# AN EXTENSION OF THE BASIC THEOREMS OF CLASSICAL WELFARE ECONOMICS

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## 1. Summary

The classical theorem of welfare economics on the relation between the price system and the achievement of optimal economic welfare is reviewed from the viewpoint of convex set theory. It is found that the theorem can be extended to cover the cases where the social optima are of the nature of corner maxima, and also where there are points of saturation in the preference fields of the members of the society. The first point is related to an item in the Hicks-Kuznets discussion of real national income. The assumptions underlying the analysis are briefly reviewed and criticized.

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## 2. Introduction

In regard to the distribution of a fixed stock of goods among a number of individuals, classical welfare economics asserts that a necessary and sufficient condition for the distribution to be optimal (in the sense that no other distribution will make everyone better off, according to his utility scale) is that the marginal rate of substitution between any two commodities be the same for every individual.[1] Similarly, a necessary and sufficient condition for optimal production from given resources (in the sense that no other organization of production will yield greater quantities of every commodity) is stated to be that the marginal rate of transformation for every pair of commodities be the same for all firms in the economy.[2]

Let it be assumed that for each consumer and each firm there is no divergence between social and private benefits or costs, that is, a given act of consumption or production yields neither satisfaction nor loss to any member of the society other

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[1] By marginal rate of substitution between any commodity $A$ and commodity $B$ is meant the additional amount of commodity $A$ needed to keep an individual as well off as he was before losing one unit of $B$, the amounts of all other commodities being held constant. If the preference scale for commodity bundles is expressed by means of a utility indicator, then the marginal rate of substitution between $A$ and $B$ equals the marginal utility of $A$ divided by the marginal utility of $B$. See, for example, [8, pp. 19–20].

[2] The marginal rate of transformation between commodities $A$ and $B$ is the amount by which the output of $A$ can be increased when the output of $B$ is decreased by one unit, all other outputs remaining constant. In this definition, an input is regarded as a negative output. See [8, pp. 79–81].

than the consumer or producer in question. Then, it is usually argued, equality of the marginal rates of substitution between different commodities will be achieved if each consumer acts so as to maximize his utility subject to a budget restraint of a fixed money income and fixed prices, the same for all individuals. Similarly, equalization of the marginal rates of transformation will be accomplished if each firm maximizes profits, subject to technological restraints, where the prices paid and received for commodities are given to each firm and the same for all. Possible wastage of resources by producing commodities which are left unsold is avoided by setting the prices so that the supply of commodities offered by producers acting under the impulse of profit maximization equals the demand for commodities by utility maximizing consumers. So, perfect competition, combined with the equalization of supply and demand by suitable price adjustments, yields a social optimum.[3]

There is, however, one important point on which the proofs which have been given of the above theorems are deficient. The choices made by an individual consumer and the range of possible social distributions of goods to consumers are restricted by the condition that negative consumption is meaningless. Social optimization or the utility maximization of the individual must therefore be carried out subject to the constraint that all quantities be nonnegative. Now all the proofs which have been offered, whether mathematical in form, such as Professor Lange's, or graphical, such as Professor Lerner's, implicitly amount to finding maxima or optima by the use of the calculus [14, pp. 162–165]. Since the problem is one of maximization under constraints, the method of Lagrange multipliers in its usual form is employed. Implicitly, then, it is assumed that the maxima are attained at points at which the inequality conditions that consumption of each commodity be nonnegative are ineffective, all maxima are interior maxima.

Let us illustrate by considering the distribution of fixed stocks of two commodities between two individuals. Let the preference system of individual $i$ be represented by the utility indicator $U_i(x_1, x_2)$, where $x_1$ and $x_2$ are quantities of the two commodities, respectively. Let $X_1$ and $X_2$ be the total stocks of the two goods available for distribution. Then, if individual 1 receives quantities $x_1$ and $x_2$, individual 2 receives quantities $X_1 - x_1$ and $X_2 - x_2$ of the two goods, respectively. Then an optimal point can be defined by finding the distribution which will maximize the utility of individual 1 subject to the condition that the utility of individual 2 be held constant, that is, we maximize $U_1(x_1, x_2)$ subject to the condition that $U_2(X_1 - x_1, X_2 - x_2) = c$. The second relation implicitly defines $x_2$ as a function of $x_1$. Taking the total derivative with respect to $x_1$ and setting it equal to zero yields the relation

$$\frac{dx_2}{dx_1} = -\frac{\dfrac{\partial U_2}{\partial x_1}}{\dfrac{\partial U_2}{\partial x_2}},$$

the partial derivatives being evaluated at the point $(X_1 - x_1, X_2 - x_2)$. We can then differentiate $U_1(x_1, x_2)$ totally with respect to $x_1$, if we consider $x_2$ as a func-

[3] For a compact summary presentation of the proofs of the theorems sketched above, see O. Lange [12] and the earlier literature referred to there, particularly the works of Pareto and Professors Lerner and Hotelling.

tion of $x_1$. The total derivative is

$$\frac{\partial U_1}{\partial x_1} - \frac{\partial U_1}{\partial x_2} \frac{\frac{\partial U_2}{\partial x_1}}{\frac{\partial U_2}{\partial x_2}}.$$

If we ignore the additional conditions that $x_1 \geqq 0$, $x_2 \geqq 0$, $X_1 - x_1 \geqq 0$, $X_2 - x_2 \geqq 0$, a necessary condition for a maximum is that this total derivative be zero. It then easily follows that the marginal rate of substitution for the two commodities is the same for both individuals.

If we introduce the restraints on the ranges of $x_1$ and $x_2$, however, it can happen that the maximum value of $U_1$ as a function of $x_1$, where $x_2$ is considered not as an independent variable but as a function of $x_1$, is attained at one endpoint of the range, for example, when $x_1 = 0$. For such a maximum, all that is required is that the value of $U_1$ when $x_1 = 0$ is greater than that for slightly larger values of $x_1$, but not necessarily for values of $x_1$ slightly smaller than 0; indeed, $U_1$ is not even defined for such values. Then all we can assert is that the total derivative of $U_1$ with respect to $x_1$ at the optimal point is nonpositive; it may be negative. Then it would follow that the marginal rate of substitution between commodities 1 and 2 is less for individual 1 than for individual 2.[4]

It therefore follows that the condition of equality of marginal rates of substitution between a given pair of commodities for all individuals is not a necessary condition for an optimal distribution of goods in general. The classical theorem essentially considers only the case where the optimal distribution is an *interior* maximum, that is, every individual consumes some positive quantity of every good, so that the restraint on the ranges of the variables are ineffective. Now if commodities are defined sharply, so that, for example, different types of bread are distinguished as different commodities, it is empirically obvious that most individuals consume nothing of at least one commodity. Indeed, for any one individual, it is quite likely that the number of commodities on the market of which he consumes nothing exceed the number which he uses in some degree. Similarly, the optimal conditions for production, as usually expressed in terms of equality of marginal rates of substitution, are not necessarily valid if not every firm produces every product, yet it is even more apparent from casual observation that no firm engages in the production of more than a small fraction of the total number of commodities in existence.

On the face of it, then, the classical criteria for optimality in production and consumption, have little relevance to the actual world. From the point of view of policy, the most important consequence of these criteria was the previously mentioned theorem that the use of the price system under a regime of perfect competition will lead to a socially optimum allocation of economic resources. The question

[4] The importance of such *corner* maxima has been stressed in the "linear programming" approach to production theory, developed by J. von Neumann [15], T. C. Koopmans [9], [10], M. K. Wood [22], and G. B. Dantzig [4]. As was pointed out by Professor von Neumann and by the authors of several of the papers in [9] the corner maxima occurring in the formulation of linear programming are closely related to the optimal strategies of zero sum two person games; see J. von Neumann and O. Morgenstern [16, chap. 3]. A generalization of linear programming closely related in spirit to the ideas of the present paper is contained in a paper in this volume by H. W. Kuhn and A. W. Tucker, which also relates corner maxima to the saddle points of a suitably chosen function.

is naturally raised of the continued validity of this theorem when the classical criteria are rejected.

It turns out that, broadly speaking, the optimal properties of the competitive price system remain even when social optima are achieved at corner maxima. In a sense, the role of prices in allocation is more fundamental than the equality of marginal rates of substitution or transformation, to which it is usually subordinated. From a mathematical point of view, the trick is the replacement of methods of differential calculus by the use of elementary theorems in the theory of convex bodies in the development of criteria for an optimum.[5]

These results have a bearing on one aspect of the recent controversy between Professors Hicks and Kuznets over the concept of real national income. Professor Kuznets [11, pp. 3–4] argues essentially that if an individual does not consume anything of a certain commodity, his marginal valuation of the commodity is, in general, less than that of someone who consumes a positive quantity of that commodity. The redistributions which Professor Hicks has made use of in his treatment of real national income are therefore imperfect. Professor Hicks, in his reply, essentially accepts the point [7, pp. 163–164]. But if the argument of the present paper is correct, it is the prices and not the marginal utilities which are in some sense primary. What Professor Kuznets is getting at is the valid statement that the Hicks criterion may lead to the assertion that one situation is both better and worse than another, for example, [18], [17, pp. 2–3]. But this possibility has no special connection with the existence of corner maxima in individual utility maximization or social welfare optimization.

It develops as a byproduct of the main investigation, that the use of convex set methods also enables the criteria for optimality to cover the cases where there are goods which are unwanted or which are positive nuisances. The assumption usually implicit in past studies has been that any individual would prefer to have more of any one commodity, holding all other commodity flows constant, to less. Providing we consider negative and zero as well as positive prices, the theorem on the optimality of the competitive price system is still valid for commodities such that additional quantities are useless or worse.

It should be noted, however, that there is an exceptional case in which an optimal distribution is not achievable through the use of prices. This case seems not to have been noted previously.

