

# WHY "SHOULD" STATISTICIANS AND BUSINESSMEN MAXIMIZE "MORAL EXPECTATION"?

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## 1. Introduction

1.1. The word "should" in the title of this paper has the same meaning as in the following sentences: "In building a house, why should one act on the assumption that the floor area of a room is the product and not the sum of its length and width?"; "If all  $A$  are  $B$  and all  $B$  are  $C$ , why should one avoid acting as if all  $C$  were  $A$ ?" People may often act contrary to these precepts or norms but then we say that they do not act reasonably. To discuss a set of norms of reasonable behavior (or possibly two or more such sets, each set being consistent internally but possibly inconsistent with other sets) is a problem in logic, not in psychology. It is a normative, not a descriptive, problem.

1.2. The phrase "moral expectation" stems from the early students of probability who applied probabilities in their study of reasonable behavior of players in games of chance. Let the "prospect"  $P$ , that is, the probability distribution  $P(X)$  of a random "outcome"  $X$ , depend upon a man's decision ("strategy")  $S$ :

$$(1.2:1) \quad P = P(X) = P(X; S).$$

Let the set  $\mathcal{X}$  of all possible outcomes  $X$  be completely ordered by a relation  $g$  ("read: as good as or better than"). Define a scalar function  $u(X)$  on the set  $\mathcal{X}$  as follows: for any pair,  $X_1$  and  $X_2$ , in  $\mathcal{X}$ ,

$$(1.2:2) \quad u(X_1) \geq u(X_2) \quad \text{if } X_1 g X_2.$$

Then  $u(X)$  is called the utility of  $X$ . It is a random variable whose distribution depends on the distribution  $P$  and hence on the strategy  $S$ . Its expected value,

$$(1.2:3) \quad Eu(X) | P(X; S) = \mu_u(S), \quad \text{say,}$$

is called the moral expectation of  $X$ . Define a space  $\mathcal{S}$  whose elements  $S$  represent possible strategies. The title of the paper asks whether it is reasonable always to choose as one's strategy an element  $S^*$  of  $\mathcal{S}$  whenever

$$(1.2:4) \quad \mu_u(S^*) > \mu_u(S')$$

where  $S'$  is any element of  $\mathcal{S}$  distinct from  $S^*$ .

1.3. The "precept," always (that is, for any space  $\mathcal{S}$ ) to maximize moral expectation, leads to inconsistent results unless all the utility functions considered

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are linear transforms of each other (in which case utility is sometimes said to be "measurable"). This can be easily shown for the case of discrete probability distributions. Suppose  $X$  can only take values  $X_0, \dots, X_N$ , and denote the corresponding probabilities by  $p_0, \dots, p_N$ . Let  $u$  and  $v$  be two utility functions as defined in (1.2:2), and suppose the space  $\mathcal{S}$  is such as to include a strategy for every distribution. In particular, let  $S'$  result in probabilities  $p'_0, \dots, p'_N$ ; and  $S^*$  in  $p^*_0, \dots, p^*_N$ . Suppose that, following the "precept,"  $S^*$  is not chosen in preference to  $S'$ ; that is, by (1.2:4),

$$(1.3:1) \quad \mu_u(S^*) \leq \mu_u(S') ; \quad \mu_v(S^*) \leq \mu_v(S') .$$

Suppose, in addition, that neither is  $S'$  chosen in preference to  $S^*$ . Then (1.3:1) becomes

$$\mu_u(S^*) = \mu_u(S') ; \quad \mu_v(S^*) = \mu_v(S') ;$$

$$\sum_0^N (p_n^* - p_n') \cdot u(X_n) = 0 = \sum_0^N (p_n^* - p_n') \cdot v(X_n) = \sum_0^N (p_n^* - p_n') \cdot 1 .$$

This must remain true for prospects such that  $p_n^* \neq p_n'$  for  $n \leq 2$  and  $p_n^* = p_n' = 0$  for  $n > 2$ . Then the three equations

$$\sum_0^2 (p_n^* - p_n') \cdot u(X_n) = \sum_0^2 (p_n^* - p_n') \cdot v(X_n) = \sum_0^2 (p_n^* - p_n') \cdot 1$$

form a homogeneous linear system in the three  $(p_n^* - p_n')$ ,  $n = 0, 1, 2$ . Hence the  $u(X_n), v(X_n), 1$  are linearly dependent, for any three arbitrarily chosen values of  $n$ . Therefore, there exist  $\alpha, \beta$  such that

$$v(X_n) = \alpha + \beta u(X_n) , \quad n = 0, \dots, N .$$

The linear dependence of the utility functions follows thus from the linear nature of the operator  $E$  in (1.2:3). The following illustration may be useful. Suppose  $\mathcal{X}$  consists of three alternative sums of money: \$ - 1, 0, 1. Let  $w = M(u)$  and  $v = L(u) = 1 + 2u$  be, respectively, a nonlinear and a linear monotone increasing transform of a utility function  $u(X)$ . Let  $S'$  and  $S''$  be two strategies resulting, respectively, in two different probability distributions of  $X$ ,  $P'(X)$  and  $P''(X)$ . In the following table, the moral expectations,  $\mu(S')$  and  $\mu(S'')$ , are computed for the

