# CONTINUOUS PARAMETER MARTINGALES

J. L. DOOB
UNIVERSITY OF ILLINOIS

## 1. Introduction

Let $\Omega$ be an abstract space of points $\omega$, and let $Pr\{\cdot\}$ be a probability measure defined on some Borel field of $\omega$ sets. The sets of this field will be called *measurable*. A family of (real or complex valued) random variables, that is of measurable $\omega$ functions, is called a *stochastic process*. We shall use the notation $\{x(t), t \in T\}$ to denote a stochastic process. Here $T$ is the parameter set of the process, and $x(t)$ is for each $t \in T$ a random variable, taking on the value $x(t, \omega)$ for given $t$, $\omega$. For fixed $\omega$, $x(t, \omega)$ determines a function $x(\cdot, \omega)$ of $t \in T$. The functions of $t$ determined in this way are called the *sample functions* (or sample sequences if $T$ is finite or denumerable) of the process. The random variable $x(t)$ can also appropriately be denoted by $x(t, \cdot)$, but the latter notation will not be used. The phrase *almost all sample functions* will mean *for almost all* $\omega$.

Suppose that our old friend Peter is playing a fair game with his old friend Paul (or suppose that the classical situation is modernized, so that a SCIENTIST plays NATURE). Suppose that at time $t$ our protagonist has fortune $x(t)$. One mathematical version of a fair game is obtained by supposing that $x(t)$ is a random variable, and that our protagonist's expected fortune at time $t$, in view of his previous fortunes up to time $s < t$, is simply $x(s)$. More precisely our mathematical version of a fair game is a stochastic process $\{x(t), t \in T\}$ for which $T$ is a simply ordered set, for which

$$E\{|x(t)|\} < \infty, \qquad t \in T,$$

and for which

$$E\{x(t) | x(r), \quad r \leqq s\} = x(s)$$

with probability 1, if $s < t$. A stochastic process satisfying these conditions is called a *martingale*.

If $x$ is a random variable, it will be convenient to denote the $\omega$ set where $x(\omega) \in A$ by $\{x \in A\}$. Here and in the following, in this connection, it will be understood that $A$ is a linear set if the random variable $x$ is real and a plane set if $x$ is complex. The $\omega$ measure of the indicated set will be denoted by $Pr\{x \in A\}$, if this $\omega$ set is measurable. The corresponding conventions are made if more than one random variable is involved. The integral of a random variable $x$ on a measurable set $\Lambda$ will be denoted by

$$\int_\Lambda x \, dPr .$$

This research was connected with a probability project at Cornell University under an Office of Naval Research contract.

Let $\{x(t), t \in T\}$ be a stochastic process, let $t$ be a parameter value, and let $A$ be a Borel set. The $\omega$ set $\{x(t) \in A\}$ is then a measurable set. Every $\omega$ set in the Borel field of $\omega$ sets generated by those defined in this way, for all $t \in T$ and all $A$, or which differs from a set in this field by at most an $\omega$ set of probability 0, will be said to be a set *determined by conditions on the* $x(t)$'s.

We can rephrase the definition of a martingale as follows, using the terminology of the preceding paragraph. *A martingale is a stochastic process* $\{x(t), t \in T\}$, *for which T is a simply ordered set, for which*

$$E\{|x(t)|\} < \infty, \qquad t \in T, \tag{1.1}$$

*and for which*

$$\int_\Lambda x(t)\,dPr = \int_\Lambda x(s)\,dPr, \qquad s < t, \tag{1.2}$$

*for every* $\omega$ *set* $\Lambda$ *determined by conditions on the* $x(r)$'s *for* $r \leqq s$. In this version of the definition it is obvious that if the random variables of a martingale are complex valued, their real and imaginary parts determine stochastic processes which are also martingales. Using this fact it is possible to reduce theorems on complex martingales to theorems on real martingales.

The concept of fairness of a game is of course rather vague, and the martingale equality (1.2) cannot be expected to embody all our notions of fairness, or even to be consistent with all of them. However the intuitive notion of a fair game is helpful in the theoretical analysis of martingales.