In section 3, the problem of optimal economic systems is posed formally, and certain assumptions about the functions entering therein are made. Some mathematical tools are presented in section 4. The necessary and sufficient conditions for the achievement of optimal situations are then developed in sections 5 and 6. The case where it can be assumed that unwanted goods are disposable without cost is discussed in section 7 and related to linear programming in its present form. Diagrammatic representations of the conclusions are presented in section 8. An assessment of the economic meaning and probable validity of the assumptions made in section 3 is presented in section 9. Finally, the relevant portions of the theory of convex sets are quickly sketched in section 10.

[5] A sketch of the relevant parts of the theory of convex bodies is given in the last section of this paper.

## 3. Formulation of the problem of optimal distribution

We suppose that we have $m$ individuals and $n$ commodities in the society. By a *commodity bundle* will be meant a vector of $n$ components expressing the quantity some individual will receive of each of the $n$ commodities, the $i$-th component designating the quantity of the $i$-th commodity.

ASSUMPTION 1. *All quantities consumed must be nonnegative.*

The behavior and desires of each individual are assumed to be expressed by a system of rules specifying for each pair of commodity bundles either a preference for one over the other or indifference between them. This preference pattern is assumed to possess the usual properties of a (weak) complete ordering[6] and also suitable continuity properties. The pattern therefore can be represented by a *utility indicator* $U(x)$ defined for all commodity bundles $x$ in the nonnegative octant of Euclidean space and continuous in its domain of definition, with the property that bundle $x$ is preferred to bundle $y$ if and only if $U(x) > U(y)$; see, for example, Wold [21].

By a *distribution* is meant an assignment of the $n$ commodities among the $m$ individuals. A distribution $X$ is thus an array of $mn$ numbers $X_{ij}$, designating the amount of commodity $i$ to be given to individual $j$. For fixed $j$, the numbers $X_{1j}, \ldots, X_{nj}$ form the commodity bundle to be given to individual $j$; for a given $X$, this bundle will be designated by $X_j$. Implicit in the above notation for utility is the following important assumption:

ASSUMPTION 2. *The desirability of a distribution $X$ to individual $j$ is solely dictated by the desirability to him of the commodity bundle $X_j$.*

This is the assumption that individuals act selfishly. Hence, for any given distribution $X$, the desirabilities to individuals $1, \ldots, m$ are represented by the numbers $U_1(X_1), \ldots, U_m(X_m)$, respectively.

If $x$ and $y$ are commodity bundles and $t$ a real number between 0 and 1, we shall understand by the notation $tx + (1 - t)y$ the commodity bundle whose $i$-th component is $tx_i + (1 - t)y_i$. If $x$ and $y$ are indifferent in the judgment of an individual, then it is usually assumed in economic theory that the *convex combination* $tx + (1 - t)y$ is preferred to either $x$ or $y$ if $t$ is different from 0 to 1.

ASSUMPTION 3. *For all $j$, if $U_j(x) = U_j(y)$, and $0 < t < 1$, then $U_j[tx + (1 - t)y] > U_j(x)$.*

Naturally, the possibilities for a social choice among alternative distributions are limited by the limitations on production. Such limitations can be phrased by saying that the social commodity bundle $\sum_{j=1}^{m} X_j$ must lie in a set $T$, where by the notation $\sum_{j=1}^{m} X_j$ is meant a bundle whose $i$-th component is the sum of the $i$-th components of the bundles $X_1, \ldots, X_m$. The set $T$ will be known as the *transformation set.*

[6] That is, (1) for any two commodity bundles $A$ and $B$, either $A$ is preferred to $B$ or $B$ to $A$ or the two are indifferent; (2) if $A$ is preferred or indifferent to $B$ and $B$ is preferred or indifferent to $C$, then $A$ is preferred or indifferent to $C$ (transitivity).

ASSUMPTION 4. *The transformation set $T$ is nonnull, convex and compact;*[7] *further, if $x$ is a bundle in $T$, $x_i \geqq 0$ for every component of $x$.*

DEFINITION. *A distribution $X^*$ is said to be* optimal *in $T$ if (a) $\sum_{j=1}^{m} X_j^*$ belongs to $T$; and (b) if there is no other distribution $X$ such that $\sum_{j=1}^{m} X_j$ belongs to $T$ and $U_j(X_j) \geqq U_j(X_j^*)$ for all $j$, with the strict inequality holding for at least one $j$.*

It is clear that for any distribution which is nonoptimal, there is another distribution in which everybody is at least as well off and at least one person better off. The optimal distribution of a fixed stock of goods is the special case where $T$ consists of a single point.

## 4. Some preliminary lemmas

An elementary mathematical consequence of the assumptions and other statements from the elementary theory of convex sets will be presented here for later use.

LEMMA 1. *For given $j$ and given number $U$, the set of vectors $x$ for which $U_j(x) \geqq U$ is closed and convex; further, if $x$ and $y$ belong to the set and $0 < t < 1$, then $U_j[tx + (1 - t)y] > U$.*

PROOF. Let $x$ and $y$ both belong to the indicated set. Without loss of generality, we may suppose

$$U \leqq U_j(x) \leqq U_j(y) . \tag{1}$$

Define $f(t) = U_j[tx + (1 - t)y]$; this is a continuous function on the closed interval (0, 1) and so has a minimum there at some point $t_0$. Suppose $0 < t_0 < 1$; then we can obviously choose $t_1$, $t_2$ so that $0 < t_1 < t_0 < t_2 < 1$, $f(t_1) = f(t_2) \geqq f(t_0)$. But from the definition of $f(t)$ and assumption 3, $f(t_0) > f(t_1)$ under these circumstances. Hence, it must be that $t_0 = 0$ or $t_0 = 1$; since $f(0) \geqq f(1)$ by (1), $f(t) > f(1) \geqq U$ for all $t$ such that $0 < t < 1$, so that the set is convex.

That this set is closed follows immediately from the continuity of the function $U_j(x)$.

LEMMA 2. *Let $A$ be any closed convex set and $x^*$ a boundary point of $A$. Then there is a vector $(p_1, \ldots, p_n)$, $p_i \neq 0$ for some $i$, such that for all $x$ in $A$, $\sum_{i=1}^{n} p_i x_i \leqq \sum_{i=1}^{n} p_i x_i^*$.*

LEMMA 3. *Let $A$ and $B$ be closed convex sets such that $A$ has at least two points and no internal point of $A$ is also a point of $B$. Then there is a vector $p = (p_1, \ldots, p_n)$, not all of whose components are zero, and a number $c$ such that,*

$$\sum_{i=1}^{n} p_i x_i \geqq c \text{ for all } x \text{ in } A\ ,$$

$$\sum_{i=1}^{n} p_i x_i \leqq c \text{ for all } x \text{ in } B\ .$$

[7] A set is said to be nonnull if it contains at least one element. It is said to be convex if for any two bundles $x$ and $y$ in the set and any $t$ such that $0 \leqq t \leqq 1$, the bundle $tx + (1 - t)y$ also belongs to the set. Finally, a compact set is a bounded set such that no sequence of points in the set converges to a point outside the set.

These lemmas and the definitions of internal and external points are discussed in section 10 below.

### 5. The case of a single individual

If $m = 1$, the distribution $X$ reduces to a single vector or commodity bundle $x$. Then $x^*$ is optimal in $T$ if $U(x^*) \geqq U(x)$ for all $x$ in $T$, that is, if $x^*$ maximizes $U(x)$ for $x$ in $T$. [Here, the subscript 1 on $U_1(x)$ has been omitted.] This case is of some interest because in certain respects the general case can be reduced to it.

THEOREM 1. *There is a unique optimal point $x^*$ in any $T$.*

PROOF. Since $U(x)$ is continuous and $T$ is nonnull and compact, there is at least one maximum of $U(x)$ and therefore at least one optimal point. Suppose $x^*$ and $y^*$ are both optimal; then $U(x^*) \geqq U(y^*)$, $U(y^*) \geqq U(x^*)$ and therefore $U(x^*) = U(y^*)$. Let $z^* = \frac{1}{2}x^* + \frac{1}{2}y^*$. Since $T$ is convex and $x^*$ and $y^*$ both belong to $T$, $z^*$ belongs to $T$. By assumption 3, $U(z^*) > U(x^*)$, contrary to the assumption that $x^*$ and $y^*$ are both optimal. Hence, the optimal point is unique.

DEFINITION. *The bundle $x^*$ is said to be a* point of bliss *if $U(x^*) \geqq U(x)$ for all $x$.*

Clearly, if the point of bliss belongs to the transformation set $T$, it is optimal. Usually, an optimal point is not a point of bliss; that is, the optimal point for a given set of production restraints is not the best point the individual would wish for were he unrestrained.

LEMMA 4. *If $x^*$ is optimal in $T$ but not a point of bliss, then there is a vector $p$ such that* (a) $\sum_{i=1}^{n} p_i x_i \geqq \sum_{i=1}^{n} p_i x_i^*$ *for all $x$ such that $U(x) \geqq U(x^*)$;* (b) $\sum_{i=1}^{n} p_i x_i \leqq \sum_{i=1}^{n} p_i x_i^*$ *for all $x$ in $T$; $p_i \neq 0$ for at least one $i$.*

PROOF. Let $V$ be the set of vectors $x$ such that $U(x) \geqq U(x^*)$. From theorem 1 and the definition of an optimal point, $V$ and $T$ have only the point $x^*$ in common. Suppose $x^*$ were an internal point of $V$. Then, by definition, in some linear subspace of the commodity $n$-space, $x^*$ would be surrounded by a neighborhood of points all in $V$, and therefore there would exist two points $x$ and $y$ in $V$ such that $x^* = tx + (1 - t)y$, where $0 < t < 1$. By lemma 1, $U(x^*) > U(x^*)$, a contradiction. Therefore, $x^*$ is an external point of $V$, and hence no internal point of $V$ belongs to $T$. Since $x^*$ is not a point of bliss, $V$ contains at least one point besides $x^*$. By lemma 1 and assumption 4, $V$ and $T$ are closed convex sets. Lemma 4 then follows from lemma 3, since $x^*$ belongs to both $V$ and $T$, so that $\sum_{i=1}^{n} p_i x_i^* = c$.

