementary properties of the 2 f.i.c. designs, for 5 to 15 factors, are set forth in table VII. The number of runs required N_R and the "degree-of-freedom efficiency" E are of some interest. This efficiency gives the fraction of the total degrees of freedom used in estimating main effects or 2 f.i. For the most-used designs (for p greater than seven), this efficiency is seen to vary from 56 to 31 per cent; it is locally maximal for the 2^{8-2}, the 2^{11-4}, and for the 2^{15-7}, requiring 64, 128, and 256 runs, respectively.

By its construction, it is clear that a fractional replicate, 2^{p-q}, corresponds to a single block of a full factorial, 2^p confounded in 2^q blocks. The members of the alias subgroup, that is, the interactions confounded with the population mean in the fractional replicate, are the confounded interactions in the full factorial if done in 2^q blocks. In an unpublished note, A. Birnbaum has pointed out that, following R. A. Fisher [15], and R. C. Bose [16], it must be possible to derive balanced incomplete block designs which confound only *five*-factor interactions (and higher) with blocks. If as an important example it proves practicable to derive the design for the 2^{20} in blocks of (hopefully) 2^9, each of these blocks could be used as a 2^{20-11}. This design, requiring 512 runs, will be put to wide use as soon as it is available.

3. Calculations

Yates [7] has given a compact means of computing simultaneously the desired $(N_R - 1)$ effect-contrasts from a 2^p design. This can be adapted to the 2^{p-q} series by ignoring q letters in carrying out the calculation. The ignored letters must be ones occurring in only one alias subgroup generator. The details of this calculation are more easily available in Kempthorne [4], in Bennett and Franklin [8], or in Davies [14].

With N_R as large as 32, Yates' computational form may be split into two forms of size 16, using sums and differences of pairs over the last factor, instead of the original single results. This subdivision may be continued further for N_R larger than 32.

4. Error estimation

Confronted with the large number of contrasts provided by a 2^{p-q} design, the statistician must have some means of deciding which of the contrasts represent real effects, and which are "error contrasts." If none of the factors varied produces any effect then the whole collection of contrasts, $Q = (\bar{x}_+ - \bar{x}_-)$, will behave like a random sample of $(N_R - 1)$ independent drawings from a normal population (by central limit theorem) with mean zero and variance $4\sigma_R^2/N_R$. Since the signs of the effects are the result of the arbitrary definitions of the two levels of each factor, the absolute values of the Q contain all the information available about the standard deviation of the distribution. This "half-normal," or $\mathcal{N}/2$, distribution can be plotted on "arithmetic normal probability paper" with the probability scale rewritten as $|100\text{-}2P|$ where P is the printed probability in per cent.[1]

A collection of plots of actual experiments is offered in figure 1. Plot a is that of Yates' 2^5 factorial on beans [7]. The four largest contrasts were judged real in that paper. Plot b shows the results of varying eight factors thought to influence the yield of gasoline in a certain catalyst system [9]. Plot c uses data taken from an experiment published by Bennett and Franklin [8]. Plot d is from a 2^{15-7} (unpublished).

For comparison, figure 2 gives the empirical cumulative distribution for the first four sets of 30 random normal deviates (one from each column) given in Dixon and Massey [10]. The arbitrary practice has been followed of calculating cumulative per cents as $100\ (i - \frac{1}{2})/n$, where n is the number of degrees of freedom as i runs from

[1] The writer had been graphing this distribution as a set of ranges-of-pairs until the much more convenient half-normal plot was suggested by A. Birnbaum.