The results of this paper are for the most part contained in a forthcoming book, and the justification for their publication here is the infinite time required to write the book. Previous work on martingales has been done by Lévy [1], [2], Ville [3], and the author [4]. The "known theorems" referred to below can all be found in the last reference.

## 2. Examples

(a) Let $\{x(t), a \leqq t \leqq b\}$ be a stochastic process with independent increments, that is, it is supposed that whenever

$$a \leqq t_0 < \ldots < t_n \leqq b,$$

the random variables

$$x(t_1) - x(t_0), \ldots, x(t_n) - x(t_{n-1})$$

are mutually independent. We make the following assumptions:

$$E\{|x(t) - x(s)|\} < \infty$$
$$E\{x(t) - x(s)\} = 0. \qquad a \leqq s, \quad t \leqq b$$

Then the process

$$\{x(t) - x(a), \quad a \leqq t \leqq b\}$$

has independent increments and is a martingale.

(b) Let $\{x(t), a \leqq t \leqq b\}$ be a real process with independent increments. Let $\Phi$ be the characteristic function of the random variable $x(t) - x(a)$,

$$\Phi(t, \lambda) = E\{e^{i\lambda[x(t)-x(a)]}\},$$

and define

$$\mathring{x}(t) = \frac{e^{i\lambda[x(t)-x(a)]}}{\Phi(t, \lambda)}.$$

This definition assumes that $\lambda$ is chosen so that $\Phi(t, \lambda) \neq 0$. If $\delta$ is chosen positive, and so small that

$$\Phi(b, \lambda) \neq 0, \qquad |\lambda| \leqq \delta,$$

the equation

$$\Phi(b, \lambda) = \Phi(t, \lambda) E\{e^{i\lambda[x(b)-x(t)]}\}$$

shows that

$$\Phi(t, \lambda) \neq 0, \qquad \begin{matrix} |\lambda| \leqq \delta \\ a \leqq t \leqq b \end{matrix}.$$

Thus for each $\lambda$ with $|\lambda| \leqq \delta$ there is an $\mathring{x}(t)$ process and this process is readily checked to be a martingale.

Examples (a) and (b) show how it is possible to reduce properties of processes with independent increments to properties of martingales. The following example shows how it is possible to reduce properties of Markov processes to properties of martingales.

(c) Let $\{x(t), a \leqq t \leqq b\}$ be a Markov process, and let $A$ be a Borel set. Fix $u$ in the parameter interval, and define

$$\mathring{x}(t) = Pr\{x(u) \in A \mid x(t)\}, \qquad a \leqq t \leqq u,$$

that is $\mathring{x}(t)$ is the conditional probability that $x(t, \omega) \in A$, for the past of the process given up to time $t$. Then $\mathring{x}(t)$ is a random variable and the $\mathring{x}(t)$ process is a martingale.

(d) Let $T$ be a linear set, unbounded on the right, and let

$$\{x_i(t), t \in T\}, \qquad i = 1, 2$$

be two stochastic processes (not necessarily defined on the same $\omega$ space). Let $\mathring{\omega}$ be a function of $t \in T$, real if the processes are, and complex otherwise, and let $\mathring{\Omega}$ be the space of all functions $\mathring{\omega}$. Let $t_1, \ldots, t_n$ be any finite parameter set, and let $A_1, \ldots, A_n$ be any Borel sets. Consider the class of functions $g$, with argument $t \in T$, satisfying the relations

$$g(t_j) \in A_j, \qquad j = 1, \ldots, n. \tag{2.1}$$

This class is an $\mathring{\omega}$ set. Let $\mathring{F}_t$ be the Borel field of $\mathring{\omega}$ sets generated by these sets, with $t_j$'s $\leqq t$. We define two measures of $\mathring{\omega}$ sets as follows. In the first place if $\mathring{\Lambda}$ is the $\mathring{\omega}$ set determined by the relations (2.1) we define

