WIENER'S RANDOM FUNCTION, AND OTHER LAPLACIAN RANDOM FUNCTIONS

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1. An application of a formula of Wiener

1.1. Let $X(t)$ be Wiener's well known random function, defined up to an additive constant by the condition

$$X(t) - X(t_0) = \xi \sqrt{t - t_0}, \qquad t > t_0, \tag{1.1.1}$$

ξ being a real and normalized Laplacian (often called Gaussian) random variable. Suppose $0 \leqq t \leqq 2\pi$, $X(0) = 0$, and put

$$X(t) = \frac{t}{2\pi} X(2\pi) + U(t). \tag{1.1.2}$$

The Laplacian function $U(t)$ is completely characterized by its covariance

$$E\{U(t)\, U(t')\} = \frac{u(2\pi - v)}{2\pi} \tag{1.1.3}$$

[$u = \min(t, t')$; $v = \max(t, t')$; $0 \leqq u \leqq v \leqq 2\pi$]. We may conclude that it may be represented by the almost surely convergent Fourier series

$$U(t) = \sum_1^\infty \frac{1}{n\sqrt{\pi}} [\xi_n (\cos nt - 1) + \xi_n' \sin nt], \tag{1.1.4}$$

and that

$$X(t) = \frac{\xi' t}{\sqrt{2\pi}} + \sum_1^\infty \frac{1}{n\sqrt{\pi}} [\xi_n (\cos nt - 1) + \xi_n' \sin nt], \tag{1.1.5}$$

the Greek letters indicating normalized Laplacian random variables, all independent of each other. To prove this, it is sufficient to verify that the Laplacian function (1.1.4) has the covariance (1.1.3).

Thus, the same random function may be defined by (1.1.1) or by (1.1.5). This theorem was proved by N. Wiener [9] in 1924 and, ten years later, formula (1.1.5) was used as a definition by Paley and Wiener. Starting from one or the other point of view, it is easy to prove that $X(t)$ is almost surely a well defined and continuous function; $\delta X(t)$ is generally $O(\sqrt{dt})$ $(dt > 0)$, and not $O(dt)$. Thus $X(t)$ is not differentiable.

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The explicit representation of $X(t)$, given by (1.1.5), is often very useful. Yet, during more than twenty years, the author and other mathematicians did not have the idea of using it. We shall now apply it to the study of the Brownian plane curve.

1.2. Let us now put

$$
(1.2.1)\qquad \begin{cases} X(t) = \dfrac{\xi' t}{\sqrt{2\pi}} + \displaystyle\sum_{1}^{\infty} \frac{1}{n\sqrt{\pi}} [\xi_n (\cos nt - 1) + \xi'_n \sin nt] \\[2ex] Y(t) = \dfrac{\eta' t}{\sqrt{2\pi}} + \displaystyle\sum_{1}^{\infty} \frac{1}{n\sqrt{\pi}} [\eta_n (\cos nt - 1) + \eta'_n \sin nt] , \end{cases}
$$

and consider the curve

$$
(\mathrm{C}) \qquad x = X(t) , \qquad y = Y(t) , \qquad 0 \leqq t \leqq 2\pi ,
$$

and its chord D. The area included by C and D may be *formally* defined by the formula

$$
(1.2.2) \qquad S = \frac{1}{2} \int_0^{2\pi} [X(t)\, dY(t) - Y(t)\, dX(t)] ,
$$

and a formal calculation leads to the formula

$$
(1.2.3) \qquad S = \sum_{1}^{\infty} \frac{1}{n} [\xi_n (\eta'_n - \eta' \sqrt{2}) - \eta_n (\xi'_n - \xi' \sqrt{2})] .
$$

Ten years ago, the author proved that, if one considers only the classical theory of integration, the integral (1.2.2) has no sense [5]. But it is easy to give a stochastic definition of that integral. Another definition results from the formula (1.2.3), in which the series is almost surely convergent. We shall now start from this new definition, and shall in 1.6 come back to the definition (1.2.2).

1.3. Consider first the conditional probability, in the case $\eta'\sqrt{2} = h$, $\xi'\sqrt{2} = k$ (h and k being given numbers). From

$$
E\{ e^{iu\xi} \} = e^{-u^2/2} ,
$$

putting $u = (\eta - h)z$, we deduce that the characteristic function of $\xi(\eta - h)$ is

$$
\omega(z, h) = E\{ e^{i\xi(\eta-h)z} \} = E\{ e^{-(\eta-h)^2 z^2/2} \}
$$

$$
= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-[(y-h)^2 z^2 + y^2]/2} dy .
$$