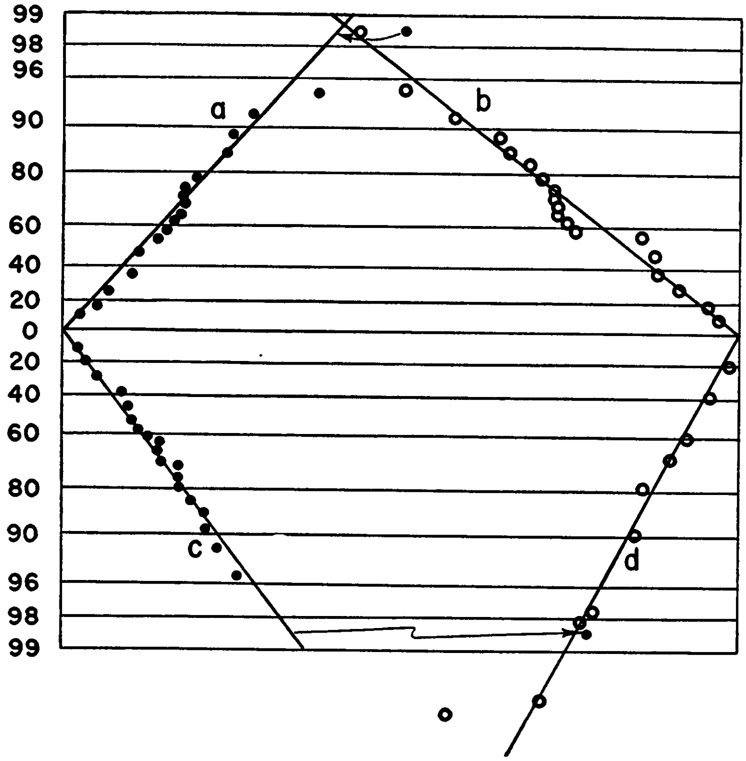


FIGURE 1

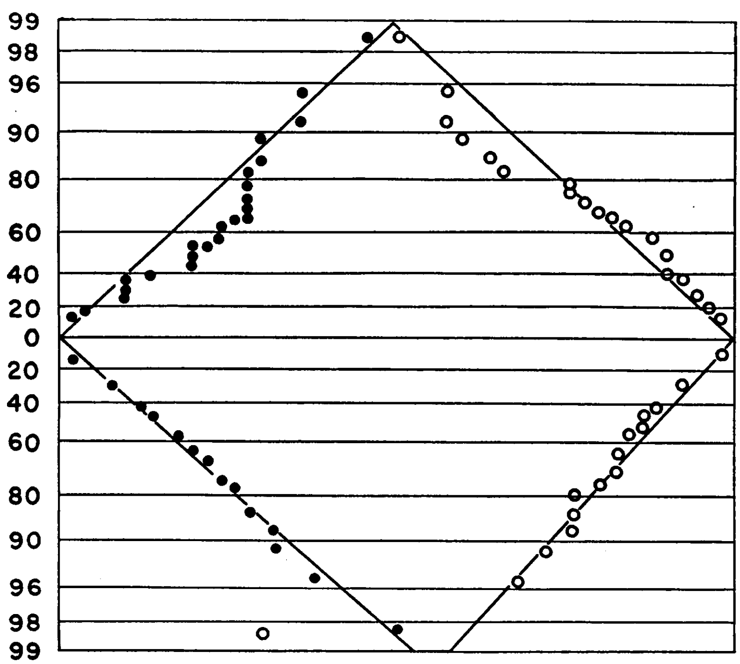


FIGURE 2

1 to n. The estimated standard error, $\hat{\sigma}_Q$, of the quantity Q, whose absolute values are plotted, is read off at the 68% point of the best line forced through the origin.

Generalizing somewhat prematurely from twenty such plots, it seems that if only the largest contrast appears to be off the line, then one must be quite conservative in judging its reality—about 1.5 $\hat{\sigma}_Q$ being a rather common deviation. (Such a deviation occurred twice in twenty plots.) Equally roughly, two contrasts are judged real only when they are both off the line by 1.0 $\hat{\sigma}_Q$; and if three or four are off the line plotted, the required minimum deviation should be 0.5 $\hat{\sigma}_Q$. Larger sampling experiments are in process and will be reported shortly. (See Note at the end of this paper.)

If more than one of the contrasts estimated appear to be real, then the assumption being made, that few effects are large, is somewhat vitiated. A re-plotting of the "error contrasts" is suggested on the same graph, to correct for the deflation of the ordinates by the excessive denominator.

5. Limitations

Four difficulties in using fractional replication have appeared repeatedly. Inoperable sets of conditions may destroy the required balance. No unbiased error estimate is obtained. The 2^{p-q} series are almost intrinsically first-order designs, that is, curvature of response surfaces cannot be measured. Sequences of fractional replicates are not easy to integrate to arrive at general conclusions.

The risk of running into inoperable sets of conditions may be reduced by preliminary experimentation. It has often proved practicable to assemble all of the experimenter's judgments about risky sets of conditions into a fraction of the intended fractional replicate, which is then run first.