LEMMA 5. *For a given $x^*$, let $p$ be such that $\sum_{i=1}^{n} p_i x_i \geqq \sum_{i=1}^{n} p_i x_i^*$ for all $x$ for which $U(x) \geqq U(x^*)$ and such that $p_k x_k^* \neq 0$ for some $k$. Then $x^*$ uniquely maximizes $U(x)$ subject to the condition that $\sum_{i=1}^{n} p_i x_1 \leqq \sum_{i=1}^{n} p_i x_1^*$.*

PROOF. Suppose the conclusion is false. Then, for some $x \neq x^*$,

$$(1) \qquad \sum_{i=1}^{n} p_i x_i \leqq \sum_{i=1}^{n} p_i x_i^*,$$

$$U(x) \geqq U(x^*). \tag{2}$$

From the hypothesis, (2) implies that $\sum_{i=1}^{n} p_i x_i \geqq \sum_{i=1}^{n} p_i x_i^*$, so that, from (1), $\sum_{i=1}^{n} p_i x_i = \sum_{i=1}^{n} p_i x_i^*$. Let $y = \frac{1}{2}x + \frac{1}{2}x^*$. Then,

$$\sum_{i=1}^{n} p_i y_i = \sum_{i=1}^{n} p_i x_i^*. \tag{3}$$

By hypothesis, $x_k^* > 0$; therefore, $y_k > 0$. Define the vector $z$, as follows: $z_i = y_i$ for $i \neq k$, $z_k = y_k + \epsilon$. For all $\epsilon$ sufficiently close to 0, $z_i \geqq 0$ for all $i$. From (2) and lemma 1, $U(y) > U(x^*)$; therefore, for all $\epsilon$ sufficiently close to 0,

$$U(z) \geqq U(x^*). \tag{4}$$

By hypothesis, $p_k \neq 0$. Choose $\epsilon$ sufficiently close to 0 to satisfy (4) and of a sign opposite to $p_k$. Then, by (3)

$$\sum_{i=1}^{n} p_i z_i = \sum_{i=1}^{n} p_i y_i + \epsilon p_k < \sum_{i=1}^{n} p_i x_i^*. \tag{5}$$

But the existence of a vector $z$ with properties (4) and (5) contradicts the hypotheses of the lemma.

DEFINITION. *The "price" vector $p$ is said to equate supply and demand at $x^*$ if* (a) *$x^*$ uniquely maximizes $U(x)$ subject to the condition $\sum_{i=1}^{n} p_i x_i \leqq \sum_{i=1}^{n} p_i x_i^*$,* (b) *$x^*$ maximizes $\sum_{i=1}^{n} p_i x_i$ subject to the condition that $x$ belongs to $T$.*

If we interpret the vector $p$ as a set of prices, one for each commodity, then clearly (a) states that $x^*$ constitutes the quantities demanded of each commodity at the given price levels, provided sufficient income is supplied to purchase $x^*$ but no more, under conditions of perfect competition, while (b) states that $x^*$ will also be the quantities supplied under the assumption of profit maximization under competitive conditions. It is not implied that the price vector is unique, nor, of course, need there exist a price vector with the above properties for every $x$ in $T$. Indeed, such price vectors will only exist for optimal points, as we shall see.

THEOREM 2. *If there is a vector $p$ which equates supply and demand at $x^*$, then $x^*$ is an optimal point.*

PROOF. Suppose $x^*$ is not optimal. Then for some $y$ in $T$, $U(y) > U(x^*)$. Since $x^*$ uniquely maximizes $U(x)$ subject to the conditions $\sum_{i=1}^{n} p_i x_i \leqq \sum_{i=1}^{n} p_i x_i^*$, it must be that $\sum_{i=1}^{n} p_i y_i > \sum_{i=1}^{n} p_i x_i^*$. But this contradicts the hypothesis that $x^*$ maximizes the linear function $\sum_{i=1}^{n} p_i x_i$ for $x$ in $T$.

Theorem 2 states that if a set of prices can be found which equate supply and demand, then the resulting situation is optimal. The triviality of the reasoning leading to this sufficient condition for optimality is in contrast with the more complicated proof leading to the converse theorem, which in fact is not valid in complete generality. The precise statement follows.

THEOREM 3. *For any optimal point $x^*$, there is a vector $p$ with at least one nonzero component with the following properties:* (a) $\sum_{i=1}^{n} p_i x_i \geqq \sum_{i=1}^{n} p_i x_i^*$ *for all $x$ such that* $U(x) \geqq U(x^*)$; (b) *there is a commodity bundle $y^*$, where $y_i^* \geqq x_i^*$ for all $i$, which maximizes the profit function $\sum_{i=1}^{n} p_i x_i$ subject to the condition that $x$ be in $T$;* (c) *if $x^*$ is not a point of bliss, then $y_i^* = x_i^*$ for all $i$ in* (b); (d) *if either $p_k x_k^* \neq 0$ for some $k$ or $x^*$ is a point of bliss, then $x^*$ uniquely maximizes $U(x)$ subject to the condition that* $\sum_{i=1}^{n} p_i x_i \leqq \sum_{i=1}^{n} p_i x_i^*$.

PROOF. If $x^*$ is not a point of bliss, then statements (a–c) are made in lemma 4. Suppose $x^*$ is a point of bliss which is optimal in $T$. Let $\tau$ be the least upper bound of values of $t$ for which $tx^*$ belongs to $T$. Since $T$ is a closed set, $\tau x^*$ belongs to $T$; also, clearly, $\tau \geqq 1$. Let $y^* = \tau x^*$. Since $x_i^* \geqq 0$ for each $i$, by assumption 4, $y_i^* \geqq x_i^*$ for each $i$. For $t > \tau$, $tx^*$ does not belong to $T$. Therefore every neighborhood of $y^*$ contains points not in $T$, so that $y^*$ is a boundary point of $T$. By lemma 2, there is a price vector $p$ such that $\sum_{i=1}^{n} p_i x_i \leqq \sum_{i=1}^{n} p_i y_i^*$ for all $x$ in $T$, establishing (b). Now suppose for some $x \neq x^*$, $U(x) \geqq U(x^*)$. Then if $y = \frac{1}{2}x + \frac{1}{2}x^*$, $U(y) > U(x^*)$ by lemma 1, which contradicts the assumption that $x^*$ is a point of bliss. Therefore, the set of points for which $U(x) \geqq U(x^*)$ contains just the point of bliss $x^*$, so that (a) is trivial. Part (c) is irrelevant if $x^*$ is a point of bliss. Finally, the previous argument shows that $U(x^*) > U(x)$ for all $x \neq x^*$, so that (d) follows trivially in case $x^*$ is a point of bliss.

If $x^*$ is not a point of bliss, then (d) follows from (a) by lemma 5.

Parts (a–c), particularly, characterize optimal points. For a point of bliss, it is possible to set prices so that at least as much will be produced, under the assumption of profit maximization, of each commodity as is used at the point of bliss. Then, if enough income is given the consumer so that he can purchase the quantities at the point of bliss evaluated at the prices just set, he will in fact purchase them. The more interesting case is that in which the point of bliss, if any, is not contained within the available production possibilities. Then prices can be set so that simultaneously the optimal point will maximize profits to producers and minimize the cost of achieving the associated (optimal) utility level to consumers. Apart from an exceptional case, that is, when the optimal bundle contains positive quantities only of those goods with zero price, it is also true this minimum cost property is equivalent to the proposition that the individual maximizing his utility subject to the constraint that his expenditures at given prices not exceed a quantity sufficient to purchase the optimal bundle, will in fact choose that bundle.

Theorem 3 says, in effect, that an optimal point can be achieved by suitable choice of prices under a competitive system. By itself, this hardly distinguishes the price system from others. For example, obviously any point in $T$, and in particular the optimal point, can be achieved by rationing. Theorem 2 adds, however, the important quality that once a bundle has been chosen by means of the price system, we know that it is optimal, a quality not shared in as direct a way, at least, by direct controls. Of course, the validity of these theorems is dependent upon the validity of assumptions 1–4.

It is to be noted that no assumptions or conclusions as to the signs of the prices were made or drawn. At this stage, the presence of unwanted commodities in consumption or of goods whose production is made easier rather than harder by the employment of resources to produce other goods is not excluded.

## 6. The case of many individuals

We will now return to the general case, where $m$, the number of individuals may be more than 1.

DEFINITION. *For a given optimal distribution $X^*$ and a given individual $k$, let $T_k$ be the set of all vectors $x$ for which there exists a distribution $X$ such that* (a) $x = X_k$; (b) $U_j(X_j) \geqq U_j(X_j^*)$ *for all* $j \neq k$; (c) $\sum_{j=1}^{m} X_j$ *belongs to* $T$.