Thus

$$
(1.3.1) \qquad \omega(z, h) = \frac{1}{\sqrt{1 + z^2}} e^{-h^2 z^2/2(1+z^2)} ,
$$

and the characteristic function of $\xi_n(\eta'_n - h) - \eta_n(\xi'_n - k)$ is

$$
(1.3.2) \qquad \omega(z, h)\, \omega(z, k) = \frac{1}{1 + z^2} e^{-\rho^2 z^2/(1+z^2)} , \qquad h^2 + k^2 = 2\rho^2
$$

which depends only on z and ρ. Then, the conditional characteristic function of S, under the condition $\xi'^2 + \eta'^2 = \rho^2$, is

$$(1.3.3) \qquad \phi(z, \rho) = \prod_1^\infty \frac{n^2}{n^2+z^2} e^{-\rho^2 \sum_1^\infty z^2/(n^2+z^2)},$$

and may be written in the form

$$(1.3.4) \qquad \phi(z, \rho) = \frac{\pi z}{\operatorname{sh} \pi z} e^{\rho^2[1-\pi z \operatorname{ch}\pi z/\operatorname{sh}\pi z]/2}.$$

Now, to obtain the characteristic function $\phi(z)$ of S, we have to integrate with respect to ρ. Since

$$Pr\{\xi'^2+\eta'^2 < \rho^2\} = 1 - e^{-\rho^2/2},$$

we have

$$\phi(z) = \int_0^\infty \phi(z, \rho)\, e^{-\rho^2/2} \frac{d\rho^2}{2} = \frac{\pi z}{\operatorname{sh} \pi z} \int_0^\infty e^{-\rho^2 \pi z \operatorname{ch} \pi z/2 \operatorname{sh} \pi z} \frac{d\rho^2}{2},$$

and finally

$$(1.3.5) \qquad \phi(z) = E\{e^{izS}\} = \frac{1}{\operatorname{ch} \pi z}.$$

Thus $\phi(z)$ and $\phi(z, 0) = \pi z/\operatorname{sh} \pi z$, which is the conditional characteristic function of S in the case where C is known to be a closed curve, are very simple functions. Changing the unit, we shall consider the simpler functions $\phi_1(z) = 1/\operatorname{ch} z$ and $\phi_2(z) = z/\operatorname{sh} z$, and deduce some properties of the corresponding laws.

1.4. Let us first calculate the frequency functions

$$f_i(x) = \frac{1}{\pi} \int_0^\infty \cos xz\, \phi_i(z)\, dz, \qquad i = 1, 2.$$

Using the formula

$$\phi_1(z) = \frac{1}{\operatorname{ch} z} = \frac{2e^{-z}}{1+e^{-2z}} = 2\sum_0^\infty (-1)^n e^{-(2n+1)z}, \qquad z > 0,$$

we have

$$f_1(x) = \frac{1}{\pi} \int_0^\infty \sum_0^\infty [e^{-(2n+1+ix)z} + e^{-(2n+1-ix)z}]\, dz.$$

It is easy to prove that we may interchange the integration and summation signs. Then

$$f_1(x) = \frac{1}{\pi} \sum_0^\infty (-1)^n \left[\frac{1}{2n+1+ix} + \frac{1}{2n+1-ix}\right],$$

and finally

$$(1.4.1) \qquad f_1(x) = \frac{1}{2 \operatorname{ch} \frac{\pi x}{2}}.$$

By the same method, starting from

$$\phi_2(z) = \frac{z}{\operatorname{sh} z} = \frac{2ze^{-z}}{1-e^{-2z}} = 2z \sum_0^\infty e^{-(2n+1)z}, \qquad z > 0,$$

we get

$$(1.4.2) \qquad f_2(x) = \frac{\pi}{4 \operatorname{ch}^2 \frac{\pi x}{2}}.$$

These formulas are not new. Both may be found in the tables of Fourier transforms by Campbell and Forster (formulas 609.0 and 614).

From (1.4.2), we deduce that, conversely: to the characteristic function $\phi_3(z) = 1/\text{ch}^2 z$ corresponds the frequency function

$$f_3(x) = \frac{x}{2\,\text{sh}\,\frac{\pi x}{2}}. \tag{1.4.3}$$

By the same method as was used to prove (1.4.1) and (1.4.2), it is easy to calculate the moments of the functions $f_i(x)$. The results are

$$c_p = \int_{-\infty}^{+\infty} x^{2p} f_1(x)\,dx = \frac{2^{2p+2}(2p)!}{2p+1} \sum_0^{\infty} \frac{(-1)^n}{(2n+1)^{2p+1}}, \tag{1.4.4}$$

$$c'_p = \int_{-\infty}^{+\infty} x^{2p} f_2(x)\,dx = (4^p - 2) B_p, \tag{1.4.5}$$

$$c''_p = \int_{-\infty}^{+\infty} x^{2p} f_3(x)\,dx = \frac{4^{p+1}(4^{p+1}-1)}{2p+2} B_{p+1}, \tag{1.4.6}$$

where the B_n's are the Bernoulli numbers. They are valid even for $p = 0$, if we define $B_0 = -1$. From (1.4.5) and (1.4.6), we deduce that c'_p and c''_p are rational numbers. From

$$\phi_1(iz) = \frac{1}{\cos z} = \sum_0^{\infty} c_p \frac{z^{2p}}{(2p)!}, \qquad |z| < \frac{\pi}{2}, \tag{1.4.7}$$

we deduce that c_p is also a rational number.