$X$	$u$	$w$	$v$	$P'(X)$	$P''(X)$
-1	-2	-1	-3	$\frac{1}{2}$	0
0	0	0	1	0	1
1	1	2	3	$\frac{1}{2}$	0
$\mu(S') =$	$-\frac{1}{2}$	$\frac{1}{2}$	0	$EX =$	0
$\mu(S'') =$	0	0	1	$E(X - EX)^2 =$	1
Best $S =$	$S''$	$S'$	$S''$		0

three different utility functions,  $u, w$  and  $v$ . Thus, of the two strategies  $S'$  and  $S''$ ,  $S''$  (resulting in a smaller variance of  $X$ ) is chosen when the utility function is  $u$  or the linear transform  $v$  of  $u$ . But when the utility function is the nonlinear monotone

transform  $w$  of  $u$ , a different strategy may be chosen—although the man maximizes his moral expectation.

1.4. Pascal's [9] immortality wager was an early application of the precept to maximize moral expectation. Essentially, Pascal made four propositions, with the first three of which we shall not quarrel. *First*, assume that, since chances for and against immortality are unknown, they are equal. *Second*, assume that, if there is immortality, then good life is followed by eternal bliss and bad life by eternal damnation. Regard these two sequences as outcomes  $X'_1$  and  $X''_1$ , respectively, and denote the outcomes "good life followed by nothing" and "bad life followed by nothing" by  $X'_2$  and  $X''_2$ , respectively. Consider "good life" and "bad life" as two strategies,  $S'$  and  $S''$ . In effect, Pascal computes the following two expected values:

$$\begin{aligned}\mu_u(S') &= \frac{1}{2}u(X'_1) + \frac{1}{2}u(X'_2), \\ \mu_u(S'') &= \frac{1}{2}u(X''_1) + \frac{1}{2}u(X''_2).\end{aligned}$$

Hence,

$$\mu_u(S') > \mu_u(S'') \quad \text{if} \quad u(X'_1) - u(X''_1) > u(X''_2) - u(X'_2).$$

For Pascal, the difference between the advantages of eternal bliss (following a short period of possibly tedious good life) and the disadvantages of eternal damnation (following a short though possibly not unpleasant bad life) exceeds the difference between the possible pleasures of sin and the possible inconveniences of virtue. These valuations (the utility function  $u$ ) can be regarded as his *third* proposition and may be accepted. We shall be concerned with his *fourth* proposition: that, because  $\mu_u(S')$  exceeds  $\mu_u(S'')$ , it is reasonable to choose  $S'$ .

1.5. In the particular case when the space  $\mathcal{X}$  of outcomes consists of alternative sums of money (as in 1.3, table), the moral expectation of the gain,  $Eu(X)$ , is contrasted with its "mathematical expectation,"  $EX$ , which also depends on  $S$ . In the Petersburg game, there exists a strategy  $S'$ , say, which makes  $EX$  infinite, yet a reasonable player would not choose  $S'$ . To explain the paradox, Daniel Bernoulli stated that a reasonable man maximizes  $Eu(X)$  and not  $EX$  and that the function  $u(X)$  has certain properties. See Menger [7].

1.6. In section 2 of this paper, the precept to maximize  $Eu(X)$  will be related to problems facing statisticians and businessmen and, in fact, to human decisions in general. Section 3 gives as a necessary condition for the precept a postulate which may be called the *Postulate of Substitution between Indifferent Prospects*. In an earlier paper [4], this postulate (jointly with certain other postulates) was shown to be a sufficient condition for the precept of maximizing  $Eu(X)$ , valid for a non-empty class of utility functions, each element of which is a linear transform of any of the others. This postulate appears, thus, to be logically equivalent to the moral expectation precept (provided the other postulates are admitted). It is also possibly equivalent to certain postulates of von Neumann and Morgenstern [8]. For a comparison, see [4, section 7]. Finally, it is also equivalent to a postulate which Samuelson [11] recently formulated very succinctly and which he proposed to call *Special Independence Assumption*. A comparison between the postulate and the economist's concept of "independence between consumption goods" is contained in section 3 of the present paper.

1.7. In section 4 a very rough outline of a different approach will be attempted.

A rule of *Long Run Success* is formulated (“in the long run, it pays to be reasonable”) by considering a strategy as a sequence of rules of action to be taken in response to future situations. It seems that the rule of Long Run Success is not equivalent to the precept to maximize moral expectation unless some further conditions are imposed upon the utility functions. No definitive results are available so far.

1.8. Note that the space  $\mathcal{S}$  of strategies can be conceived as including among its elements, strategies consistent with the ordinary rules of logic and mathematics, and strategies not consistent with these rules. The distribution  $P$  of outcomes  $X$  and, therefore, the quantity  $Eu(X)$  will depend on whether the decision maker is a good or a bad logician and arithmetician, on what kind of geometry he applies, etc. This kind of justification of a set of behavior norms including norms of thinking and counting was, I believe, occasionally attempted by pragmatist and evolutionary writers with some rule of long run success in mind: “If you act on the assumption that 2 times 2 equals 5, you (or your tribe or species) will, in some sense, fare worse in the long run than if you act on the assumption that 2 times 2 equals 4.”