$$Pr_i\{\mathring{\Lambda}\} = Pr\{x_i(t_j) \in A_j, \quad j = 1, \ldots, n\}, \qquad i = 1, 2,$$

and the $i$-th set function is then extended in the usual way, following Kolmogorov, to the Borel field of sets generated by those for which we have already defined it. We thus have two probability measures defined on the same field of sets of function space $\mathring{\Omega}$. Consider the field $\mathring{F}_t$ of $\mathring{\omega}$ sets, and suppose that on this field $Pr_1$ measure is absolutely continuous with respect to $Pr_2$ measure, that is there is a random variable $\mathring{y}(t)$, defined on $\mathring{\Omega}$ and measurable with respect to the field $\mathring{F}_t$, whose expectation exists, such that

$$Pr_1\{\mathring{\Lambda}\} = \int_{\mathring{\Lambda}} \mathring{y}(t)\, dPr_2, \qquad \mathring{\Lambda} \in \mathring{F}_t.$$

The stochastic process $\{\mathring{y}(t), t \in T\}$ is a martingale relative to $Pr_2$ measure. In fact

$$E\{\mathring{y}(t)\} = \int_{\mathring{\Omega}} \mathring{y}(t)\, dPr_2 = Pr_1\{\mathring{\Omega}\} = 1 < \infty,$$

and more generally, if $\mathring{\Lambda} \in \mathring{F}_s$, and if $s < t$, so that $\mathring{\Lambda} \in \mathring{F}_t$ also,

$$\int_{\mathring{\Lambda}} \mathring{y}(t)\, dPr_2 = Pr_1\{\mathring{\Lambda}\} = \int_{\mathring{\Lambda}} \mathring{y}(s)\, dPr_2.$$

Since in particular $\mathring{\Lambda}$ can be taken as any $\mathring{\omega}$ set determined by conditions on the $\mathring{y}(r)$'s with $r \leqq s$, the latter equality means that the $\mathring{y}(t)$ process satisfies the martingale equality (1.2). When $T$ is the set of positive integers, $\mathring{y}(n)$ is called the $n$-th likelihood ratio, and plays an important role in statistics.

## 3. Martingale inequalities

The following two theorems are basic in the study of martingales.

THEOREM 3.1. *Let $\{x(t), t \in T\}$ be a real martingale with an enumerable parameter set $T$, and suppose that $T$ has a last point $b$. Then for every real $c$*

$$cPr\{\sup_{t \in T} x(t) \geqq c\} \leqq E\{|x(b)|\}. \tag{3.1}$$

This theorem is known, and is due essentially to Ville. It is proved for finite parameter sets $T$ first, and the general case is then obtained by the obvious limiting procedure. Applying the theorem to the martingale $\{-x(t), t \in T\}$ we obtain

$$cPr\{\inf_{t \in T} x(t) \leqq c\} \geqq -E\{|x(b)|\} \tag{3.1'}$$

for all real $c$, and combining the two inequalities we obtain

$$cPr\{\sup_{t \in T} |x(t)| \geqq c\} \leqq 2E\{|x(b)|\} \tag{3.2}$$

for all real $c$.

Let $c_1, \ldots, c_n$ be any real numbers, and let $r_1, r_2$ be real numbers with $r_1 < r_2$. The *number of upcrossings of the interval* $[r_1, r_2]$ by $c_1, \ldots, c_n$ is defined as the number of times the sequence $c_1, \ldots, c_n$ passes from below $r_1$ to above $r_2$. More precisely let $c_{n_1}$ be the first $c_i$ if any for which $c_i \leqq r_1$, and in general let $c_{n_j}$ be the first $c_i$ if any after $c_{n_{j-1}}$ for which

$$c_i \geqq r_2, \quad j \text{ even},$$
$$c_i \leqq r_1, \quad j \text{ odd},$$

so that

$$c_{n_1} \leqq r_1, \quad c_{n_2} \geqq r_2, \quad c_{n_3} \leqq r_1, \ldots.$$

Then the number of upcrossings is $\beta$, where $2\beta$ is the largest even integer $j$ for which $c_{n_j}$ is defined, and $\beta = 0$ if $c_{n_2}$ is not defined.