A more interesting suggestion, due to H. Robbins [11], is to choose a set of, say, 25 random points in the proposed (bounded) factor space for a preliminary set of "operability runs." The proportion of these points that proves operable estimates the operable fraction of the region chosen with standard error $\leqq 0.1$. It may be reasonable to inflate the convex region determined by the points found operable, to the extent indicated by the operable fraction, thus getting an indication of the actual boundary of the operable region.

The second shortcoming mentioned is the absence of an unbiased error-estimate. This lack appears more serious in the short run than it does when we remember that a fractional replicate is generally only the first stage in two simultaneous campaigns. The first is the campaign of the engineers and technical men to locate and estimate the influential factors in a new field. The other is the campaign of the consulting statistician, teaching the research men the elements of statistical design. From the point of view of the former, the 2^{p-q} designs are far more precise than the work to which they are accustomed, often by a factor of p. From the statistician's standpoint, the inflated error that he reports often serves the beneficial—and provocative—purpose of persuading the engineers to run some real duplicates, in order to prove the statistician wrong. The revisions downward of the inflated estimate are rarely very great, and so both campaigns are advanced at the same time.

For more sophisticated engineers, who will want an unbiased estimate of the error built directly into the first set of runs, a compromise was suggested to the writer by W. A. Griffith [12]. It consists in duplicating some runs of the balanced design but not all. Unpublished work by H. Scheffé, extended by A. Birnbaum, has shown that the resulting normal equations are quite manageable in full replicates of factorial designs, and for small numbers of factors. It is hoped to extend these results to produce designs for the "partial duplication of fractional replicates" that will have easily computable analyses. In the meantime, it appears, judging from work of G.E.P. Box [13], that little information will be lost if the duplicated values are simply averaged and treated like single values. It is obvious that a considerable number of duplicates must be run, never less than 16, and it is plausible that they be chosen to comprise a "block" of the fractional replicate so as to guarantee a tolerably uniform spreading over the region of factor space being explored.

The criticism of the 2^{p-q} series implied in calling them intrinsically first-order designs may be partially covered by including some factors at four levels. This is done by reserving two two-level factors, now called pseudo-factors, *and their interaction* for the three degrees of freedom required. The column headed n_4 in table VII shows the number of four-level factors that can be crammed into a 2 f.i.c. design.

The difficulty in organizing sequences of fractional replications will be discussed in the next section.

6. Sequences of fractional replicates

If the effects of some factors are unexpectedly large, it will be well to learn this as early as possible. This desideratum is not taken into account in the designs discussed so far, since the whole sequence must be completed before any conclusions are drawn. It is possible, however, to plan a sequence of fractional replicates so that the initial fraction, and the final 2 f.i.c. design if it turns out to be required, are both "best." There are then some restrictions on the path taken through the intermediate fractions that may be used.

To study the main effects of seven factors, one might start with the 2^{7-4} in eight runs. The alias structure can be deduced from table VI, column 3, by dropping all letters after G. The generators of the alias subgroup may be taken as $-ABG$, $-ACF$, $ADFG$, and $BDEG$. Taking all products and arranging in alphabetic order, give

$-ABCDEFG$	$ACEG$	$-BCE$	$CEDF$
$ABCD$	$-ACF$	$BCFG$	$-CDG$
$ABEF$	$-ADE$	$BDEG$	$-EFG$
$-ABG$	$ADFG$	$-BDF$	I

This alias subgroup is well balanced in the sense that three 2 f.i. are confounded with each main effect.

The corresponding set of eight runs is advantageous in the further sense that all are from one half replicate. In choosing fractions to separate effects that need separating it is necessary to remember that it may prove desirable to go all the way to the 2^{7-1}, since the latter is the smallest 2 f.i.c. design for $p = 7$. It will generally be best then to retain the interaction $-ABCDEFG$ in the alias subgroup. This prevents the use of some intermediate fractions that would appear, by themselves, to be best. For example, the 2^{7-3} generated by $-ACF$, $ADFG$, and $BDEG$ has three 3 f.i. in its alias subgroup and so appears inferior to that generated by $ABCD$, $ADFG$, and $BDEG$, which has only 4 f.i. in its alias subgroup. But the former subgroup still contains the 7 f.i. and so its runs are all in one half-replicate. Such is not the case for the latter subgroup, and this fact must be weighed in deciding between the two. If sufficient sensitivity is going to be achieved with 16 runs, then the second alternative will usually be preferred. If the sensitivity still leaves much to be desired, and if several large contrasts not clearly unconfounded have already turned up, then the first, more conservative, alternative should be chosen.