If we start from a given optimal distribution $X^*$, then $T_k$ is the set of all possible bundles which individual $k$ can secure for himself if he is given complete charge of the distribution of goods subject only to the conditions that the distribution be compatible with the production possibilities and at the same time not bring any other individual to a position in which the latter is worse off than he would be under the given optimal distribution. In order that $X^*$ be in fact optimal, it is clear that individual $k$ must find that the best vector $x$ in $T_k$ with respect to his utility function be $x_k^*$, for otherwise there would be another distribution compatible with the production possibilities in which no individual other than $k$ is worse off than he would be at $X^*$, while individual $k$ would have a way of being better off. For an optimal distribution, then, $X_k^*$ must maximize $U_k(x)$ subject to the condition that $x$ belongs to $T_k$; this must hold for each $k$. The set $T_k$ then plays, in effect, the role of the production possibilities open to individual $k$, and the results of the previous one individual case can be used here. Incidentally, there is one difference between the sets $T_k$ and the set $T$; the former, but not the latter, may (and usually will) include bundles with negative components. Their inclusion in $T_k$ is in fact essential to the proof following. However, their exclusion from $T$ was not in fact made use of in the proofs of section 5, so that the theorems there proved are still applicable where relevant.

LEMMA 6. *The sets $T_k$ are nonnull, closed and convex.*

PROOF. By definition, the bundle $X_k^*$ belongs to $T_k$, so that it is nonnull.

Let $x^n$ be a sequence of points in $T_k$ which converge to a given point or bundle $x$. For each $x^n$, then, by definition of $T_k$, there is a distribution $X^n$ such that

(1) $$\sum_{j=1}^{m} X_j^n \text{ belongs to } T ,$$

(2) $$U_j (X_j^n) \geqq U_j (X_j^*) \text{ for all } j \neq k ,$$

(3) $$x^n = X_k^n \text{ for each } n\,.$$

Since by assumption 1, $X_{ij}^n \geqq 0$ for each $n$, $i$, and $j \neq k$, it follows that

(4) $$0 \leqq X_{ij}^n \leqq \sum_{j=1}^{m} X_{ij}^n - x_i^n \text{ for } j \neq k\,.$$

Since $T$ is a bounded set by assumption 4 and $x_i^n$ a bounded sequence, it follows from (4) and (2) that the sequence $X_{ij}^n$ is bounded for each fixed $i$ and $j \neq k$. Then we can choose a subsequence of the integers, $n_1, \ldots, n_r, \ldots$, such that each of the sequences $X_{ij}^{n_r}$ converge; by (3), the sequences $X_{ik}^{n_r}$ also converge. Let $X_{ij} = \lim_{r\to\infty} X_{ij}^{n_r}$. From (3), it follows that, for each $i$,

(5) $$X_{ik} = x_i\,, \quad \text{or} \quad x = X_k\,.$$

From (2) and the continuity of $U_j(x)$,

(6) $$U_j(X_j) \geqq U_j(X_j^*) \text{ for all } j \neq k\,.$$

From (1), $\sum_{j=1}^{m} X_j^{n_r}$ belongs to $T$ for each $r$. Since $T$ is a closed set, it follows that in the limit,

(7) $$\sum_{j=1}^{m} X_j \text{ belongs to } T\,.$$

From (5–7), $x$ belongs to $T_k$, so that $T_k$ is closed.

Now let $x$ and $y$ be any two elements of $T_k$. Then, there exist distributions $X$ and $Y$ such that,

(8) $$x = X_k\,, \quad y = Y_k$$

(9) $$U_j(X_j) \geqq U_j(X_j^*)\,, \quad U_j(Y_j) \geqq U_j(X_j^*) \text{ for all } j \neq k\,,$$

(10) $$\sum_{j=1}^{m} X_j \text{ and } \sum_{j=1}^{m} Y_j \text{ both belong to } T.$$

Let $z = tx + (1-t)y$. Define the distribution $Z$ so that $Z_j = tX_j + (1-t)Y_j$. Then, by (8),

(11) $$z = Z_k\,.$$

Assume $0 \leqq t \leqq 1$. By lemma 1, the set of all bundles for which $U_j(x) \geqq U_j(X_j^*)$ is convex. From (9).

(12) $$U_j(Z_j) \geqq U(X_j^*) \text{ for all } j \neq k\,.$$

Finally, since $T$ is convex by assumption 4, it follows from (10) that

(13) $$\sum_{j=1}^{m} Z_j \text{ belongs to } T\,.$$

From (11–13), $z$ belong to $T_k$, so that $T_k$ is convex.

Lemma 6 formally establishes that $T_k$ has all the relevant properties of $T$.

LEMMA 7. *If $X^*$ is optimal in $T$, then $X_j^*$ is optimal in $T_j$ for each $j$.*

PROOF. Suppose not. Then there is some individual $k$ and some bundle $x$ in $T_k$ such that $U_k(x) > U_k(X_k^*)$, and hence a distribution $X$ such that,

$$U_k(X_k) > U_k(X_k^*)\,, \tag{1}$$

$$U_j(X_j) \geqq U_j(X_j^*) \text{ for all } j \neq k\,, \tag{2}$$

$$\sum_{j=1}^{m} X_j \text{ belongs to } T\,. \tag{3}$$

(1–3) contradict the statement that $X^*$ is optimal in $T$.

Lemma 7 formally reduces the optimality problem for the society to that for single individuals. Without further argument, it could be deduced that there is a set of prices for each individual such that utility maximization under a budget constraint would lead him to choose the given optimal point [subject to the minor exception noted in theorem 3(d)]. However, a stronger statement can be made; the same set of prices will do for all individuals.

THEOREM 4. *If $X^*$ is optimal in $T$, there is a vector $p$, for which $p_i \neq 0$ for some $i$, with the following properties:* (a) *for each $j$, $\sum_{i=1}^{n} p_i x_i \geqq \sum_{i=1}^{n} p_i X_{ij}^*$ for all $x$ such that $U_j(x) \geqq U_j(X_j^*)$*; (b) *there is a vector $y^*$ such that $\sum_{j=1}^{m} X_{ij}^* \leqq y_i^*$ for all commodities $i$ and $\sum_{i=1}^{n} p_i x_i \leqq \sum_{i=1}^{n} p_i y_i^*$ for all $x$ in $T$;* (c) *if, for some $j$, $X_j^*$ is not a point of bliss, then $y^* = \sum_{j=1}^{m} X_j^*$ in* (b); (d) *for any individual $j$ for whom either $p_k X_{kj}^* \neq 0$ for some $k$ or $X_j^*$ is a point of bliss, $X_j^*$ uniquely maximizes $U_j(x)$ subject to the budget condition $\sum_{i=1}^{n} p_i x_i \leqq \sum_{i=1}^{n} p_i X_{ij}^*$.*

PROOF. Let $V_j$ be the set of all bundles $x$ for which $U_j(x) \geqq U_j(X_j^*)$. We will consider two cases in proving the theorem.

*Case* 1. For some $j$, $X_j^*$ is not a point of bliss. Let $k$ be the value of $j$ in question. By lemma 7, $X_k^*$ is optimal in $T_k$. By lemma 4,

$$\sum_{i=1}^{n} p_i x_i \leqq \sum_{i=1}^{n} p_i X_{ik}^* \text{ for all } x \text{ in } T_k\,, \tag{1}$$

$$\sum_{i=1}^{n} p_i x_i \geqq \sum_{i=1}^{n} p_i X_{ik}^* \text{ for all } x \text{ in } V_k\,. \tag{2}$$

Choose an individual $q$ distinct from $k$. Let $x$ belong to $V_q$, which is nonnull since it contains $X_q^*$. Then $U_q(x) \geqq U_q(X_q^*)$. Define the distribution $X$ as follows: $X_k = X_k^* + X_q^* - x$, $X_q = x$, $X_j = X_j^*$ for $j$ distinct from both $q$ and $k$. Then, it is easy to see that $X_k$ belongs to $T_k$; by (1),

$$\sum_{i=1}^{n} p_i (X_{ik}^* + X_{iq}^* - x_i) \leqq \sum_{i=1}^{n} p_i X_{ik}^*\,,$$

or

(3) $$\sum_{i=1}^{n} p_i x_i \geqq \sum_{i=1}^{n} p_i X_{iq}^* .$$

(3) holds for any $x$ in $V_q$, where $q$ is any individual distinct from $k$. From (2), then,

(4) $$\sum_{i=1}^{n} p_i x_i \geqq \sum_{i=1}^{n} p_i X_{ij}^* \text{ for all } j \text{ and all } x \text{ in } V_j ,$$

which is (a).

Now let $x$ be any element of $T$. Define the distribution $Y$ as follows: $Y_j = X_j^*$ for all $j \neq k$, $Y_k = x - \sum_{j \neq k} X_j^*$. Clearly $Y_k$ belongs to $T_k$. By (1),

$$\sum_{i=1}^{n} p_i \left( x_i - \sum_{j \neq k} X_{ij}^* \right) \leqq \sum_{i=1}^{n} p_i (X_{ik}^*) ,$$

or

(5) $$\sum_{i=1}^{n} p_i x_i \leqq \sum_{i=1}^{n} p_i \left( \sum_{j=1}^{n} X_{ij}^* \right) \text{ for all } x \text{ in } T ,$$

which establishes (b) and (c). Finally, (d) follows from (a) by lemma 5.

*Case* 2. $X_j^*$ is a point of bliss for all $j$. Let $\tau$ be the least upper bound of the values of $t$ such that $t\left(\sum_{j=1}^{m} X_j^*\right)$ belongs to $T$, $y^* = \tau \left(\sum_{j=1}^{m} X_j^*\right)$. Then $\tau \geqq 1$. As in theorem 3, $y^*$ is a boundary point of $T$, so that, by lemma 2, we can choose $p$ so that

(6) $$\sum_{i=1}^{n} p_i x_i \leqq \sum_{i=1}^{n} p_i y_i^* \text{ for all } x \text{ in } T .$$

Clearly,

(7) $$\sum_{j=1}^{m} X_{ij}^* \leqq y_i^* \text{ for all } i ,$$

so that (b) is valid. As shown in the proof of theorem 3, (a) and (d) are trivial in this case.