Let us notice that from $\phi_3(z) = \phi_1^2(z)$ results $f_3(x) = f_1(x) * f_1(x)$, or $f_3\left(\frac{2x}{\pi}\right) = \frac{2}{\pi} f_1\left(\frac{2x}{\pi}\right) * f_1\left(\frac{2x}{\pi}\right)$, that is,

$$\int_{-\infty}^{+\infty} \frac{dy}{\text{ch}\,y\,\text{ch}\,(x-y)} = \frac{2x}{\text{sh}\,x}. \tag{1.4.8}$$

1.5. We shall now prove that the three distributions considered are infinitely divisible and have no Laplacian term. It is clearly sufficient to prove that $\log \phi_i(z)$ $(i = 1, 2, 3)$ may be written in the form

$$\log \phi_i(z) = \int_0^{\infty} (\cos zu - 1)\, n_i(u)\,du, \qquad n_i(u) > 0.$$

This is a consequence of the formulas

$$-\log\left(1 + \frac{z^2}{a^2}\right) = 2\int_0^{\infty} (\cos zu - 1)\, e^{-au} \frac{du}{u}, \qquad a > 0, \tag{1.5.1}$$

$$\log \phi(z) = \log \phi_1(\pi z) = \tfrac{1}{2} \log \phi_3(\pi z) = -\sum_0^{\infty} \log\left[1 + \frac{4z^2}{(2n+1)^2}\right],$$

$$\log \phi(z, 0) = \log \phi_2(\pi z) = -\sum_0^{\infty} \log\left(1 + \frac{z^2}{n^2}\right).$$

From these formulas, we deduce immediately

$$\log \phi(z) = \int_0^\infty \frac{\cos zu - 1}{\operatorname{sh}\frac{u}{2}} \frac{du}{u}, \tag{1.5.2}$$

$$\text{og } \phi(z, 0) = \int_0^\infty \frac{\cos zu - 1}{e^u - 1} \frac{du}{u}, \tag{1.5.3}$$

and we obtain the values of $\log \phi_1(z)$ and $\log \phi_2(z)$ if we change z to z/π and u to πu.

Let us also notice that, from (1.3.1), (1.5.1), and

$$\int_0^\infty (\cos zu - 1)\, e^{-u} du = \frac{z^2}{1+z^2}$$

we deduce

$$\log \omega(z, h) = \int_0^\infty (\cos zu - 1)\left(1 + \frac{h^2}{2} u\right) e^{-u} \frac{du}{u}. \tag{1.5.4}$$

Consequently $\omega(z, h)$ is also the characteristic function of an infinitely divisible distribution, and, from

$$\phi(z, \rho) = \prod_1^\infty \omega^2\left(\frac{z}{n}, \rho\right),$$

we deduce that the same conclusion holds for $\phi(z, \rho)$.

1.6. Let us now come back to the definition of S by the integral (1.2.2). It is not difficult to prove that the stochastic definition of this integral leads almost surely to the same value as the definition of S by the series (1.2.3). But it is perhaps of greater interest to show that the fundamental formula (1.3.5) may be deduced directly from the stochastic definition of the integral (1.2.2).

Let us put

$$I = \int_0^1 X^2(t)\, dt, \qquad J = \int_0^1 Y^2(t)\, dt,$$

and consider the area s analogous to S, but limited by the arc $0 \leqq t \leqq 1$ of the curve C. In his first paper concerning this curve [4], the author proved that

$$s = \tfrac{1}{2} \xi \sqrt{I+J},$$

ξ being again a normalized Laplacian random variable, independent of $I + J$. As $2s$ and S/π depend on the same law, we have to prove that

$$E\{ e^{iz\xi\sqrt{I+J}} \} = \frac{1}{\operatorname{ch} z}.$$

Again putting

$$\frac{1}{\cos z} = \sum_0^\infty c_n \frac{z^{2n}}{(2n)!}, \qquad |z| < \frac{\pi}{2},$$

