A REMARK ON CHARACTERISTIC FUNCTIONS

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Let $F_1(t), F_2(t), \ldots, F_n(t), \ldots$ be a sequence of distribution functions, and let

$$\varphi_n(x) = \int_{-\infty}^{+\infty} e^{ixt} dF_n(t)$$

be the corresponding characteristic functions. If the sequence $\{\varphi_n(x)\}$ converges over every finite interval, and if the limit is continuous at the point $x = 0$, then, as is very well known, the sequence $\{F_n(t)\}$ converges to a distribution function $F(t)$ at every point of continuity of the latter (see, for example, [1, p. 96]. It is also very well known that in this theorem convergence over every finite interval cannot be replaced by convergence over a fixed interval containing the point $x = 0$.

The situation is different if the random variables whose distribution functions are the F_n are uniformly bounded below (or above). Without loss of generality we may assume that the random variables in question are positive, so that all $F_n(t)$ are zero for t negative. The purpose of this note is to prove the following theorem.

THEOREM. *Let $F_1(t), F_2(t), \ldots, F_n(t), \ldots$ be a sequence of distribution functions all vanishing for $t \leqq 0$, and let*

$$\varphi_n(x) = \int_0^{+\infty} e^{ixt} dF_n(t), \qquad -\infty < x < +\infty.$$

If the functions $\varphi_n(x)$ tend to a limit in an interval around $x = 0$, and if the limiting function is continuous at $x = 0$, then there is a distribution function $F(t)$ such that $F_n(t)$ tends to $F(t)$ at every point of continuity of F.

PROOF. Let $z = x + iy$, and let us consider the functions

$$\varphi_n(z) = \int_0^{+\infty} e^{izt} dF_n(t) = \int_0^{+\infty} e^{ixt} e^{-yt} dF_n(t).$$

Each $\varphi_n(z)$ is regular for $y > 0$, continuous for $y \geqq 0$, and is of modulus $\leqq 1$ there. For z real, $\varphi_n(z)$ coincides with the characteristic function $\varphi_n(x)$. It is easy to see that the sequence $\{\varphi_n(z)\}$ converges in the half plane $y > 0$, and that the convergence is uniform over any closed and bounded set of this half plane. For let $z = \lambda(\zeta)$ be a conformal mapping of the half plane $y > 0$ onto the unit circle $|\zeta| < 1$, and let us consider the functions

$$\varphi_n^*(\zeta) = \varphi_n[\lambda(\zeta)]. \tag{1}$$

These functions are regular for $|\zeta| < 1$, are numerically $\leqq 1$ there and their boundary values converge to a limit on a set of positive measure situated on the circumference $|\zeta| = 1$ (this set is actually an arc). By the theorem of Khintchine [2] and Ostrowski [3], the sequence $\{\varphi_n^*(\zeta)\}$ converges for $|\zeta| < 1$, and the convergence is uniform in every circle $|\zeta| \leqq \rho$, $\rho < 1$. Going back to the half plane $y > 0$, we see that the functions $\varphi_n(z)$ converge there to a regular function $\varphi(z)$, and that the convergence is uniform over any closed and bounded set in that half plane. In particular, the convergence is uniform over any finite segment of any line

$$y = y_0\,, \qquad y_0 > 0\,.$$

We shall now show that

$$\varphi(iy) \to 1 \quad \text{as} \quad y \to +0\,. \tag{2}$$

It will again be slightly easier to consider the functions $\varphi_n^*(\zeta)$ defined by (1). They tend to a function $\varphi^*(\zeta)$ regular in $|\zeta| < 1$ and numerically $\leqq 1$ there. This function has nontangential boundary values $\varphi^*(e^{i\theta})$ for almost every θ and (as a bounded harmonic function) is the Poisson integral of $\varphi^*(e^{i\theta})$. Let us assume for simplicity that the mapping function $z = \lambda(\zeta)$ makes correspond $z = 0$ and $\zeta = 1$. If we can prove that in the neighborhood of $\theta = 0$ the function $\varphi^*(e^{i\theta})$ coincides almost everywhere with a function continuous at $\theta = 0$ and taking the value 1 at that point, then [since the values of $\varphi^*(e^{i\theta})$ in a set of measure zero are immaterial for the Poisson integral] the function $\varphi^*(\zeta)$ will tend to 1 as ζ approaches 1 along any nontangential path. This will immediately lead to relation (2).