## 2. Some concrete cases

2.1. In recent years, the theory of statistical inference has taken a remarkably “economic” turn. In choosing a rule for making observations (the design of a sample), money cost ( $C$ ) is subtracted from what may be called the gross gain ( $G$ ) derived by the statistician or his “employer” from the knowledge acquired from the observations.  $G$  is conceived as a sum of money and is, in a simple case, the larger the smaller the error of the estimation based on the sample. The money sum  $G - C = X$  is thus the outcome of the statistician’s decision to choose a certain sample design.  $X$  is called the net gain or profit.  $X$  is to be maximized with respect to the variable under the statistician’s control, that is, with respect to the design of the sample.

2.2. In a particular case when the sample designs under consideration differ only with respect to the size of the sample (number of observations)  $S$ , the best value  $S = S^*$  must satisfy the approximate rule,

$$\frac{dG}{dS} = \frac{dC}{dS},$$

(provided  $G$  and  $C$  can be approximated by differentiable functions of  $S$ ). This is the familiar rule of the economists: to equalize the marginal monetary product and the marginal monetary cost of the “input”  $S$ . More generally one defines the space  $\mathcal{S}$  of all possible sample designs and maximizes the scalar function  $X(S)$  over this space.

2.3. However, the profit  $X$  is a random scalar since the gross gain  $G$  depends on the values that the observed random variables happen to take. (In addition, the cost  $C$ , too, may depend on observed values, as for example, when  $C$  depends on the location of individuals that happen to fall into a social survey sample or when the number of observations depends on observed values as in sequential sampling.) One cannot maximize the random variable  $X$  but one can maximize some quantity depending on its distribution, for example, its mean  $EX$ .

2.4. Both concepts, the sample design and the monetary profit, can be replaced by wider ones. As regards the first: the statistician can recommend not only the rule of making observations but also the decision to be taken after having collected them. This decision may be the choice of an estimate or of a hypothesis. More generally, it may be any decision that will influence the probability distribution of the gross gain  $G$ , for example, the decision to buy a certain quantity of a commodity. Generalizing the notation of 2.2, one defines  $\mathcal{S}$  as the space of all possible "strategies"  $S$ , each strategy being a certain rule for making observations *and* for taking decisions on the basis of these observations. The distribution of the random profit  $X$ — to be denoted by  $P(X)$ —will depend on  $S$  and on the true distribution  $F$  of the observables. We can write

$$(2.4:1) \quad P(X) \equiv P(X; S, F), \quad EX \equiv EX | (S, F) \equiv \mu_X(S, F), \text{ say.}$$

(In the simple case of 2.1,  $X$  depended on the estimation error, that is, the difference between a point estimate and the true value of the estimated parameter of the distribution of observables. This was obviously a special case of the one now stated.) The negative of the function  $\mu_X$  just defined is identical with Wald's [12] "risk function," with two differences: first, Wald always has  $G \geq 0$  (regarding  $-G$ , the "loss suffered by the statistician," as a nonpositive quantity)—a trivial difference; second, Wald does not necessarily regard  $G$ ,  $C$  and  $X$  as monetary quantities and presumably accepts the generalization that we are going to make now.

2.5. With the "statistician" taking over entrepreneurial decisions, it becomes necessary to reconsider what is of concern to the businessman. To begin with, "A full purse is not as good as an empty one is bad." There exists a certain quantity  $K$  (possibly zero) which depends on the firm's reserves and is such that, if  $X \leq K$ , the firm is bankrupt and must be dissolved. It is reasonable that the probability of the occurrence of this situation should be made as small as possible. This objective may not be reached if the strategy chosen maximizes  $EX$ . On the other hand, suppose the firm tries to maximize the expression  $Eu(X)$ , where the "utility function"  $u(X)$  is defined as follows:

$$(2.5:1) \quad \begin{aligned} u(X) &= -\nu && \text{when } X \leq K, \\ u(X) &= X && \text{when } X > K, \end{aligned}$$

where  $\nu$  is a positive constant. Let the probability density function of  $X$ ,  $X \neq K$ , for a given strategy  $S$  be  $p(X; S)$ . Then the expected value of  $u(X)$  given  $S$  is

$$(2.5:2) \quad Eu(X; S) = -\alpha(\nu + \beta) + \beta,$$

where

$$\begin{aligned} \alpha &= \int_{-\infty}^K p(X; S) dX = \text{probability of bankruptcy,} \\ \beta &= \frac{\int_K^{\infty} X p(X; S) dX}{\int_K^{\infty} p(X; S) dX} \\ &= \text{profit averaged over all cases other than bankruptcy.} \end{aligned}$$

Both  $\alpha$  and  $\beta$  depend on  $S$ , and  $\beta$  is usually nonnegative. It follows from (2.5:2)

that, for a given  $\beta$ , the firm's moral expectation is the larger the smaller is  $\alpha$ , the probability of bankruptcy. The maximization of  $Eu(X)$  would thus seem to describe reasonable behavior better than the maximization of  $EX$ .