as s depends clearly on a symmetric distribution, it is sufficient to prove that

$$E\{\xi^{2n}(I+J)^n\} = c_n, \qquad n = 1, 2, \ldots. \tag{1.6.1}$$

Now, Cameron and Martin proved, six years ago [1], that

$$E\{ e^{izI} \} = (\cos \sqrt{2iz})^{-1/2}. \tag{1.6.2}$$

Hence

$$E\{e^{iz(I+J)}\} = \frac{1}{\cos\sqrt{2iz}} = \sum_0^\infty c_n \frac{(2iz)^n}{(2n)!}, \tag{1.6.3}$$

$$E\{(I+J)^n\} = \frac{2^n n!}{(2n)!} c_n, \tag{1.6.4}$$

$$E\{\xi^{2n}(I+J)^{2n}\} = E\{\xi^{2n}\}E\{(I+J)^{2n}\} = \frac{(2n)!}{2^n n!}\frac{2^n n!}{(2n)!} c_n = c_n, \tag{1.6.5}$$

which was to be proved.

1.7. Let us now end this first section by remarks which are connected with the work of M. Kac and J. F. Siegert [4]. Let ϕ be a quadratic functional of $X(t)$ and $Y(t)$. By (1.2.1), it becomes a quadratic function of the variables ξ', η', ξ_n, ξ'_n, η_n, η'_n ($n = \pm 1, \pm 2, \dots$), and may be written as a finite or infinite sum

$$\sum \lambda_n \frac{\xi_n^2}{2}, \tag{1.7.1}$$

the ξ_n being normalized Laplacian variables, independent of each other, and not the same as in (1.2.1). Consequently,

$$E\{e^{iz\phi}\} = \prod (1 - i\lambda_n z)^{-1/2}. \tag{1.7.2}$$

Three particular cases may be considered: 1°. ϕ has a symmetric distribution; 2°. ϕ is the sum of two independent quadratic functionals having the same distribution; 3°. The two preceding circumstances are both realized. In these three cases, the sum (1.7.1) becomes, respectively,

$$\sum \frac{\lambda_n}{2}(\xi_n^2 - \xi_n'^2), \quad \sum \frac{\lambda_n}{2}(\xi_n^2 + \eta_n^2), \quad \sum \frac{\lambda_n}{2}(\xi_n^2 - \xi_n'^2 + \eta_n^2 - \eta_n'^2), \tag{1.7.3}$$

and the characteristic function (1.7.2) becomes

$$\prod \frac{1}{\sqrt{1+\lambda_n^2 z^2}}, \quad \prod \frac{1}{1-\lambda_n iz}, \quad \prod \frac{1}{1+\lambda_n^2 z^2}. \tag{1.7.4}$$

The last case is realized by the area S, and also by the area of a triangle having as vertices three points of C. For that triangle, the initial proof of the author [5, p. 521–522], is unnecessarily complicated. The area of such a triangle is

$$\frac{c}{2}(\xi\eta' - \eta\xi') = \frac{\lambda}{2}(\xi_1^2 - \xi_1'^2 + \eta_1^2 - \eta_1'^2), \qquad \lambda = \frac{c}{2}.$$

Hence its characteristic function is $1/(1 + \lambda^2 z^2)$ and its frequency function is $[1/(2\lambda)] \exp(-|x|/\lambda)$.

As for the integral I, we shall give a proof which is simpler than those given by R. H. Cameron and W. T. Martin [1], or by P. Erdös and M. Kac [2], or even by R. Fortet [3]. If $X(t)$ is written in the form

$$X(t) = \int_0^t \xi_u \sqrt{du},$$

the integral I becomes

$$I = \int_0^1 dt \int_0^t \xi_u \sqrt{du} \int_0^t \xi_v \sqrt{dv} = \int_0^1 \int_0^1 [1 - \max(u, v)]\, \xi_u \xi_v \sqrt{du\,dv},$$

or, changing u and v to $1 - t$ and $1 - u$, and writing again ξ_t and ξ_u instead of ξ_{1-t} and ξ_{1-u},

$$I = \int_0^1 \int_0^1 \min(t, u)\, \xi_t \xi_u \sqrt{dt du}. \tag{1.7.5}$$

This integral may be considered as a quadratic form in Hilbert space. The λ_n are the fundamental values of the integral equation

$$2\lambda \int_0^1 \min(t, u)\, f(u)\, du = f(t), \qquad 0 \leqq t \leqq 1. \tag{1.7.6}$$

Now, we shall end the proof exactly as R. Fortet does. This equation may be written

$$2\lambda \int^t u f(u)\, du + 2\lambda t \int_0^1 f(u)\, du = f(t). \tag{1.7.7}$$

By two successive derivations, we have

$$2\lambda \int_t^1 f(u)\, du = f'(t) \tag{1.7.8}$$

$$-2\lambda f(t) = f''(t). \tag{1.7.9}$$

From (1.7.7) and (1.7.8), we deduce

$$f(0) = f'(1) = 0.$$

Then the values of the λ_n's are

$$\lambda_n = (2n+1)^2 \frac{\pi^2}{8}, \qquad n = 0, 1, \ldots, \tag{1.7.10}$$

and we obtain finally

$$I = \frac{4}{\pi^2} \sum_0^\infty \frac{\xi_n^2}{(2n+1)^2}, \tag{1.7.11}$$

$$E\{e^{izI}\} = \prod_0^\infty \left[1 - \frac{8iz}{(2n+1)^2 \pi^2}\right]^{-1/2} = \frac{1}{\sqrt{\cos\sqrt{2iz}}}, \tag{1.7.12}$$

which was to be proved.