THEOREM 3.2. *Let $\{x_j, j \leqq n\}$ be a real martingale, and let $\beta$ be the random variable whose value at $\omega$ is the number of upcrossings of $[r_1, r_2]$ by the sample sequence corre-*

*sponding to* $\omega$. *Then*

$$E\{\beta\} \leqq \frac{1}{r_2 - r_1}\int_{\{x_n \geqq r_1\}} (x_n - r_1)\, dPr \leqq E\{\,|x_n - r_1|\,\}. \tag{3.3}$$

To prove the theorem define $\omega$ functions $\eta_1, \ldots, \eta_n$ in terms of the $x_1, \ldots, x_n$ sample sequences as described above in the definition of $\beta$, defining $\eta_j = n + 1$ if the above definitions have not already defined $\eta_j$. For example

$$\eta_1 = \ldots = \eta_n = n + 1$$

if $\min_j x_j > r_1$. The $\eta_j$'s and $\beta$ are now random variables, and we have

$$\int_{\{x_n \geqq r_1\}} (x_n - r_1)\, dPr \geqq \sum_{i \geqq 1} \int_{\{\eta_{2i} \leqq n, \eta_{2i+1} > n\}} (x_n - r_1)\, dPr \tag{3.4}$$

$$= \sum_{i \geqq 1} \int_{\{\eta_{2i} \leqq n\}} (x_n - r_1)\, dPr - \sum_{i \geqq 1} \int_{\{\eta_{2i+1} \leqq n\}} (x_n - r_1)\, dPr .$$

Now using the martingale equality and the definition of the $\eta_i$'s,

$$\int_{\{\eta_{2i} = j\}} (x_n - r_1)\, dPr = \int_{\{\eta_{2i} = j\}} (x_j - r_1)\, dPr \geqq (r_2 - r_1)\, Pr\{\eta_{2i} = j\}, \quad j \leqq n$$

and

$$\int_{\{\eta_{2i+1} = j\}} (x_n - r_1)\, dPr = \int_{\{\eta_{2i+1} = j\}} (x_j - r_1)\, dPr \leqq 0, \qquad j \leqq n .$$

Hence (3.4) implies that

$$\int_{\{x_n \geqq r_1\}} (x_n - r_1)\, dPr \geqq \sum_{i \geqq 1} \sum_{j=1}^{n} \int_{\{\eta_{2i} = j\}} (x_n - r_1)\, dPr$$

$$- \sum_{i \geqq 1} \sum_{j=1}^{n} \int_{\{\eta_{2i+1} = j\}} (x_n - r_1)\, dPr$$

$$\geqq \sum_{i \geqq 1} \sum_{j=1}^{n} (r_2 - r_1) Pr\{\eta_{2i} = j\}$$

$$= (r_2 - r_1) \sum_{i \geqq 1} Pr\{\eta_{2i} \leqq n\}$$

$$= (r_2 - r_1) \sum_{i \geqq 1} Pr\{\beta \geqq i\}$$

$$= (r_2 - r_1) E\{\beta\},$$

as was to be proved.

## 4. Regularity properties of martingales

Let $\{x(t), t \in T\}$ be a martingale. Then it is known that $\mathbf{E}\{x(t)\}$ is independent of $t$, and that $\mathbf{E}\{|x(t)|\}$ is monotone nondecreasing in $t$. Suppose now that $T$ is the set of positive integers, and let $\beta_n(r_1, r_2)$ be the random variable whose value for any $\omega$ is the number of upcrossings of $[r_1, r_2]$ be the sample sequence of