2.6. Another example of the utility function of a random sum of money and of the effect of properties of this function upon the choice of strategy was given in the table in 1.3. A much discussed case has been that of a function  $u(X)$  that is differentiable at least twice. If  $u'(X) > 0$  and  $u''(X) < 0$  for all values of  $X$  (the case of "decreasing and positive marginal utility of income"), then the individual who maximizes  $Eu(X)$  will prefer, given the value of the mean profit  $EX$ , a prospect with a low variance to a prospect with a high variance of  $X$ . This is seen by expanding  $Eu(X)$  into a Taylor series. See [6] and, for a somewhat more general case, [2].

2.7. More generally, the businessman is concerned not with a scalar quantity but with a vector: with the sequence of profits to be earned in successive years, the more distant ones being possibly of less import to the firm's decision (even if the joint probability distribution of the sequence is known exactly) than the more immediate ones; or, if he is a farmer, with a set of quantities earned, respectively, in the form of food, housing accommodation, etc. The fact that nonmonetary earnings and future earnings of any kind can be converted into current money at prices and interest rates prevailing in (perfect) markets does not dispose of the complication since these prices and interest rates themselves have to be explained by the strategies of the people that transact in these markets. However, the generalization from a scalar to a vector  $X = \{x_i\}$  does not present difficulties. In particular, if  $u(X)$  is twice differentiable, the results mentioned in 2.6 are easily generalized. Write vector

$$\mu \equiv \{Ex_i\}$$

and matrix

$$\sigma \equiv \|\sigma_{ij}\| \equiv \|E(x_i - \mu_i)(x_j - \mu_j)\|.$$

Then, if

$$u_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_j}$$

exists for all  $i, j$ , we have by expanding about  $\mu$ ,

$$Eu(X) = u(\mu) + \frac{1}{2} \sum_{i,j} u_{ij} \sigma_{ij} + \dots$$

It follows that if  $u_{ii} < 0$  (decreasing marginal utility of the  $i$ -th kind of commodity earned or of the  $i$ -th year's money profit), then the individual will try to make the variance  $\sigma_{ii}$  as small as possible, given the other elements of  $\mu$  and  $\sigma$ . A high correlation between the  $i$ -th and the  $j$ -th elements of the profit vector will be feared if  $u_{ij} < 0$  (the case of goods with negative complementarity); a high correlation between  $i$  and  $j$  will be desired if  $u_{ij} > 0$  (the case of positively complementary goods).

2.8. Note that the definition of complementarity just used is only possible because, as a corollary of the requirement to maximize moral expectation  $Eu(X)$ , all utility functions form a group of linear transforms. If a nonlinear transform of

$u(X)$ —say the function  $w(X) = f[u(X)]$ ,  $f' > 0$ —were admitted as a utility function, then, since a positive  $u_{ij}$  might be consistent with a negative  $w_{ij}$ , the sign of complementarity as just defined could not be ascertained.

2.9. In [5], the desirability of large or small variances of, or correlations between, inputs (production factors) of various kinds was studied on lines similar to 2.7. Rational advantages of “pooling the risks” and of either specialization or diversification of production (depending on the sign of complementarity between factors of production) can be derived—always assuming that it is rational to maximize  $Eu(X)$ .

2.10. Certain things desired by the businessman—such as power or social position or reputation—are not quantities at all. It is therefore necessary to generalize further the concept of what is being maximized. This generalization was presented in 1.2. If the space  $\mathcal{X}$  of “outcomes” and the space  $\mathcal{S}$  of strategies are defined, this permits us to take care of all human decisions, transcending conventional economics and including the private man’s choice of profession or wife, the legislator’s choice of election tactics or national policies or military and administrative decisions. This is of some interest to statisticians who, after all, are not all employed by profit making organizations.

2.11. Our question is, then, whether the following rule is reasonable: always choose a strategy  $S^*$  so as to obtain a prospect (a probability distribution)

$$P(X) \equiv P(X; S^*, F),$$

for which

$$\mu_u(S^*, F) \equiv Eu(X) | (S^*, F)$$

is larger than or equal to

$$\mu_u(S', F) \equiv Eu(X) | (S', F),$$

for any  $S'$ ; where the scalar function  $u(X)$  is defined over the space  $\mathcal{X}$  of outcomes,  $S^*$  and  $S'$  are elements of the space  $\mathcal{S}$  of strategies and  $F$  is the true distribution of the observables. In general,  $F$  is not (or not completely) known—in the case of the statistician as well as in the case of any other decision makers. The economist F. H. Knight [3] called the case of unknown  $F$  “presence of uncertainty” and the case of known  $F$  (as in games of chance) “presence of risk.” This terminology has been widely accepted among economists although it is not in line with general scientific usage of the word uncertainty. It would be better to speak of “incomplete” versus “complete” information. For the purpose of this paper, it is sufficient to deal with the case of complete information, that is, to assume  $F$  known. It is of no relevance to us here whether, in the case of  $F$  unknown, probabilities of the several alternatives have to be assumed equal before maximizing  $\mu_u(S, F)$  with respect to  $S$  as was done by Pascal (1.4 above), or whether, following Wald and the authors of the *Theory of Games*, one should assume the least favorable distribution  $F$ , that is, minimize  $\mu_u(S, F)$  with respect to  $F$  before maximizing it with respect to  $S$ . It is the latter, the maximization of  $\mu_u$  with respect to  $S$ , that concerns us here: a problem common to the case of complete and to that of incomplete information and arising with Pascal as with Wald.