2. A general Laplacian process

2.1. *The fundamental stochastic infinitesimal equation.* We shall now consider a complex Laplacian process, beginning at the time t_0 ($t_0 \geqq -\infty$), and suppose it may be defined, up to an additive constant, by the equation

$$\delta X(t) \sim dt \int_{t_0}^t F(t, u)\, dX(u) + \zeta \sigma(t) \sqrt{dt}, \tag{2.1.1}$$

$t \geqq t_0$, $dt > 0$, $\zeta = \dfrac{\xi + i\eta}{\sqrt{2}}$ is a complex normalized Laplacian random variable. The symbol $\sim$ means that, as $dt \to 0$, the two first moments of the difference of the two sides are $o(dt)$ or $o[d\omega(t)]$ [if $\sigma^2(t)dt$ is replaced by $d\omega(t)$].

The first term on the right hand is the value of $E'\{\delta X(t)\}$, that is, the expected value of $\delta X(t)$ which we may estimate at the time t if all the past values of the considered function are known. The expected value estimated at the time t_0 will be written $E\{\delta X(t)\}$.

Although equation (2.1.1) is very general, it implies two important restrictions: 1°. $E'\{\delta X(t)\}$ may be written in the form $\mu dt + o(dt)$; 2°. the random term may be written in the form $\zeta\sigma(t)\sqrt{dt} + o(\sqrt{dt})$.

Clearly, it would be possible to write this second term in the form $\zeta\sqrt{d\omega(t)} + o[d\omega(t)]$, $\omega(t)$ being a never decreasing function. In this case, at least if $\omega(t)$ is everywhere increasing, we could change the variable and reduce $d\omega(t)$ to the form $\sigma^2(t)dt$ [even to the form dt; we shall suppose $\sigma^2(t) < M < \infty$, but not necessarily $= 1$; thus we keep the intervals in which $\omega(t)$ has a constant value]. Then, the restriction implied by (2.1.1) is that simultaneously the first term may be written in the form $\mu dt + o(dt)$ and the second in the form $\zeta\sigma(t)\sqrt{dt} + o(\sqrt{dt})$.

Now, if the process is a Laplacian process, μ is a linear functional, and, if the process is not modified by the addition of a constant term, μ may be written, at least formally, in the form of a Stieltjes integral. But generally $X(u)$ is not a function of bounded variation, and μ will be expressed as a Young integral or as a stochastic integral.

If the function $F(t, u)$ is real, the process will be called a really correlated process. In that case, the two terms of $X(t) = X_0(t) + iX_1(t)$ are independent of each other.

2.2. *The integration of the fundamental equation.* We shall put

$$\omega(t) = \int_{t_0}^{t} \sigma^2(u)\, du\,,$$

$$Z(t) = \frac{X(t) + iY(t)}{\sqrt{2}}\,,$$

$$W(t) = Z[\omega(t)]\,,$$

$X(t)$ and $Y(t)$ being the same functions as in section 1. Then $Z(t)$ is the complex Wiener function, and $W(t)$ is a generalized complex Wiener function. Now, $X(t)$ and $Y(t)$ will no longer have the same meaning as in section 1, and $X(t)$ will be every (necessarily complex) solution of (2.1.1). If we put

$$X(t) = W(t) + V(t)\,, \tag{2.2.1}$$

$$\int_{t_0}^{t} F(t, u)\, dW(u) = \phi(t)\,, \tag{2.2.2}$$

equation (2.1.1) is transformed into

$$\delta V(t) = dt \int_{t_0}^{t} F(t, u)\, dV(u) + \phi(t)\, dt + o(dt)\,.$$

Thus $V(t)$ has a derivative $V'(t)$, which is a solution of Volterra's integral equation

$$V'(t) - \int_{t_0}^{t} F(t, u)\, V'(u)\, du = \phi(t)\,. \tag{2.2.3}$$

Now, we have to use the well known resolving kernel, which may be expressed as the sum of a rapidly convergent series, and satisfies the equations

$$(2.2.4)\qquad \begin{cases} F(t, u) + R(t, u) = \int_u^t F(t, v) R(v, u)\, dv \\ \qquad\qquad\qquad\quad = \int_u^t R(t, v) F(v, u)\, dv, \qquad t_0 < u < t. \end{cases}$$