$x(1), \ldots, x(n)$ corresponding to the given $\omega$. Then by theorem 3.1,

$$cPr\{\max_{j \leqq n} |x(j)| \geqq c\} \leqq 2E\{|x(n)|\}, \tag{4.1}$$

and by theorem 3.2

$$E\{\beta_n(r_1, r_2)\} \leqq \frac{E\{|x(n) - r_1|\}}{r_2 - r_1}. \tag{4.2}$$

Now suppose that

$$\lim_{n \to \infty} E\{|x(n)|\} = K < \infty.$$

Then when $n \to \infty$ in (4.1) we find

$$cPr\{\sup_j |x(j)| \geqq c\} \leqq 2K.$$

This inequality implies that almost all sample sequences of the $x(n)$ process are bounded, with probability 1. Moreover

$$\beta_1(r_1, r_2) \leqq \beta_2(r_1, r_2) \leqq \ldots \to \beta_\infty(r_1, r_2),$$

and when $n \to \infty$ in (4.2) we find

$$E\{\beta_\infty(r_1, r_2)\} \leqq \frac{K + |r_1|}{r_2 - r_1}. \tag{4.3}$$

Since

$$\{\limsup_{n \to \infty} x(n) > \liminf_{n \to \infty} x(n)\} = \bigcup_{r_1, r_2} \{\limsup_{n \to \infty} x(n) > r_2 > r_1 > \liminf_{n \to \infty} x(n)\}$$
$$(r_1, r_2 \text{ rational})$$

and since almost all sample sequences of the $x(n)$ process are bounded sequences, it follows that unless $\lim_{n \to \infty} x(n)$ exists and is finite with probability 1 there is a pair $r_1, r_2$ such that

$$Pr\{\limsup_{n \to \infty} x(n) > r_2 > r_1 > \liminf_{n \to \infty} x(n)\} > 0.$$

With this choice of $r_1, r_2$,

$$Pr\{\beta_\infty(r_1, r_2) = \infty\} > 0$$

[because on an $\omega$ set of positive probability $x(n, \omega) > r_2$ and $x(n, \omega) < r_1$ for infinitely many values of $n$], and this contradicts (4.3). Thus we have proved the theorem that *if*

$$\lim_{n \to \infty} E\{|x(n)|\} < \infty,$$

*then the sequence* $\{x(n)\}$ *is convergent (to a finite limit) with probability* 1. We have given the proof of this known theorem in detail in order to illustrate the application of theorems 3.1 and 3.2, and to exhibit the close connection between this convergence theorem and the fundamental theorem on the sample functions of continuous parameter martingales (hitherto unproved) which we now proceed to prove.

THEOREM 4.1. *Let* $\{x(t), a \leqq t \leqq b\}$ *be a martingale, and let R be any denumerable subset of the interval* $[a, b]$, *everywhere dense in this interval. Then almost all sample functions of this martingale coincide on R with functions defined on* $[a, b]$ *which have finite left and right hand limits everywhere on* $[a, b]$.

The usual transition from denumerable to nondenumerable parameter sets (or rather any one of the usual transitions) shows that this statement becomes, if stochastic process measures are defined suitably, the statement that almost all sample functions of a martingale $\{x(t),\ a \leqq t \leqq b\}$ have finite left and right hand limits everywhere on $[a, b]$. The strongest previous result in this direction [4] is the existence of the left hand limit described in the theorem.

In proving the theorem we shall suppose that the martingale is real. If it is not real, the theorem can be applied to the martingales determined by the real and imaginary parts of the random variables of the given martingale. It is no restriction to assume that $R$ contains the point $b$, and we shall do so. According to theorem 3.1, as applied to get (3.2),

$$cPr\{\sup_{t\in R} |x(t)| \geqq c\} \leqq 2E\{|x(b)|\}.$$

Hence almost all sample functions are bounded on $R$. Let $t_1, t_2, \ldots$ be the points of $R$, enumerated in some order. We shall suppose that $t_1 = b$. Let $r_1, r_2$ be any real numbers with $r_1 < r_2$. Let $t_1^{(n)}, \ldots, t_n^{(n)}$ be the first $n$ $t_j$'s, ordered so that $t_1^{(n)} < \ldots < t_n^{(n)}$, and let $M_{nk}$ be the $\omega$ set corresponding to the sample sequences of $x(t_1^{(n)}), \ldots, x(t_n^{(n)})$ for which the number of upcrossings of $[r_1, r_2]$ is $\geqq k$. Then