### 3. The rule of substitution between indifferent prospects

3.1. In [4], a certain set of behavior postulates, numbered I–IV, was shown by the author to imply the proposition that there exists a class of utility functions, linear transforms of each other, and such that for every utility function the expected value of utility is maximized. We shall presently restate these postulates and show that, if postulates I–III (which appear rather mild) are accepted, then postulate IV follows from the condition that  $Eu(X)$  is maximized. A joint result of the two papers is, then, that under certain weak conditions, postulate IV and the rule of maximizing moral expectation are equivalent. Postulate IV will be called the “rule of substitution between indifferent prospects.”

3.2. As in [4], we define a space  $\rho$  of prospects  $P$  (= probability distributions) for the case that the space  $\mathcal{X}$  of outcomes consists of a finite number of elements,  $X_0, X_1, \dots, X_N$ . (The case of  $N$  infinite was discussed by Rubin in [10] for the problem of [4].) Regard the probability that the particular outcome  $X_n$  will occur,  $p(X_n)$ , as a coordinate of the point  $P$  in the Euclidean  $N$ -space,  $n = 1, \dots, N$ . Since

$$(3.2:1) \quad 0 \leq \sum_{n=1}^N p(X_n) = 1 - p(X_0) \leq 1,$$

the space  $\rho$  of prospects is the domain of the  $N$ -space bounded by and including the surface of an  $(N + 1)$ -hedron whose vertices are the origin  $(0, 0, \dots, 0)$  and the ends  $(1, 0, 0, \dots, 0)$ ,  $(0, 1, 0, \dots, 0)$ ,  $\dots$ ,  $(0, 0, 0, \dots, 1)$  of the  $N$  unit vectors. These vertices represent the “sure” prospects, each promising with certainty one particular outcome. Sure prospects will be denoted by  $P^{(0)}, P^{(1)}, \dots, P^{(N)}$ . Using letters  $P, Q, R, \dots$  for prospects in general, we state the following postulates:

I. *The space  $\rho$  is completely ordered by the relation  $g$  (read “as good as or better than”).* Note: Whenever  $PgQ$  we shall also write  $u(P) \geq u(Q)$ . This defines a scalar “utility function”  $u(P)$  on the space  $\rho$ . It is related in a simple way to the function  $u(X)$  on the space  $\mathcal{X}$ , defined in 1.2 above, that is, we have  $u(X_n) = u(P^{(n)})$ ,  $n = 0, 1, \dots, N$ , by definition. Furthermore, whenever  $PgQ$  and not  $QgP$ , we write  $PpQ$  (read “ $P$  preferred to  $Q$ ”) and have  $u(P) > u(Q)$ . Whenever  $PgQ$  and  $QgP$ , we write  $PiQ$  (read “ $P$  and  $Q$  are indifferent”) and have  $u(P) = u(Q)$ .

II.  *$g$  is continuous*, that is, if  $PgQgR$ , then there exists a number  $r$ ,  $0 \leq r \leq 1$ , such that if  $Q' = rP + (1 - r)R$  then  $QiQ'$ . Note: We are using the symbols for addition, multiplication and equality in their ordinary meaning.  $Q'$  is the result of ordinary multiplication and addition performed on two vectors,  $P$  and  $R$ . Geometrically,  $Q'$  is a point on a straight line connecting  $P$  and  $R$ .

III. *There exist  $P$  and  $Q$  not on the boundary of  $\rho$ , such that not  $PiQ$ .* Note: This is postulate III\* of [4]. We refer to [4] for an alternative postulate and for a stronger one.

IV. *If  $PiQ$  and  $0 < r < 1$  and if  $P' = rR + (1 - r)P$  and  $Q' = rR + (1 - r)Q$ , then  $P'iQ'$ .* Note:  $P'$  is the prospect of having either prospect  $P$  or prospect  $R$  with certain probabilities,  $1 - r$  and  $r$ .  $Q'$  is the prospect of having either  $Q$



or  $R$  also with probabilities  $1 - r$  and  $r$ . Postulate IV states that if  $P$  and  $Q$  are indifferent, so are  $P'$  and  $Q'$ . In a special case  $P$  and  $P'$  may coincide [4, postulate IVa]. Also,  $P$  or  $Q$  or  $R$  may be sure prospects.

3.3. In the language of the last section—see note to postulate I—the rule of maximizing  $Eu(X)$  can be stated thus [writing for brevity  $u(P^{(n)}) = u^{(n)}$ , which is independent of the prospects considered, and  $P(X_n) = p_n, Q(X_n) = q_n$ , etc.].