DEFINITION. *The vector $p$ is said to equate supply and demand for the distribution $X^*$ if* (a) *for each $j$, $X_j^*$ uniquely maximizes $U_j(x)$ under the constraint* $\sum_{i=1}^{n} p_i x_i \leqq \sum_{i=1}^{n} p_i X_{ij}^*$; (b) *for all $x$ in $T$,* $\sum_{i=1}^{n} p_i x_i \leqq \sum_{i=1}^{n} p_i \left( \sum_{j=1}^{m} X_{ij}^* \right)$.

THEOREM 5. *If there is a vector $p$ which equates supply and demand for $X^*$, then $X^*$ is optimal.*

PROOF. Suppose not. Then there is a distribution $X$ such that

(1) $$U_k (X_k) > U_k (X_k^*) \text{ for some } k ,$$

(2) $$U_j (X_j) \geqq U_j (X_j^*) \text{ for all } j \neq k ,$$

(3) $$\sum_{j=1}^{m} X_j \text{ belongs to } T .$$

From (1) and condition (a) of the preceding definition, it follows that

$$\sum_{i=1}^{n} p_i X_{ik} > \sum_{i=1}^{n} p_i X_{ik}^* . \tag{4}$$

From (2) and condition (a), for each $j \neq k$, either $\sum_{i=1}^{n} p_i X_{ij} > \sum_{i=1}^{n} p_i X_{ij}^*$, or $X_j = X_j^*$, so that,

$$\sum_{i=1}^{n} p_i X_{ij} \geqq \sum_{i=1}^{n} p_i X_{ij}^* \text{ for all } j \neq k . \tag{5}$$

From (4) and (5),

$$\sum_{i=1}^{n} p_i \left( \sum_{j=1}^{m} X_{ij} \right) > \sum_{i=1}^{n} p_i \left( \sum_{j=1}^{m} X_{ij}^* \right). \tag{6}$$

But (3) and condition (b) imply that (6) is false. Hence, the theorem is true.

Theorems 4 and 5 together define the role of the price system in the same way as for a single individual in section 5. Disregarding the existence of points of bliss (for all too good empirical reasons) leaves only the case where $p_i X_{ij}^* = 0$ for all $i$ for some individual $j$ as an exceptional case of an optimal point unreachable by the price system. In this case, the optimal situation requires that individual $j$ consume only free goods. The individuals for whom the conclusion of theorem 4(d) is valid have the right to consume as much of the free goods as they wish in maximizing their utility under the budget constraint. Hence, it must be that at the bundles to which they are entitled under the optimal situation they are saturated with respect to the free goods; either an increase or a decrease in the quantity of any of the free goods, holding the quantities of other goods constant, would decrease satisfaction. The reason that the price system fails is that the prices of the goods consumed by individual $j$ must be zero to permit other individuals to become saturated with those goods, but at the same time there is no restraint compelling individual $j$ to stick to the quantity of free goods allotted to him under the optimal conditions, since, of course, he could, under the price system, consume as much of these as he pleases. Only by coincidence would he also be saturated with those free goods at a zero level of other goods.

## 7. The case of free disposal

It is common in discussions of production to make implicitly or explicitly the following

ASSUMPTION 5. *If $x$ belongs to $T$ and $y$ is a vector such that $0 \leqq y_i \leqq x_i$ for every commodity $y$, then $y$ belongs to $T$.*

The argument generally runs that, if necessary, one could always produce $y$ by producing $x$ and then discarding the quantities $x_i - y_i$ of the commodities $i$. This amounts to assuming that there is a method of disposal of surplus products which is costless to producers. Under these conditions, it turns out that we can confine ourselves to nonnegative prices.

For the following lemma, let us define, for a given integer $q \leqq n$, the *projection* of an $n$-vector $x$ to be the $q$-vector whose components are $x_1, \ldots, x_q$. The *projec-*

*tion* of a set $T$ will be the set of all points in $q$-dimensional space which are projections of points of the set $T$.

LEMMA 8. *Let $x^*$ belong to the transformation set $T$, and suppose $x_i^* > 0$ for $i = 1, \ldots, q$, $x_i^* = 0$ for $i = q + 1, \ldots, n$. Let $x^{*\prime}$ and $T'$ be the projections of $x^*$ and $T$, respectively. Then, if $x^{*\prime}$ is a boundary point of $T'$ in q-dimensional space and if assumption 5 holds, there is a vector $p$ such that $x^*$ maximizes $\sum_{i=1}^{n} p_i x_i$ for $x$ in $T$, and such that $p_i \geqq 0$ for all $i$, $p_i > 0$ for some $i$.*

PROOF. Clearly, $T'$ is closed, convex, and nonnull. By lemma 2, there exists a vector $(p_1, \ldots, p_q)$ such that

$$(1) \qquad \sum_{i=1}^{q} p_i x_i \leqq \sum_{i=1}^{q} p_i x_i^* \text{ for all } x \text{ in } T',\ p_i \neq 0 \text{ for some } i,$$

since $x^{*\prime}$ is a boundary point of $T'$. For a given $r$ such that $1 \leqq r \leqq q$, define the $n$-vector $y$ in $T$ so that $y_i = x_i^*$ for $i \neq r$, $y_r = 0$. By assumption 5, $y$ belongs to $T$. Let $y'$ be the projection of $y$; by (1), it follows that $p_r x_r^* \geqq 0$. By construction $x_r^* > 0$, so that $p_r \geqq 0$. This holds for all $r$ between 1 and $q$, inclusive. From (1), then, $p_i > 0$ for at least one value of $i$. Define $p_i = 0$ for $i = q + 1, \ldots, n$. If $x$ belongs to $T$ and $x'$ is the projection of $x$, then $\sum_{i=1}^{n} p_i x_i = \sum_{i=1}^{q} p_i x_i$. The lemma then follows from (1).

LEMMA 9. *Let $X^*$ be an optimal distribution in which, for some individual $k$, $X_k^*$ is not a point of bliss. If $p$ satisfies the conclusions of theorem 4 and if assumption 5 holds, then $p_i > 0$ for at least one value of $i$.*

PROOF. Suppose $p_i \leqq 0$ for all $i$. Then $\sum_{i=1}^{n} p_i X_{ij}^* \leqq 0$ for all $j$, and $\sum_{i=1}^{n} p_i \left( \sum_{j=1}^{m} X_{ij}^* \right) \leqq 0$. But by assumption 5, the point $(0, \ldots, 0)$ belongs to $T$; since the point $\sum_{j=1}^{m} X_j^*$ maximizes $\sum_{i=1}^{n} p_i x_i$ for $x$ in $T$, $\sum_{i=1}^{n} p_i \left( \sum_{j=1}^{m} X_{ij}^* \right) \geqq 0$, and therefore $\sum_{i=1}^{n} p_i \left( \sum_{j=1}^{m} X_{ij}^* \right) = 0$, so that for each $j$, $\sum_{i=1}^{n} p_i X_{ij}^* = 0$. Since $p$ satisfies theorem 4(a),

$$(1) \qquad \sum_{i=1}^{n} p_i x_i \geqq 0 \text{ for all } x \text{ in } V_k .$$

By hypothesis, there is a vector $y$ for which

$$(2) \qquad U_k(y) > U_k(X_k^*) .$$

Since $p_i \neq 0$ for some $i$, there is an $r$ such that $p_r < 0$. Let $z$ be any vector for which $z_r > 0$; let $w = ty + (1 - t)z$. For all $t < 1$, $w_r > 0$; but for $t$ sufficiently close to 1, it follows by continuity from (2) that $U_k(w) \geqq U_k(X_k^*)$, so that

$\sum_{i=1}^{n} p_i w_i \geqq 0$ by (1). But since $p_i \leqq 0$ for all $i$, $p_r < 0$, $\sum_{i=1}^{n} p_i w_i < 0$, a contradiction.

THEOREM 6. *If $X^*$ is an optimal distribution and if assumption 5 is valid, then there is a set of prices $p$ satisfying the conclusions of theorem 4 for which $p_i \geqq 0$ for all $i$, $p_i > 0$ for at least one $i$.*

PROOF. First suppose that $X_j^*$ is a point of bliss for all $j$. Let $y^*$ be the point in $T$ which enters into theorem 4(b). By renumbering the commodities, it may be supposed that $y_i^* > 0$ for $i = 1, \ldots q$, $y_i^* = 0$ for $i = q + 1, \ldots, n$. Let $y^{*\prime}$ and $T'$ be the projections of $y^*$ and $T$ respectively. Suppose that for some $t > 1$, the point $ty^{*\prime}$ belongs to $T'$. Then there is a point $z$ in $T$ such that $z_i = ty_i^*$ for $i = 1, \ldots, q$, $z_i \geqq 0$ for $i = q + 1, \ldots, n$. By construction, the point $ty^*$ has the same first $q$ coordinates as $z$, while the last $n - q$ are zero, so that $ty^*$ belongs to $T$ by assumption 5 contrary to the construction of $y^*$. Hence, for all $t > 1$, $ty^{*\prime}$ does not belong to $T'$, and $y^{*\prime}$ is a boundary point of $T'$. Conclusion (b) of theorem 4 then follows from lemma 8. Conclusions (a) and (d) follow trivially in this case, as before.