$$M_{1k} \subset M_{2k} \subset \ldots, \tag{4.4}$$

and, using the majorant of the expected number of upcrossings provided by theorem 3.2, we find

$$Pr\{M_{nk}\} \leqq \frac{E\{|x(b) - r_1|\}}{k(r_2 - r_1)}. \tag{4.5}$$

Now suppose that a bounded sample function $g(t)$ corresponding to some $\omega$ does not coincide on $R$ with a function defined on $[a, b]$ which has finite left and right hand limits everywhere on $[a, b]$. Then there is a point $s$ such that for some rational pair $r_1, r_2$ either

$$\limsup_{t\uparrow s} g(t) > r_2 > r_1 > \liminf_{t\uparrow s} g(t), \qquad t \in R,$$

or the same inequality is true with $t \downarrow s$. Then with this choice of $r_1, r_2$ the number of upcrossings of $[r_1, r_2]$ by $g(t_1^{(n)}), \ldots, g(t_n^{(n)})$ becomes infinite when $n \to \infty$. Thus if M is the $\omega$ set corresponding to the sample functions $g(t)$

$$M \subset \bigcup_{r_1, r_2} \bigcap_k \bigcup_n M_{nk}, \qquad r_1,\ r_2 \text{ rational}. \tag{4.6}$$

Now, according to (4.4) and (4.5),

$$Pr\{\bigcup_n M_{nk}\} \leqq \frac{E\{|x(b) - r_1|\}}{k(r_2 - r_1)},$$

so that

$$Pr\{\bigcap_k \bigcup_n M_{nk}\} = 0.$$

It then follows from (4.6) that

$$Pr\{M\} = 0,$$

as was to be proved.

## 5. Discussion of the examples of section 2

(a), (b) In examples (a), (b) of section 2 we are dealing with processes with independent increments. Lévy has shown that if $\{x(t), a \leqq t \leqq b\}$ is such a process, there is a $t$ function $f$ (not depending on $\omega$) such that if

$$x_1(t) = x(t) - f(t),$$

the limit random variables

$$\lim_{s \uparrow t} x_1(s) = x_1(t-), \qquad \lim_{s \downarrow t} x_1(s) = x_1(t+)$$

exist and are finite in the following sense: the limits are to exist with probability 1 whenever $s$ approaches $t$ from below or above as the case may be, if $s$ approaches $t$ along a sequence of values. Moreover, neglecting values on $\omega$ sets of zero probability, these limits are independent of the sequence of values along which $s \to t$. The $x_1(t)$ process obviously also has independent increments. If we can take $f(t) \equiv 0$, we shall call the $x(t)$ process *centered*. For example the $x_1(t)$ process is centered. Lévy proved that if $\{x(t), a \leqq t \leqq b\}$ is a centered process with independent increments, and if $R$ is any denumerable subset of $[a, b]$, dense in $[a, b]$, then almost all sample functions of the process coincide on $[a, b]$ with functions defined on this interval which have finite left and right hand limits everywhere on the interval. In the special case of example (a) we have supposed that

$$\mathbf{E}\{x(t) - x(s)\} \equiv 0,$$

and in this case theorem 3.2 implies the Lévy result, and shows that the original process is already centered. In the general case discussed in example (b) the process may not be centered. Hence although the $\mathring{x}(t)$ process sample functions [see example (b)] are subject to theorem 3.2, we cannot come to any conclusions about the $x(t)$ process sample functions without making hypotheses insuring regularity of $\Phi(t, \lambda)$ in $t$. However if the process is centered, $\Phi(t, \lambda)$ must have right and left hand limits at each $t$, and the Lévy theorem is then easily derived from theorem 3.2.