PROPOSITION 1.  $QgR$  if and only if  $\sum_0^N u^{(n)} q_n \geq \sum_0^N u^{(n)} r_n$ .

We shall prove that this rule implies

PROPOSITION 2. If  $PpQpR$  then the indifference sets  $J(P), J(Q), J(R)$  are segments of three parallel  $(N - 1)$ -dimensional hyperplanes, contained in  $\rho$  and stacked in the order  $P, Q, R$ , or its reverse,

where for any prospect  $P$  the indifference set  $J(P)$  is defined as consisting of all prospects  $P'$  for which  $P'iP$ .

In fact, if  $PpQpR$  then, by the definition of the relations  $p$  and  $i$ , proposition 1 implies that

$$(3.3:1) \quad \sum_0^N u^{(n)} p_n > \sum_0^N u^{(n)} q_n > \sum_0^N u^{(n)} r_n.$$

If, in addition,  $PiP', QiQ'$  and  $RiR'$ , proposition 1 requires that

$$\sum_0^N u^{(n)} p_n = \sum_0^N u^{(n)} p'_n = a^* \quad \text{say,}$$

$$\sum_0^N u^{(n)} q_n = \sum_0^N u^{(n)} q'_n = b^* \quad \text{say,}$$

$$\sum_0^N u^{(n)} r_n = \sum_0^N u^{(n)} r'_n = c^* \quad \text{say;}$$

where  $a^* > b^* > c^*$ . Write  $v^{(n)} = u^{(n)} - u^{(0)}$ ,  $n = 1, 2, \dots, N$ . Then the above equations become upon replacing  $p_0$  by  $1 - \sum_1^N p_n$ , etc. and upon replacing  $a^* - u^{(0)}$  by  $a$ , etc.

$$(3.3:2) \quad \sum_1^N v^{(n)} p_n = \sum_1^N v^{(n)} p'_n = a,$$

$$\sum_1^N v^{(n)} q_n = \sum_1^N v^{(n)} q'_n = b,$$

$$\sum_1^N v^{(n)} r_n = \sum_1^N v^{(n)} r'_n = c.$$

The last two terms in the first line of (3.3:2) give the equation of a  $(N - 1)$ -dimensional hyperplane which contains all  $P'$  such that  $P'iP$ . This hyperplane is therefore identical with the indifference set  $J(P)$ . Similarly, the other two lines in

(3.3:2) provide the equations for the hyperplanes  $J(Q)$  and  $J(R)$ . The three hyperplanes are parallel, each having the  $N$  direction cosines proportional to  $v^{(n)}$ ,  $n = 1, 2, \dots, N$ . The distances of the three hyperplanes from the origin are proportional to  $a$ ,  $b$  and  $c$ , respectively. Hence they are stacked in the order  $P$ ,  $Q$ ,  $R$  or its reverse.

3.4. Proposition 2 implies, in turn, postulate IV, provided postulates I–III are granted. In fact, in the notation used in the statement of postulate IV, prospects  $P$  and  $Q$  must both lie on  $J(P)$ , one of the parallel indifference planes revealed in proposition 2. Moreover, the indifference planes  $J(P')$  and  $J(Q')$  must be parallel to  $J(P)$ . On the other hand,  $P'$  and  $Q'$  must both lie on a plane parallel to  $J(P)$  because  $r = \overline{P'P'}/\overline{P'R} = \overline{Q'Q'}/\overline{Q'R}$ . And since in an Euclidean space there is only one plane through  $P'$  parallel to  $J(P)$ , we have  $P'iQ'$ .

The reasoning of this and the preceding sections presupposes the validity of postulates I–III. Postulate I excludes the case in which neither  $PgQ$  nor  $QgP$  and this would rule out the existence of a function  $u(P)$  on  $\rho$ . Postulate II excludes the possibility of “holes” in the indifference planes. Postulate III excludes the possibility that the whole interior of  $\rho$  might form a single indifference set.

We conclude that, given postulates I–III, postulate IV is not only a sufficient condition for proposition 1 (as was shown in [4]), but also a necessary condition. Thus, the two are equivalent, provided postulates I–III are granted. Postulates I–III seem indeed weak enough as a description of reasonable behavior. Postulate IV also seems reasonable to me and some further remarks in 3.5 may convince others. If, in addition, this postulate is taken to be intuitively more convincing than its logical equivalent, proposition 1, then the question asked in the title is answered.