Now suppose that for some $j$, $X_j^*$ is not a point of bliss for individual $j$. By lemma 9, there is a vector $p'$ satisfying the conclusions of theorem 4 with $p_i' > 0$ for at least one value of $i$. It will be shown that any negative price in this vector can be replaced by a zero price without changing the conclusions of this theorem. Suppose $p_r' < 0$. Define $p$ so that $p_i = p_i'$ for $i \neq r$, $p_r = 0$; it will be shown that $p$ has the same properties as $p'$. By theorem 4(b) and (c),

$$\sum_{i=1}^{n} p_i' x_i \leqq \sum_{i=1}^{n} p_i' \Big( \sum_{j=1}^{m} X_{ij}^* \Big) \text{ for all } x \text{ in } T. \tag{1}$$

In (1), first let $x$ be such that $x_i = \sum_{j=1}^{m} X_{ij}^*$ for $i \neq r$, $x_r = 0$. By assumption 5, $x$ belongs to $T$. Then $p_r' \Big( \sum_{j=1}^{m} X_{rj}^* \Big) \geqq 0$; since $p_r' < 0$, $\sum_{j=1}^{m} X_{rj}^* = 0$, so that

$$X_{rj}^* = 0 \text{ for all } j, \tag{2}$$

$$\sum_{i=1}^{n} p_i' \Big( \sum_{j=1}^{m} X_{ij}^* \Big) = \sum_{i=1}^{n} p_i \Big( \sum_{j=1}^{m} X_{ij}^* \Big). \tag{3}$$

Now, for any $x$ in $T$, define $y$ so that $y_i = x_i$ for $i \neq r$, $y_r = 0$. By assumption 5, $y$ belongs to $T$; by (1),

$$\sum_{i=1}^{n} p_i' y_i \leqq \sum_{i=1}^{n} p_i' \Big( \sum_{j=1}^{m} X_{ij}^* \Big). \tag{4}$$

But $\sum_{i=1}^{n} p_i' y_i = \sum_{i=1}^{n} p_i x_i$; from (3) and (4),

(5) $$\sum_{i=1}^{n} p_i x_i \leqq \sum_{i=1}^{n} p_i \left( \sum_{j=1}^{m} X^*_{ij} \right) \text{ for all } x \text{ in } T ,$$

which is conclusion (c).

By theorem 4(a),

(6) $$\sum_{i=1}^{n} p'_i x_i \geqq \sum_{i=1}^{n} p'_i X^*_{ij} \text{ for all } j \text{ and all } x \text{ in } V_j .$$

From (2),

(7) $$\sum_{i=1}^{n} p'_i X^*_{ij} = \sum_{i=1}^{n} p_i X^*_{ij} .$$

Since $x_r \geqq 0$ and $p'_r < p_r$, $\sum_{i=1}^{n} p_i x_i \geqq \sum_{i=1}^{n} p'_i x_i$. From (6) and (7), then, $\sum_{i=1}^{n} p_i x_i \geqq \sum_{i=1}^{n} p_i X^*_{ij}$ for all $j$ and all $x$ such that $U_j(x) \geqq U_j(X^*_j)$, which is conclusion (a). As in the proof of theorem 4, part (d) follows from (a) by lemma 5.

If the new vector $p$ still has negative components, they may be removed by the above process. Since $p'$ had at least one positive component, which is undisturbed by subsequent operations, each of the successive price vectors has at least one nonzero component.

Definition. *A bundle $x^*$ of goods in the transformation set $T$ will be said to be* efficient *if for some set of utility functions $U_j(x)$ $(j = 1, \ldots, m)$, there is a distribution $X^*$ such that* (a) *$X^*$ is optimal in $T$;* (b) $\sum_{j=1}^{m} X^*_j = x^*$; *and* (c) *for some $j$, $X^*_j$ is not a point of bliss for individual $j$.*

This definition of efficient points is not precisely equivalent to that used in linear programming by T. C. Koopmans and others [4], [9], [10], [22] but conveys the same general meaning. The efficient points of $T$ are just those which could be used in some optimal distribution. Clause (c) is inserted to exclude trivialities. Without it, by suitable choice of the functions $U_j(x)$, every point of $T$ would be efficient. Of course, when every individual is at his point of maximum absolute satisfaction, the concept of economic efficiency becomes meaningless.

Theorem 7. *Under assumption* 5, *the following are each a necessary and sufficient condition for $x^*$ to be efficient:* (a) $\sum_{i=1}^{n} p_i x_i \leqq \sum_{i=1}^{n} p_i x^*_i$ *for all $x$ in $T$ and for some $p$ for which $p_i \geqq 0$ for all $i$, $p_i > 0$ for some $i$;* (b) *there is no $y$ in $T$ for which $x^*_i < y_i$ for all $i$.*

Proof. First, it will be shown that (a) is a necessary and sufficient condition for $x^*$ to be optimal. The necessity has already been shown in theorem 6. For the sufficiency, let $X^*$ be a distribution which gives $x^*/m$ to each individual. For each individual $j$, let $U_j(x) = -\sum_{i=1}^{n} (x_i - x^*_i/m - p_i)^2$, defined only for those values of $x$ for which $x_i \geqq 0$ for all $i$. This utility function has an absolute maximum at the point $x^*/m + p$, and its indifference surfaces are concentric spheres about that point. From the geometric picture, or algebraically with the aid of Schwarz's inequality, it is easy to see that assumption 3 is verified. By simple algebraic manipulation, it can be seen that $x^*/m$ uniquely maximizes $U_j(x)$ subject to the condition $\sum_{i=1}^{n} p_i x_i \leqq \sum_{i=1}^{n} p_i(x_i^*/m)$. From the statement of condition (a), and theorem 5, it follows that $X^*$ is an optimal distribution for the given set of utility functions and therefore that $x^*$ is an efficient point.

To show that (b) is also a necessary and sufficient condition for $x^*$ to be an efficient point, it will be shown that (b) is equivalent to (a). If there were a $y$ in $T$ such that $y_i > x_i^*$ for all $i$, then, if $p_i \geqq 0$ for all $i$, $p_i > 0$ for some $i$, $\sum_{i=1}^{n} p_i x_i^* < \sum_{i=1}^{n} p_i y_i$, contrary to (a). Hence, (a) implies (b).

For the converse, renumber the commodities so that $x_i^* > 0$ for $i = 1, \ldots, q$, $x_i^* = 0$ for $i = q + 1, \ldots, n$.

*Case* 1. For some $y$ in $T$, $x_i^* < y_i$ for $i = 1, \ldots, q$. Suppose that for each $i = q + 1, \ldots, n$, there is a vector $x^{(i)}$ in $T$ such that $x_i^{(i)} > 0$. Let $z = \sum_{i=q+1}^{n} t_i x^{(i)} + t_0 y$, where $t_i = 0$ $(i = 0, \ldots, n)$, $\sum_{i=0}^{n} t_i = 1$. Since $T$ is convex, $z$ belongs to $T$. For $t_i$ sufficiently small $(i \neq 0)$, $x_i^* < z_i$ for all $i$, contrary to (b). Hence, for some $r$ between $q + 1$ and $n$, $x_r = 0$ for all $x$ in $T$. Let $p_r = 1$, $p_i = 0$ for $i \neq r$; then $\sum_{i=1}^{n} p_i x_i = 0$ for all $x$ in $T$, and (a) holds trivially.

*Case* 2. There is no $y$ in $T$ such that $x_i^* < y_i$ for all $i \leqq q$. Let $x^{*\prime}$ and $T'$ be the projections of $x^*$ and $T$, respectively. Let $z$ be a $q$-vector with $z_i = x_i^* + \epsilon$; then for all positive $\epsilon$, $z$ does not belong to $T'$. Hence, $x^{*\prime}$ is a boundary point of $T'$, and (a) holds by lemma 8.

Assumption 5 relates to free disposal on the part of the producers. It might instead be presupposed that it is the consumers who can dispose without cost of otherwise unwanted goods.

ASSUMPTION 6. *For each $j$, if $x_i \leqq y_i$ for all $i$, $U_j(x) \leqq U_j(y)$.*

For convenience, let $x \leq y$ mean that $x_i \leqq y_i$ for all $i$, $x_i < y_i$ for some $i$. Because of assumption 3, it easily follows that increasing the stock of one commodity, holding all others fixed, actually increases the desirability of a bundle. Hence, assumption 6 really implies insatiability of wants.

LEMMA 10. *If $x \leq y$, then under assumption 6, $U_j(x) < U_j(y)$.*

PROOF. By assumption 6, $U_j(x) \leqq U_j(y)$. Suppose $U_j(x) = U_j(y)$. Let $z = \frac{1}{2}x + \frac{1}{2}y$; by assumption 3, $U_j(z) > U_j(y)$, which is impossible since $z_i \leqq y_i$ for all $i$.

If assumption 6 holds, we will understand in the definition of an efficient point that the utility functions referred to must satisfy that assumption.

THEOREM 8. *Under assumption 6, a necessary and sufficient condition that $x^*$ be an efficient point is that for some $p$ such that $p_i > 0$ for all $i$, $\sum_{i=1}^{n} p_i x_i \leqq \sum_{i=1}^{n} p_i x_i^*$ for all $x$ in $T$.*

PROOF. Suppose $x^*$ an efficient point. Then there is an optimal distribution $X^*$ and a price vector $p$ satisfying the conclusions of theorem 4. By lemma 10, $X_j^*$ is not a point of bliss for any $j$. For any $j$,

$$\sum_{i=1}^{n} p_i x_i \geqq \sum_{i=1}^{n} p_i X_{ij}^* \text{ for all } x \text{ in } V_j . \tag{1}$$

For any $r = 1, \ldots, n$, let $y_i = X_{ij}^*$ for $i \neq r$, $y_r > X_{rj}^*$. By assumption 6, $y$ belongs to $V_j$. By (1), $p_r(y_r - X_{rj}^*) \geqq 0$, so that $p_r \geqq 0$, or

$$p_i \geqq 0 \text{ for all } i . \tag{2}$$

By theorem 4, $p_s \neq 0$ for some $s$; by (2), $p_s > 0$. For any $r \neq s$, let $z_i = y_i$ for $i \neq s$, $z_s = y_s - \epsilon$. By lemma 10, $U_j(y) > U_j(X_j^*)$; by continuity, $U_j(z) \geqq U_j(X_j^*)$ for $\epsilon$ sufficiently small but positive. By (1), $p_r(y_r - X_{rj}^*) \geqq p_s\epsilon > 0$, or $p_i > 0$ for all $i$. Since $p$ also has properties (b) and (c) of theorem 4, the conclusion follows from the assumption that $x^*$ is an efficient point.