Then the solution of (2.2.3) is

$$(2.2.5)\qquad V'(t) = \phi(t) - \int_{t_0}^t R(t, u)\, \phi(u)\, du.$$

If we substitute the value (2.2.2) of $\phi(t)$, and use (2.2.4) to simplify the resulting formula, we have

$$(2.2.6)\quad V'(t) = -\int_{t_0}^t R(t, u)\, dW(u) = -\int_{t_0}^t R(t, u)\, \zeta_u \sigma(u) \sqrt{du},$$

and finally the solution of (2.1.1) which has 0 as initial value is

$$(2.2.7)\qquad X(t) = \int_{t_0}^t \left[1 - \int_u^t R(v, u)\, dv\right] \zeta_u \sqrt{d\omega(u)}$$

the ζ_u being independent values of the random variable ζ.

As to continuity, the properties of $X(t)$ result from the fact that $X(t) - W(t)$ is differentiable. It is necessary to suppose $F(t, u)$ so chosen that the integral (2.2.2) exists; it is sufficient that $F(t, u)$ be continuous and have a continuous derivative with respect to u; but less restrictive conditions are also sufficient.

2.3. *The covariance of* $X(t)$. 1°. We shall put

$$E\{X(t)\, \overline{X}(u)\} = \Gamma(t, u).$$

Under not very restrictive conditions, the derivative

$$\gamma(t, u) = \frac{\partial^2 \Gamma(t, u)}{\partial t \partial u}$$

exists, except in the case $t = u$, and we may write

$$(2.3.1)\qquad E\{\delta X(t)\, \delta\overline{X}(u)\} = \gamma(t, u)\, dt du + o(dt du), \qquad t \neq u.$$

If $t = u$, we find from (2.2.1) that

$$(2.3.2)\qquad E\{|\delta X(t)|^2\} = \sigma^2(t)\, dt + o(dt).$$

It is easy to obtain a more precise value of $E\{|\delta X(t)|^2\}$. If (2.1.1) is multiplied by $\overline{X}(t)$, we have

$$(2.3.3)\quad E\{|\delta X(t)|^2\} = \sigma^2(t)\, dt + dt^2 \int_{t_0}^t F(t, u)\, \gamma(u, t)\, du + o(dt^2).$$

If now we multiply (2.1.1) by $\overline{X}(x)(x < t)$, we obtain the fundamental integral equation

$$(2.3.4)\qquad \gamma(t, x) = \sigma^2(x) F(t, x) + \int_{t_0}^t F(t, u)\, \gamma(u, x)\, du, \qquad t_0 \leqq x < t.$$

As we shall see later this equation is very useful if $\gamma(t, x)$ and $\sigma^2(x)$ are given, and if we have to determine $F(t, x)$. But now $\gamma(t, x)$ is unknown. It would be necessary to know this function for $t \leqq x$. Then, for each given x, this equation is again a Volterra integral equation, from which we may deduce $\gamma(t, x)$ for $t > x$. If we know only the values of $\gamma(x, x)$, it is possible to deduce $\gamma(t, x)$ from (2.3.4) and from the evident Hermitian condition $\gamma(x, t) = \bar{\gamma}(t, x)$. But, to determine $\gamma(x, x)$ it is necessary to use a third condition, which may be deduced from (2.3.3). But this way is complicated. To determine $\gamma(t, x)$, it is more convenient to start from the formulas of 2.2.

2°. From (2.2.1) and (2.2.6), we deduce

$$\delta X(t) \sim -dt \int_{t_0}^{t} R(t, v)\, \zeta_v \sigma(v) \sqrt{dv} + \zeta_t \sigma(t) \sqrt{dt},$$

$$\delta \bar{X}(u) \sim -du \int_{t_0}^{u} \bar{R}(u, v)\, \bar{\zeta}_v \sigma(v) \sqrt{dv} + \bar{\zeta}_u \sigma(u) \sqrt{du}.$$

Hence, if $u < t$

$$E\{\delta X(t)\, \delta \bar{X}(u)\} = dt du \int_{t_0}^{u} R(t, v) \bar{R}(u, v)\, \sigma^2(v)\, dv - dt du\, R(t, u)\, \sigma^2(u) + o(dt du)$$

and

$$\gamma(t, u) = -R(t, u)\, \sigma^2(u) + \int_{t_0}^{u} R(t, v) \bar{R}(u, v)\, \sigma^2(v)\, dv, \quad u < t. \tag{2.3.5}$$

Then $\gamma(t, t)$ may be defined as the limit value of $\gamma(t, u)$, and, if $u > t$, $\gamma(t, u) = \bar{\gamma}(u, t)$.