3.5. In the discussion of the theory of choice between prospects, confusion seems to arise through the use of an ambiguous word, “combination.” This word naturally expresses the operation “and,” as in “ $A$  and  $C$ .” But it has also been applied to express the relation “either—or,” as in “either  $A$  or  $C$ , with probabilities  $r$  and  $1 - r$ .” Since prospects are always mutually exclusive, they cannot be “combined” into an object of choice such as “prospect  $A$  and prospect  $C$ ” to be chosen in preference to another “combination” such as “prospect  $B$  and prospect  $C$ .” But they can be “combined” in a different sense, namely, the combination “either  $A$  or  $C$ , with probabilities  $r$  and  $1 - r$ ” can be formed and may be chosen in preference to the combination “either  $B$  or  $C$ , with probabilities  $q$  and  $1 - q$ .” Suppose, on the other hand, that  $A$ ,  $B$ ,  $C$  are objects that are not mutually exclusive. For example,  $A$  = country house,  $B$  = city house,  $C$  = car. Then the following rule of behavior would *not* be reasonable: “If I like  $A$  as well as I like  $B$ , then I like  $A$  and  $C$  as well as I like  $B$  and  $C$ .” Such a rule would neglect the possibility that a car has more use in the country than in the city. But such a rule is *not* our postulate IV. The latter says, rather: “Call  $P$  the prospect of having the country house  $A$  and having, in addition, certain other things which will be held constant throughout the comparisons and which we shall call  $D$ ; call  $Q$  the prospect of having the city house  $B$  and  $D$ . Then, if I like  $P$  as well as I like  $Q$ , I like the prospect of having either  $P$  or  $R$  as well as I like the prospect of having either  $Q$  or  $R$ , provided the odds are the same in each case.” This is reasonable. The fact that the car is of

better use when possessed jointly with a country house than when possessed jointly with a city house has no bearing on the choice between the prospects discussed, of which *none promises the joint possession of a car and a house*. To exclude more completely from our mind the logically illegitimate picture of a joint occurrence of mutually exclusive events, postulate IV might be illustrated most simply as follows:  $P$  = the career of a professor,  $Q$  = the career of a church minister,  $R$  = the career of a bus driver. Or, imagine that you have to choose between drawing from one or another of two urns; in each of them, 20 per cent of all tickets are inscribed "car"; the remaining 80 per cent are inscribed "country house" in the first urn and are inscribed "city house" in the second urn. If you are indifferent between city house and country house, will you prefer one urn to the other, knowing that if you do get a car in any of the lotteries you will *not* get a house with it? The same applies naturally when  $P$ ,  $Q$  and  $R$  are themselves unsure prospects, for example, if they are tickets to three different lotteries.

3.6. It may be useful to refer to the notation of 2.7 where the outcome  $X$  was a joint occurrence of quantities  $x_1, x_2, \dots$  and was regarded as a vector  $\{x_i\}$ . The utility of  $X$  (or, in the language of prospects, the utility of the sure prospect that  $X$  occurs with certainty) is denoted by  $u(x_1, x_2, \dots)$ . Disregard, as being kept constant, all  $x_i$  for  $i > 3$ . We say that commodity 3 is more complementary with commodity 1 than with commodity 2, over some defined intervals, if

$$(3.6:1) \quad u(x_1 + h_1, x_2, x_3) = u(x_1, x_2 + h_2, x_3)$$

and

$$u(x_1 + h_1, x_2, x_3 + h_3) > u(x_1, x_2 + h_2, x_3 + h_3),$$

where the  $h_i$  are positive. This definition of complementarity is in accord with the one used in 2.7 for the case when the  $x_i$  are continuous and  $u$  is twice differentiable. This is seen by expanding the function  $u$  into a Taylor series about  $x_1, x_2, x_3$ , and inserting into (3.6:1). On the other hand, in our example of the car and the two different houses, the three  $x_i$  can take the values 0 and 1 only. If, in the continuous case, the cross derivative  $u_{13} = 0$  or if in the general case

$$u(x_1, x_2, x_3 + h_3) - u(x_1, x_2, x_3) \\ = u(x_1 + h_1, x_2, x_3 + h_3) - u(x_1 + h_1, x_2, x_3),$$

we say that there is no complementarity (positive or negative) between commodities 1 and 3 or that the two are "independent." As stated in 2.8, these definitions presuppose that utility functions are linear transforms of each other.

To apply this concept to the choice between prospects, remember that we have then, as the argument of the utility function, not the vector of quantities of commodities but the probabilities of mutually exclusive events,  $p_1, \dots, p_N$  where, in particular,  $p_1$  may be the probability that the vector of commodity quantities has a certain value and  $p_2$  may be the probability that this vector has another value. It would be misleading to say that postulate IV asserts "independence" (in the sense just stated) between any objects of choice themselves: the prospects  $P, Q, \dots$  are not (as the commodity quantities  $x_1, x_2, \dots$  are) coordinates of the space in which the indifference surfaces are drawn.

3.7. On the other hand, the probabilities  $p_1, \dots, p_N$  are indeed "independent,"

in the sense that, if the precept of maximizing expected utility is always followed, then

$$\frac{\partial^2 u(p_1, p_2, \dots, p_N)}{\partial p_m \partial p_n} = 0, \quad m, n = 1, \dots, N,$$

because  $u$  is linear in its continuous argument, the vector  $\{p_i\}$ .

#### 4. The rule of long run success

4.1. We shall now outline tentatively another proposition which may under certain conditions be implied by the rule of maximizing moral expectation and which appears (like postulate IV) intuitively more convincing than the rule itself.