Conversely, for each individual $j$, let $U_j(x) = \sum_{i=1}^{n} a_i \log (x_i + 1/m)$, defined only for those values of $x$ for which $x_i \geqq 0$ for all $i$. If we define $f(t)$ as in the proof of lemma 1, it is easy to verify that $f''(t) < 0$ for all $t$ if $a_i > 0$ for all $i$. Hence, the minimum attained by $f(t)$ over the closed interval (0, 1) must be attained at an endpoint; if $f(0) = f(1)$, then $f(t) > f(0)$ for $0 < t < 1$, so that assumption 3 is fulfilled. Choose $a_i = p_i(x_i^* + 1)/ \sum_{i=1}^{n} p_i x_i^*$. Since $U_j(x)$ increases with each variable $x_i$, its maximum under a budget constraint will occur on the boundary of the constraint. The maximum can therefore be obtained by using Lagrangian multipliers with the constraint $\sum_{i=1}^{n} p_i x_i = \sum_{i=1}^{n} p_i(x_i^*/m)$, and the maximum turns out to be attained uniquely at $x^*/m$.

THEOREM 9. *If assumption 6 holds, then a necessary and sufficient condition that $X^*$ be an optimal distribution is that there exists a set of prices $p$, with $p_i > 0$ for all $i$, such that $\sum_{j=1}^{m} X_j^*$ maximizes $\sum_{i=1}^{n} p_i x_i$ for $x$ in $T$ and such that $X_j^*$ maximizes $U_j(x)$ for all $x$ such that $\sum_{i=1}^{n} p_i x_i \leqq \sum_{j=1}^{m} p_i X_{ij}^*$.*

PROOF. This theorem follows easily from theorems 4, 5, and 8; since $p_i > 0$ for all $i$, the exceptional case in theorem 4(d) does not arise unless $X_{ij}^* = 0$ for all $i$; but in that case the theorem is trivially valid.

This theorem is, of course, the classical theorem of the applicability of the price system under insatiable wants extended to include corner maxima.

In connection with free disposal, it may be remarked that the use of the price system to achieve a distribution where all individuals are at a point of bliss, as in theorem 4(b), implied a mechanism of free disposal somewhere in the system, since producers' profit maximization will, in general, lead to an excessive supply.

## 8. Some diagrammatic representations

In the case of two individuals, two commodities, and a set $T$ containing just one point, the various theorems can be best illustrated by a diagram introduced by F. Y. Edgeworth [6]. When $T$ contains just one point, the problem is that of distribution of a fixed stock of two commodities among the two individuals. A distribution $X$ has four components, $X_{11}$, $X_{12}$, $X_{21}$, and $X_{22}$, where $X_{ij}$ is the amount of the $i$-th commodity given to individual $j$. In this case, the two sums $X_{11} + X_{12}$ and $X_{21} + X_{22}$ are given, so that a distribution can be represented by a point in a plane. In the following box diagrams, the variables $X_{11}$, $X_{21}$, $X_{12}$, and $X_{22}$ are measured along the lower, left hand, upper, and right hand axes respectively, the last two being measured in the opposite direction to the usual manner. The solid indifference curves pertain to individual 1 and so are read with respect to the lower and left hand axes, the dotted curves to individual 2 and the upper and right hand axes. Sample optimal distributions are marked with crosses. Any point within the box is a distribution. Note that the indifference curves for either individual may go outside the axes relating to the other. Dashed straight lines denote the boundaries of budget restraints.

Figure 1 illustrates the standard case of an interior maximum, both individuals

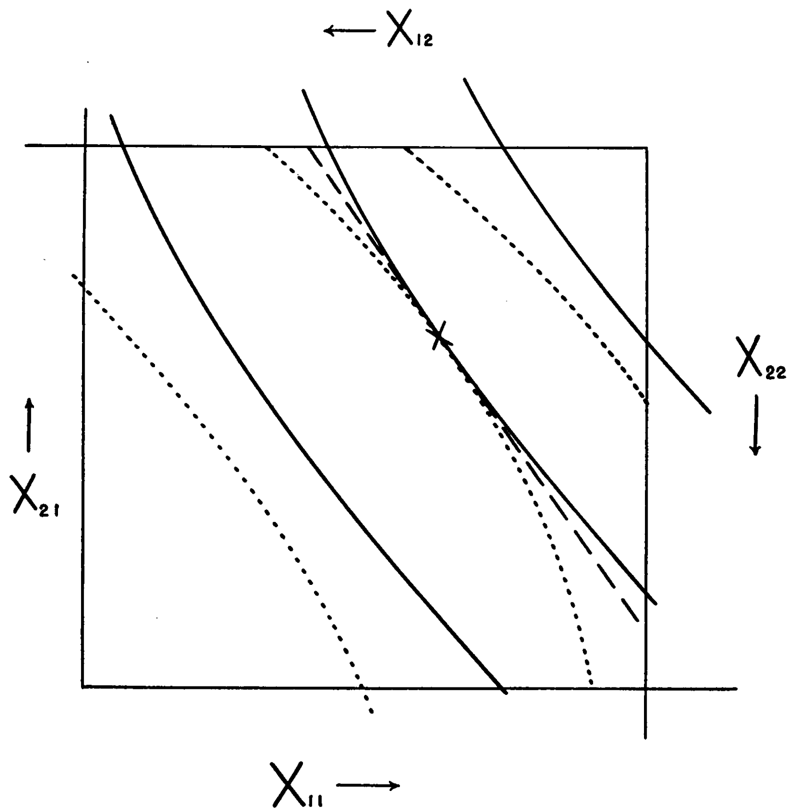


FIGURE 1

The standard case of an interior maximum, both individuals receiving positive quantities of each commodity.

receiving positive quantities of each commodity. By setting prices positive and proportional to the direction numbers of the normal to the line separating the indifference curves at their point of tangency, the indicated optimal distribution can be obtained by the workings of the price system.

In figure 2 is shown the case of a corner maximum, individual 1 getting nothing of commodity 1. The separating line is not tangent to the indifference curve of

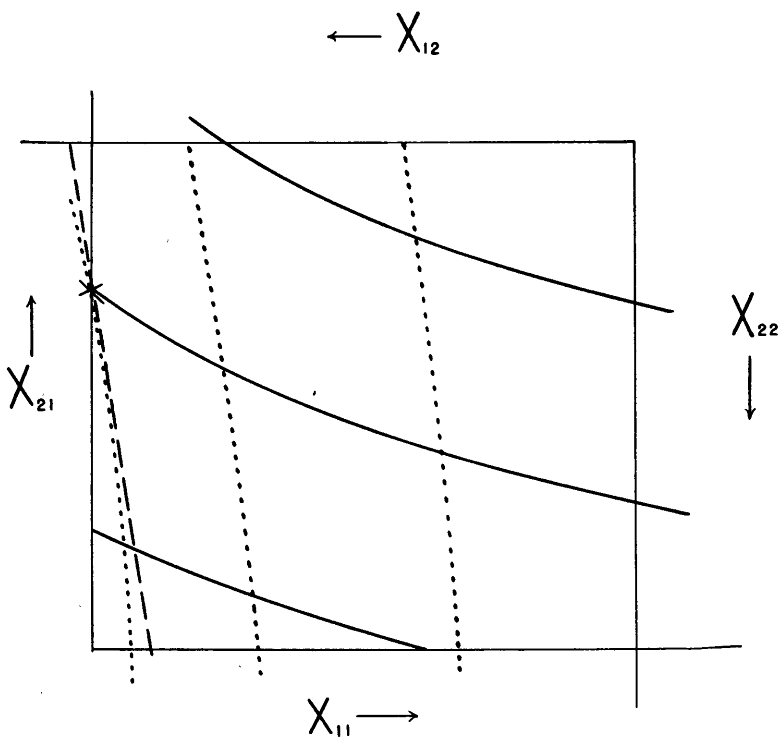


FIGURE 2

The case of corner maximum, individual 1 getting nothing of commodity 1

individual 1 through the optimal point; nevertheless there is a price vector (indeed, unique up to a factor of proportionality) which will drive both individuals to the optimal point.

Figure 3 shows the exceptional case in which conclusion (d) of theorem 4 does not hold, that is, the optimal point is not a point of utility maximization for both individuals for some pair of prices. Individual 2 will only choose the indicated point if $p_1 = 0$, but then individual 1 will seek a point further out on the $X_{11}$-axis than is consistent with the claims of individual 2.