Similarly a campaign may be planned from the 2^{15-11}, in 16 runs, to the 2^{15-7} in 256 runs, the latter being the smallest 2 f.i.c. design. It will be important to keep *all* the members of the alias subgroup of the latter design in the subgroup for the intermediate stages. This condition is met if any seven generators of the 2^{15-7} are present in the 2^{15-11} and are retained in the intermediate designs.

7. Summary

The 2^{p-q} series of fractional replicate designs is beginning to find wide application in industry. Since small effects (of the order of magnitude of the error standard deviation) are often industrially important, large numbers of runs are required in any case. The useful range so far is from five to fifteen factors, and from 16 to 256 runs. A method of deriving all the designs in this range is applied to complete the published lists of fractional replicates. Suggestions are made for simplifying the calculation and interpretation of fractional replicates.

Some limitations of these designs are: their sensitivity to missing data, their failure to provide strictly unbiased error-estimates, and their intrinsic first-orderedness. Some means of relaxing these limitations are proposed.

Certain sets of fractional replicates that have been found useful in sequential experimentation are discussed.

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Note added in proof: At the suggestion of A. Birnbaum, A. Bowker arranged to have 2500 sets of thirty-one standard normal deviates produced and ranked by machine in order of absolute magnitude. The statistic $t_1 = u_{31}/u_{22}$ was computed from each set, where u_i is the ith absolute value order statistic of a set of thirty-one. The distribution found for $\log_{10}t_1$ was noted by the writer to be very nearly normal, with mean 0.35 and standard deviation 0.11. The distances off the "half-normal line" corresponding to Type I error rates per experiment of 0.1, 0.05, and 0.01 are 0.65, 0.94, and 1.54 $\hat{\sigma}_Q$ units, respectively. Special thanks are due to G. Lieberman and to J. Carter for carrying through the machine computations. Bounds on both types of error, also for larger experiments, are being prepared by A. Birnbaum.

REFERENCES

[1] D. C. Finney, "The fractional replication of factorial experiments," *Annals of Eugenics*, Vol. 12 (1943–45), pp. 291–301.

[2] ———, "Recent developments in the design of field experiments. III. Fractional replication," *Jour. Agri. Sci.*, Vol. 36 (1946), pp. 184–191.

[3] O. Kempthorne, "A simple approach to confounding and fractional replication in factorial experiments," *Biometrika*, Vol. 34 (1947), pp. 255–272.

[4] ———, *Design and Analysis of Experiments*, New York, John Wiley and Sons, 1952.

[5] K. A. Brownlee, B. K. Kelly, and P. K. Loraine, "Fractional replication arrangements for factorial experiments with factors at two levels," *Biometrika*, Vol. 35 (1948), pp. 268–276.

[6] W. H. Clatworthy, W. S. Connor, and M. Zelen, "*Some Fractional Factorial Arrangements for Factors at Two Levels*," National Bureau of Standards Report (unpublished), July 28, 1954.

[7] F. YATES, *Design and Analysis of Factorial Experiments*, Harpenden, England, Imperial Bureau of Soil Science, Technical Communication No. 35, 1937.
[8] C. A. BENNETT and N. L. FRANKLIN, *Statistical Analysis in Chemistry and the Chemical Industry*, New York, John Wiley and Sons, 1954.
[9] C. DANIEL and E. W. RIBLETT, "A multifactor experiment," *Industrial and Engineering Chem.*, Vol. 46 (1954), pp. 1465–1468.
[10] W. J. DIXON and F. J. MASSEY, *Introduction to Statistical Analysis*, New York, McGraw-Hill, 1951.
[11] H. ROBBINS, personal communication.
[12] W. A. GRIFFITH, personal communication
[13] G. E. P. BOX, "Some theorems on quadratic forms applied in the study of analysis of variance problems. II. Effects of inequality of variance and of correlation between errors in the two-way classification," *Annals of Math. Stat.*, Vol. 25 (1954), pp. 484–498.
[14] O. L. DAVIES (Editor), *Design and Analysis of Industrial Experiments*, London and Edinburgh, Oliver and Boyd, 1954.
[15] R. A. FISHER, "The theory of confounding in factorial experiments in relation to the theory of groups," *Contributions to Mathematical Statistics*, New York, John Wiley and Sons, 1950, Paper 39.
[16] R. C. BOSE, "On a resolvable series of balanced incomplete block designs," *Sankhyā*, Vol. 8 (1947), pp. 249–256.