3°. To obtain $\Gamma(t, u)$ we may start again from formula (2.2.7). We have at once

$$\Gamma(t, u) = \int_{t_0}^{m} d\omega(x) \left[1 - \int_{x}^{t} R(v, x)\, dv\right]\left[1 - \int_{x}^{u} \bar{R}(v, x)\, dv\right], \tag{2.3.6}$$

m being the smallest of the two numbers t and u.

It may be useful to see also how $\Gamma(t, u)$ may be deduced from $\sigma^2(t)$ and $\gamma(t, u)$. Assuming again $X(t_0) = 0$, we have

$$\Gamma(t_0, u) = \Gamma(t, t_0) = 0, \qquad \Gamma'_t(t, t_0) = 0,$$

and consequently

$$\Gamma'_t(t, u) = \int_{t_0}^{u} \gamma(t, v)\, dv, \qquad t_0 \leqq u \leqq t. \tag{2.3.7}$$

From (2.3.1), we deduce

$$\Gamma(t + dt, t + dt) = E\{[X(t) + \delta X(t)][\bar{X}(t) + \delta \bar{X}(t)]\}$$
$$= \Gamma(t, t) + \sigma^2(t)\, dt + [\Gamma'_t(t, u) + \bar{\Gamma}'_t(t, u)]_{u=t} dt + o(dt)$$

and consequently

$$\frac{d\Gamma(t, t)}{dt} = \sigma^2(t) + \int_{t_0}^{t} [\gamma(t, v) + \bar{\gamma}(t, v)]\, dv,$$

and

$$\Gamma(t, t) = \int_{t_0}^{t} \sigma^2(v)\, dv \int_{t_0}^{t} dv \int_{t_0}^{v} [\gamma(v, w) + \bar{\gamma}(v, w)]\, dw. \tag{2.3.8}$$

Then, from

$$\Gamma(t, u) = \Gamma(u, u) + \int_u^t \Gamma'_v(v, u)\, dv, \qquad t_0 \leqq u \leqq t$$

using (2.3.7), we deduce

$$(2.3.9) \quad \Gamma(t, u) = \int_{t_0}^{u} dw \left[\sigma^2(w) + \int_w^{t'} \gamma(v, w)\, dv + \int_w^{u'} \bar{\gamma}(v, w)\, dv \right]$$

where $t' = \max(t, u)$ and $u' = \min(t, u)$.

4°. If $\sigma^2(t)$ and $F(t, u)$ are given, and if we want to know if the process has stationary increments, a necessary condition is clearly that $\sigma^2(t) = \sigma^2 =$ const. If this condition is satisfied, we have to calculate $\gamma(t, u)$, and a necessary and sufficient condition is $\gamma(t, u) = \phi(t - u)$.

Let us also notice that if the process is defined by an equation such as

$$(2.3.10) \qquad \delta X(t) \sim dt \int_{t_0}^{t} f(t, u)\, X(u)\, du + \zeta\sigma(t)\, \sqrt{dt},$$

the fundamental integral equation becomes

$$(2.3.11) \qquad \frac{\partial}{\partial t} \Gamma(t, x) = \int_{t_0}^{t} f(t, u)\, \Gamma(u, x)\, du, \qquad t_0 \leqq x \leqq t.$$

It is easy to solve these equations directly as we did equations (2.1.1) and (2.3.4); in this case, it is not necessary to use the intermediary of $\gamma(t, u)$.

5°. Let us now consider the inverse problem. The process is defined, up to an additive constant, by the data of $\sigma(t)$ and $\gamma(t, u)$, and we want to determine $F(t, u)$.

Clearly, $\sigma^2(t)dt$ and $F(t, u)dtdu$ are the variance and the covariance of the infinitesimal increments $\delta X(t)$. Then the data are acceptable, and determine a stochastic process in the interval (t_0, T), if and only if Loève's condition is satisfied, that is, if, for every measurable function $\phi(t)$, we have

$$(2.3.12) \quad \int_{t_0}^{T} \sigma^2(t)\, |\phi(t)|^2\, dt + \int_{t_0}^{T} \int_{t_0}^{T} \gamma(t, u)\, \phi(t)\, \bar{\phi}(u)\, dt du \geqq 0.$$

Let us at first suppose that this quadratic functional is of strictly positive type; if this hypothesis holds for the interval (t_0, T), it may be applied in (t_0, t), if $t_0 < t \leqq T$. Then: *for every given t $(t_0 \leqq t \leqq T)$, the integral equation* (2.3.4) *is a Fredholm equation, with determinant* $\Delta(t) \neq 0$, *and consequently* $F(t, x)(t_0 \leqq x \leqq t)$ *is determined by this equation.*

PROOF. If we had $\Delta(t) = 0$, there would exist a function $\phi(t)$ (not a.e. $= 0$) such that, for the considered value t_1 of t, we would have

$$\sigma^2(x)\, \phi(x) + \int_{t_0}^{t_1} \phi(u)\, \gamma(u, x)\, du = 0.$$

Then, if we multiply by $\bar{\phi}(x)$ and integrate, we see that the quadratic functional (2.3.12) should not be of strictly positive type. Consequently, in the case considered above, $\Delta(t) \neq 0$ for every $t \leqq T$.