## 9. Comments on the assumptions

Assumptions 2–4 have played a vital role in the analysis. The most critical is probably assumption 2, that the desirability of a distribution to any individual depends only on the commodity apportionment to him. If any component of $X$ entered as a variable into the utility functions of more than one individual, the whole analysis will be vitiated as it stands. Conspicuous consumption of the type envisioned by Veblen is a case where there is a negative interrelation between the consumption of one individual and the welfare of another. The drive for income

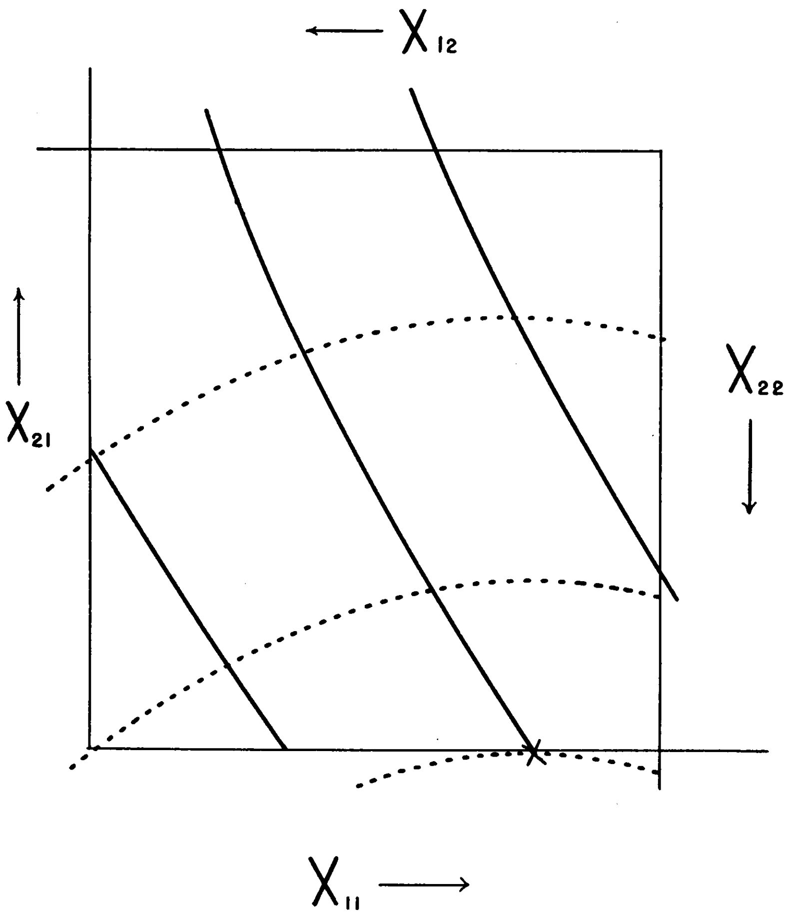


FIGURE 3

The exceptional case in which conclusion (d) of theorem 4 does not hold

equality and similar concepts of social equity, to the extent that it is shared by individuals who stand to lose from a purely individualistic viewpoint, represents another case of this type.

The empirical importance of this phenomenon has been stressed by Veblen [20] and more recently by Professor J. S. Duesenberry [5]; references to other studies are to be found in a paper by Dr. H. Leibenstein [13, especially pp. 184–186]. Some of the formal implications for the problem of optimal allocation are discussed by Professors Pigou, Meade, Reder (see the references in Leibenstein), Tintner [19], and Duesenberry [5, pp. 92–104]. The general feeling is that in these cases, optimal allocation can be achieved by a price system, accompanied by a suitable system of taxes and bounties. However, the problem has only been discussed in simple cases; and no system has been shown to have, in the general case, the important property possessed by the price system and expressed in theorem 5; not only can optimal distributions (usually) be achieved by the price system but any distribution so achieved is optimal.

I have argued elsewhere [1], [2] that if we seek distributions which are not merely optimal in the above sense but uniquely best in some social sense, then it must be assumed that the utility functions are interdependent, to the extent, at least, that each individual has standards of social equity. These imply that preferences as among distributions depend not only on the consumption of the individual but also among the distribution of welfare as related to the individual's social ideals.

The as yet unachieved hope of the type of analysis of which the present paper is a sample, the so-called "new welfare economics," is that the problems of social welfare can be divided into two parts: a preliminary social value judgment as to the distribution of welfare followed by a detailed division of commodities taking interpersonal comparisons made by the first step as given. It is in the second step that the present type of analysis may be useful. The preceding paragraphs suggest some of the difficulties.

Assumptions 3 and 4 are convexity assumptions in the field of consumption and production, respectively. Assumption 3 is invariably made in discussions of consumer's demand theory; it is a lineal descendant of the postulate of diminishing marginal utility, made when it was customary to regard utility as measurable. The justification for the assumption, however, is usually given little consideration. A common one, given for example by Hicks [8, pp. 23–24] is that the demand function is known empirically to be single valued and continuous and that for every commodity bundle there is a set of prices and an income level for which that bundle will be demanded. It may be doubted that this assumption is really empirically verifiable, and in any case, it is an assumption of a totally different logical order from that of utility maximization itself. The older discussions of diminishing marginal utility as arising from the satisfaction of more intense wants first make more sense, although they are bound up with the untenable notion of measurable utility. However, their fundamental point seems well taken. We must imagine that the individual has the choice of alternative uses of a given stock of goods to maximize his well being. The preferences for alternative bundles rest then on the *best* use that can be made of each. This preliminary maximization, so to speak, gives rise to the convexity of the indifference curves.

This argument has been given a more definite form by T. C. Koopmans (oral communication). Let $x$ and $y$ be two indifferent bundles. If it be supposed that the goods can be stored, even if only for a very short time, then a flow of goods at the rate $tx + (1 - t)y$ $(0 \leqq t \leqq 1)$ can be consumed at the rate of $x$ for fraction of time $t$ and at the rate of $y$ for fraction of time $1 - t$. Since in each part the individual is as well off as he would be with consuming at the rate $x$ (by the assumption that the two bundles are indifferent), the satisfaction from the flow $tx + (1 - t)y$ should be at least as great as that from $x$; and in general, if $0 < t < 1$, one would expect that there would be some rearrangement of the time order of consumption to yield still greater satisfaction.

Assumption 4 is usually derived from the two hypotheses of constant returns to scale (that is, for a given production process, multiplying all inputs in the same proportion will lead to a multiplication of outputs by the same proportion) and additivity of distinct production processes [that is, if process 1 yields a (vector) output $x_1$ and process 2 yields an output $x_2$, then both processes may be operated simultaneously to yield output $x_1 + x_2$]. Convexity of the transformation set may, however, hold under more general hypotheses. For example, diminishing returns to scale will not violate the convexity assumption so long as the additivity postulate holds; even if the activities are rival (that is, if performance of both will yield less the sum of the outputs of the two separately), the set $T$ will be convex provided returns to scale diminish sufficiently rapidly. Similarly, increasing returns to scale may still not violate the convexity assumption if there is sufficient complementarity among activities.[8]

## 10. Convex sets

To make the discussion self contained, some definitions and the proofs of lemmas 2 and 3 will be sketched here. For a more complete treatment see [3, especially chapter 1].

DEFINITION. *The* dimension *of a convex set $A$ is the dimension of that linear subspace of the original space containing $A$ which has the smallest dimension.*

DEFINITION. *An* external point *of a convex set is a point which is a boundary point of the set in the space of smallest dimension containing the set.*

DEFINITION. *An* internal point *of a convex set is a point of the set which is not an external point.*

DEFINITION. *By the* convex hull *of a set $S$ is meant the set of all points which belong to every convex set containing $S$.*

It is easy to see that the convex hull of $S$ is itself a convex set; it is the same as the set of all convex combinations of a finite number of elements of $S$. Also, if a convex set $A$ has at least two points, it possesses an internal point since its dimension is at least 1.

PROOF OF LEMMA 3. For any $r$, define $A_r$ as the (closed) set of all points of $A$ whose distance from the nearest external point of $A$ was at least $1/r$. Any given internal point of $A$ has a distance greater than zero from every external point; hence $A_r$ is nonnull for $r$ sufficiently large. Let $A'_r$ be the convex hull of $A_r$; it too is nonnull for $r$ sufficiently large. Clearly no external point belongs to $A_r$ for any $r$. Since no external point is a convex combination of internal points, the closed set $A'_r$ contains only internal points for every $r$, and therefore is disjoint from $B$ for every $r$. Find the shortest line segment between a point of $B$ and a point of $A'_r$, and construct a plane $\sum_{i=1}^{n} p_i^r x_i = c_r$ through the midpoint of the line segment and nor-

[8] The general problem of rivalry and complementarity among activities has been discussed in an unpublished manuscript by my colleague, Stanley Reiter.

mal to it. Clearly, by proper choice of sign,

$$\sum_{i=1}^{n} p_i^r x_i > c_r \text{ for } x \text{ in } A_r', \tag{1}$$

$$\sum_{i=1}^{n} p_i^r x_i < c_r \text{ for } x \text{ in } B, \tag{2}$$

since $A_r'$ lies on one side of the plane and $B$ on the other. Further, since for some $i$, $p_i^r \neq 0$, we can assume

$$\sum_{i=1}^{n} (p_i^r)^2 = 1. \tag{3}$$

From (3), the sequence of vectors $\{p^r\}$ is bounded; it can be inferred from (1) and (2) that the same is true of $\{c_r\}$. Hence, there is a subsequence of the integers $r$ for which the two sequences converge, say to $p$ and $c$, respectively. From (2),

$$\sum_{i=1}^{n} p_i x_i \leqq c \text{ for } x \text{ in } B. \tag{4}$$

Any given internal point of $A$ belongs to $A_r'$ for $r$ sufficiently large, so that, from (1),

$$\sum_{i=1}^{n} p_i x_i \geqq c \text{ for all internal points of } A. \tag{5}$$

Since an external point of $A$ is the limit of a sequence of internal points, (5) holds for all $x$ in $A$. Finally, from (3), by taking limits, $\sum_{i=1}^{n} p_i^2 = 1$, so that $p_i \neq 0$ for some $i$.

Proof of Lemma 2. If the dimension of $A$ is less than $n$, then all the points of $A$ lie in a hyperplane $\sum_{i=1}^{n} p_i x_i = c$. In particular, $\sum_{i=1}^{n} p_i x_i^* = c$, and the lemma is trivially true. If the dimension of $A$ equals $n$, then $A$ has at least two points. Let $B$ be the set consisting of the point $x^*$ alone. Then the conditions of lemma 3 are satisfied. Since $x^*$ belongs to both $A$ and $B$, $c = \sum_{i=1}^{n} p_i x_i^*$, and lemma 2 holds.

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