(c) Suppose that $\{x(t), a \leqq t \leqq b\}$ is a Markov chain with stationary transition probabilities, determined by initial probabilities

$$p_i = Pr\{x(0) = i\}, \qquad i = 1, 2, \ldots$$

and transitional probabilities

$$p_{ij}(t) = Pr\{x(s+t) = j \mid x(s) = i\}, \qquad i, j = 1, 2, \ldots .$$

Then, as noted in section 2, for each $u, j$ the process

$$\{p_{x(t)j}(u - t), \quad a \leqq t \leqq u\}$$

is a martingale. A weak version of theorem 3.2 has been used to derive the continuity properties of Markov chain sample functions from those of martingale sample functions. More general Markov processes will be treated in this way in a forthcoming paper by J. R. Kinney.

(d) The general likelihood ratio $\mathring{y}(t)$ defined in example (d) of section 2 determines a martingale $\{\mathring{y}(t), t \in T\}$ with nonnegative random variables, and

$$E\{|\mathring{y}(t)|\} = E\{\mathring{y}(t)\} = 1, \qquad t \in T.$$

By hypothesis $T$ is unbounded on the right. Suppose that

$$t_1 < t_2 < \ldots, \qquad t_n \to \infty, \qquad t_n \in T.$$

Then $\{x(t_n), n \geqq 1\}$ is a martingale, and by the convergence theorem proved in section 2 it follows that

$$\lim_{n\to\infty} \mathring{y}(t_n) = \mathring{y}(\infty) \tag{5.1}$$

exists and is finite with probability 1. Since this limit exists for every such parameter sequence $\{t_n\}$, and since any two such parameter sequences can be combined into a single one, the limit $x(\infty)$ must be independent of the parameter sequence involved, neglecting values taken on $\omega$ sets of probability 0. Moreover if $t_n \to \infty$, with $t_n \in T$, (5.1) remains true even if the parameter sequence is not monotone, because the $t_n$'s can be reordered to be monotone. It can then be concluded [5, theorem 1.3] that if $R$ is a denumerable subset of $T$, unbounded on the right,

$$\lim_{t\uparrow\infty} \mathring{y}(t) = \mathring{y}(\infty), \qquad t \in R,$$

with probability 1. The random variable $\mathring{y}(t)$ is not uniquely determined. Since it was defined as a density function, it can be changed arbitrarily on an $\omega$ set of probability 0. Using this fact it can even be shown that $\mathring{y}(t)$ can be defined for each $t$ in such a way that

$$\lim_{t\uparrow\infty} \mathring{y}(t) = \mathring{y}(\infty),$$

with probability 1. Thus in these various senses there is a limiting likelihood ratio $y(\infty)$. It is easily seen that, in the terminology of section 2, $Pr_1$ measure is absolutely continuous wìth respect to $Pr_2$ measure if and only if the $\mathring{y}(t)$'s are uniformly integrable; this is in turn true if and only if

$$E\{\mathring{y}(\infty)\} = 1.$$

In the general case we can only say that

$$E\{\mathring{y}(\infty)\} \leqq 1.$$

A more detailed discussion in the special case when $T$ is the set of positive integers and [using the notation of section 2 example (d)] the random variables

$$x_i(1), \quad x_i(2), \ldots$$

are mutually independent, for $i = 1, 2$ has been given by Kakutani [6].

## REFERENCES

[1] PAUL LÉVY, *Théorie de l'Addition des Variables Aléatoires*, Gauthier-Villars, Paris, 1937.

[2] ———, *Processus Stochastiques et Mouvement Brownien*, Gauthier-Villars, Paris, 1948.

[3] JEAN VILLE, *Étude Critique de la Notion de Collectif*, Gauthier-Villars, Paris, 1939.

[4] J. L. DOOB, "Regularity properties of certain families of chance variables," *Trans. Amer. Math. Soc.*, Vol. 47 (1940), pp. 455–486.

[5] ———, "Stochastic processes depending on a continuous parameter," *Trans. Amer. Math. Soc.*, Vol. 42 (1937), pp. 107–140.

[6] SHIZUO KAKUTANI, "On equivalence of infinite product measures," *Annals of Math.*, Vol. 49 (1948), pp. 214–224.