Let us revert to the Khintchine-Ostrowski theorem used above. It can be completed as follows. *If the sequence of functions $\varphi_n^*(\zeta)$ regular and of modulus $\leqq 1$ for $|\zeta| < 1$, converges in a set E of positive measure on the circumference $|\zeta| = 1$, then on almost every radius $\zeta = \rho e^{i\theta}$, $0 \leqq \rho < 1$, terminating in the set E the sequence converges uniformly* (for the proof, see [4, p. 213]). Since the function $\varphi^*(\zeta) = \lim \varphi_n^*(\zeta)$ has nontangential limit $\varphi^*(e^{i\theta})$ for almost every θ, it immediately follows that $\varphi^*(e^{i\theta}) = \lim \varphi_n^*(e^{i\theta})$ *almost everywhere in E*. In our particular case, the functions $\varphi_n^*(\zeta)$ are continuous on $|\zeta| = 1$ except at the point ζ corresponding to $z = \infty$, and converge on an arc $-\delta \leqq \theta \leqq +\delta$ to a function $\gamma(\theta)$ continuous at $\theta = 0$ and taking the value 1 there [since $\varphi_n^*(1) = 1$ for all n]. Hence at almost every point θ in $(-\delta, \delta)$ the function $\varphi^*(e^{i\theta})$ coincides with $\gamma(\theta)$. Thus the proof of (2) is complete.

Since, as seen from the formula for $\varphi_n(z)$, all the quantities $\varphi_n(iy)$ are positive for $y > 0$, the quantity $\varphi(iy) = \lim \varphi_n(iy)$ is nonnegative. On account of (2), we have $\varphi(iy_0) > 0$ for all y_0 small enough. Let us fix such a y_0 and let us consider the nonnegative and nondecreasing functions

$$G_n(t) = \frac{1}{\varphi_n(iy_0)} \int_{-\infty}^{t} e^{-uy_0}\, dF_n(u) \tag{3}$$

[thus $G_n(t) = 0$ for $t \leqq 0$]. As seen from the formula defining $\varphi_n(z)$, the characteristic function $\psi_n(x)$ of $G_n(t)$ is

$$\int_0^\infty e^{ixt}\, dG_n(t) = \frac{1}{\varphi_n(iy_0)} \int_0^\infty e^{ixt} e^{-ty_0}\, dF_n(t) = \frac{\varphi_n(x + iy_0)}{\varphi_n(iy_0)}\,.$$

Since

$$1 = \psi_n(0) = \int_0^\infty dG_n(t),$$

it follows that the G_n are distribution functions. We know that the functions $\psi_n(x) = \varphi_n(x + iy_0)/\varphi_n(iy_0)$ converge uniformly over any finite interval of the variable x. Hence the functions $G_n(t)$ converge to a distribution function $G(t)$ at the points of continuity of G.

From (3) we see that

$$F_n(t) = \varphi_n(iy_0) \int_{-\infty}^t e^{uy_0} dG_n(u).$$

The right side here can be written

$$\varphi_n(iy_0) \left\{ e^{ty_0} G_n(t) - y_0 \int_{-\infty}^t e^{uy_0} G_n(u)\, du \right\}.$$

Hence the functions $F_n(t)$ tend to a nondecreasing function $F(t)$ at every point t at which G is continuous, and

$$(4)\quad F(t) = \varphi(iy_0) \left\{ e^{ty_0} G(t) - y_0 \int_{-\infty}^t e^{uy_0} G(u)\, du \right\} = \varphi(iy_0) \int_{-\infty}^t e^{uy_0} dG(u).$$

From this formula we see that the points of discontinuity of F are the same as those of G. It remains to show that F is a distribution function, that is that

$$(5)\qquad F(+\infty) - F(-\infty) = 1.$$

That the left side here is $\leqq 1$ is obvious since $0 \leqq F_n(t) \leqq 1$ for all n. Observing that both F and G vanish for $t < 0$, we deduce from (4) that

$$F(a) - F(-0) \geqq \varphi(iy_0) \{G(a) - G(-0)\} \quad \text{for} \quad a > 0.$$

Taking first a large, and then y_0 small, and using (2), we find that $F(+\infty) - F(-0) \geqq 1$, which gives (5). This completes the proof of the theorem.

Remark 1. The theorem can be extended to nonnegative random variables in the k-dimensional space R_k. The requirement is that the characteristic functions $\varphi_n(x_1, \ldots, x_k)$ converge in the neighborhood of $(0, \ldots, 0)$ to a function continuous at that point. The proof follows the same line as for $k = 1$, and the proofs of the corresponding lemmas for functions $\varphi_n(z_1, \ldots, z_k)$ of several complex variables offer no serious difficulties. The details are omitted here.

Remark 2. It is easy to see that the condition of the theorem, namely that all of the $F_n(t)$ vanish for $t \leqq 0$ (or for $t \leqq t_0$), can be replaced by a less stringent one:

$$F_n(t) \leqq Ae^{-\epsilon|t|}, \qquad t \leqq t_0,$$

where the positive constants A, ϵ and the constant t_0 are all independent of n.

The proof of this generalization remains essentially the same as before. For, applying integration by parts in the formula defining the function $\varphi_n(z)$, we see that the $\varphi_n(z)$ are regular in the strip

$$0 < y < \epsilon,$$

and are continuous and uniformly bounded in every closed strip

$$0 \leqq y \leqq \epsilon', \qquad \epsilon' < \epsilon.$$

In the proof given above it is therefore enough to take for $\lambda(\zeta)$ the function mapping the latter strip onto the unit circle $|\zeta| \leqq 1$ and consider only the values of y_0 sufficiently small ($y_0 < \epsilon$).

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