For an integer  $T > 0$ , define a space  $\mathcal{X}_{(T)}$ , whose element  $X_{(T)}$  represents a possible time sequence of situations  $x_0, x_1, \dots, x_T$  (for example, a sequence of annual profits or of balance sheets). Define a utility function  $u_T(X_{(T)})$  such that if  $X'_{(T)}$  and  $X''_{(T)}$  are in  $\mathcal{X}_{(T)}$ , and  $X'_{(T)} \geq X''_{(T)}$  then  $u_T(X'_{(T)}) \geq u_T(X''_{(T)})$ . Define a closed space  $\mathcal{S}_T$ , whose element  $S_T$  represents a possible strategy, defined as a sequence of functions,  $s_0, s_1, \dots, s_{T-1}$ , where  $s_t = s_t(x_0, x_1, \dots, x_t)$ . Thus  $S_T$  is a sequence of rules of how to respond at given times to a given sequence of past situations. Now define the probability that the strategy  $S_T^*$  will be at least as successful as  $S'_T$ :

$$(4.1:1) \quad Pr \{ [u_T(X_{(T)}; S_T^*) \geq u_T(X_{(T)}; S'_T)] \} = \pi_T(S_T^*, S'_T), \quad \text{say}.$$

Now let  $T$  increase and consider sequences such as

$$\mathcal{S}^* = (S_1^*, S_2^*, \dots), \quad \mathcal{S}' = (S'_1, S'_2, \dots), \quad \text{etc.}$$

Suppose that there exists a limit

$$\lim_{T \rightarrow \infty} \pi_T(S_T^*, S'_T) = \pi(\mathcal{S}^*, \mathcal{S}'), \quad \text{say},$$

for any two sequences  $\mathcal{S}^*$  and  $\mathcal{S}'$  and suppose that this limit

$$(4.1:2) \quad \pi(\mathcal{S}^*, \mathcal{S}') = 1.$$

Then the rule of long run success requires that the sequence  $\mathcal{S}^*$  be chosen [or, if two or more values  $\mathcal{S}^*$  exist that satisfy (4.1:2), one of them must be chosen]. This corresponds to the common sense definition, "The best policy is the one that succeeds in the long run."

4.2. We should like to know conditions under which the rule of maximizing moral expectation implies that the rule of long run success is satisfied. As a mere example that may start a discussion among those better qualified, we shall impose a (probably unnecessarily strong) condition upon the sequence of utility functions,

$$u_1(x_1), u_2(x_1, x_2), \dots, u_T(x_1, x_2, \dots, x_T)$$

with means

$$Eu_1, Eu_2, \dots, Eu_T$$

and variances

$$\sigma_1^2, \sigma_2^2, \dots, \sigma_T^2.$$

We do not assume the successive random variables  $u_1, u_2, \dots$  to be independent

statistically. But we make the assumption that the variance  $\sigma_T^2$  tends to zero as  $T \rightarrow \infty$ . Then  $u_T$  converges in probability to  $Eu_T$ . Therefore the difference,

$$u_T(X_{(T)}; S_T^*) - u_T(X_{(T)}; S_T'),$$

converges in probability to

$$Eu_T(X_{(T)}; S_T^*) - Eu_T(X_{(T)}; S_T').$$

(See, for example, [1, especially pp. 253–255].) This difference is nonnegative if the rule of maximizing moral expectation is followed, that is, if for every  $T$  a strategy  $S_T^*$  is chosen satisfying

$$Eu_T(X_{(T)}; S_T^*) \geq Eu_T(X_{(T)}; S_T'),$$

where  $S_T'$  is another strategy in  $\mathcal{S}_T$ . Then

$$\pi(\mathcal{S}^*, \mathcal{S}') = 1$$

for every  $\mathcal{S}'$  and the rule of long run success is satisfied.

We have used here, merely to illustrate the proposed approach, a crucial assumption that is hardly plausible: that the variance of the utility function of outcomes over a horizon  $T$ , tends to zero as the horizon increases. This assumption is also unnecessarily strong as it proves more than is required: for the limiting probability  $\pi(\mathcal{S}^*, \mathcal{S}')$  to be equal to 1, it is not necessary that each of the two compared utilities converge separately to its respective expected value.

An alternative assumption might be that of statistical independence between utility functions over successive horizons, that is, between  $u_1(x_1)$ ,  $u_2(x_1, x_2)$ , . . . . But this is even less reasonable. It is possible that no plausible conditions exist under which the rule of maximizing moral expectation satisfies the rule of long run success as defined.

## 5. Summary

The rule of maximizing the expected value of utility was shown to imply that utility functions of prospects (that is, of alternative distributions of outcomes of strategies) are linear transforms of each other and are linear in the probabilities of outcomes. The rule is equivalent to the postulate that indifferent prospects are substituted for each other—provided certain other, weak postulates are granted. Finally, an attempt was made to relate the rule of maximizing the expected value of utility to a rule of aiming at a long run success. This required redefining outcomes, strategies and utilities as time sequences. The strategies discussed included those of statisticians and businessmen and can be conceived to include human decisions in general. At no point was it claimed that reasonable behavior is actually practiced by men: the paper is a study in consistent sets of norms, not an empirical study.

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