Now, if T is the smallest root of $\Delta(t)$, the preceding conclusion is valid so long as $t < T$. But, if $t = T$, there exists at least a function $\phi(t)$, not a.e. $= 0$, such that

$$\sigma^2(x)\phi(x) + \int_{t_0}^{T} \phi(u)\gamma(u, x)\,du = 0\,, \qquad t_0 \leqq x \leqq T\,.$$

Then, the integral

$$\phi = \int_{t_0}^{T} \phi(u)\,dX(u)$$

is a linear functional of the increments $\delta X(x)$, and verifies

$$E\{\phi\,\delta\bar{X}(x)\} = 0\,, \qquad t_0 \leqq x \leqq T\,,$$

and we have almost surely

(2.3.13) $$\phi = \int_{t_0}^{T} \phi(u)\,dX(u) = 0\,.$$

Now, if $t \geqq T$, the process which is defined by the equation (2.1.1) is not modified by the addition of ϕdt to the right hand. Then, the process may be deterministic or undeterministic; in every case, $\Delta(t)$ remains $= 0$ for every $t \geqq T$, and there is no hope, if the process is defined by the data $\sigma(x)$ and $\gamma(t, x)$, to deduce from the equation (2.3.4) a well defined value of $F(t, x)$.

2.4. *The explicit representation of $X(t)$ by a Fourier series.* The following results are a generalization of Wiener's formula (1.1.5), and are connected with a paper by M. Kac and A. J. F. Siegert [4].

We shall suppose the process defined for $0 \leqq t \leqq 2\pi$, and

(2.4.1) $$Pr\{X(2\pi) = X(0)\} = 1\,.$$

Let us first observe that this is not an essential restriction. In the most general case, if we put

(2.4.2) $$X(t) - X(0) = \frac{t}{2\pi}[X(2\pi) - X(0)] + X_1(t)\,,$$

$X(t)$ is reduced to a function which verifies the condition (2.4.1).

For instance, if $X(t)$ is a Wiener function, $X_1(t)$ may be considered as the Laplacian random function for which

(2.4.3) $$\Gamma(t, u) = E\{X_1(t)\bar{X}_1(u)\} = \frac{u(2\pi - t)}{2\pi}\,, \qquad 0 \leqq u \leqq t \leqq 2\pi\,,$$

or as the solution of the equation

(2.4.4) $$\delta X_1(t) \sim -\frac{dt}{2\pi - t}X_1(t) + \zeta\sqrt{dt}\,, \qquad 0 \leqq t < 2\pi,\ dt > 0\,,$$

with the condition $X_1(0) = 0$. This equation may be written

(2.4.5) $$\delta X_1(t) \sim -dt\int_0^t \frac{dX(u)}{2\pi - t} + \zeta\sqrt{dt}\,.$$

Consequently, it is a particular case of (2.1.1).

More generally, suppose that in (2.1.1) we have

$$F(t, u) = -f(t)[1 + \epsilon(t, u)]$$

with the following conditions: 1°. $f(t) > 0$, at least if $t > t_0 (0 < t_0 < 2\pi)$; 2°. The function $f(t)$ tends to $+\infty$ as $t \to 2\pi$, and the integral

$$\int_{t_0}^{2\pi} f(t)\, dt$$

is infinite. 3°. $\eta(t) = \max \epsilon(t, u)(0 \leqq u \leqq t)$ tends to 0 as $t \to 2\pi$; 4°. The integral

$$I(t) = \int_0^{2\pi} \epsilon(t, u)\, dW(u)$$

exists almost surely, at least as a stochastic integral, and

$$Pr\{I(t) \to 0 | t \to 2\pi\} = 1 .$$

If these conditions are fulfilled and if $X(t)$ satisfies the equation

$$\delta X(t) \sim -f(t)\, dt \int_{t_0}^{t} [1 + \epsilon(t, u)]\, dX(u) + \sigma(t)\, \zeta \sqrt{dt}, \tag{2.4.6}$$

$0 \leqq t < 2\pi$, $dt > 0$, it is easy to prove that conclusion (2.4.1) is valid. Then a solution of this equation may be represented by a Fourier series,

$$X(t) \sim \sum{}' A_n e^{nit}; \tag{2.4.7}$$

the summation $\sum'$ extends over all integers $\neq 0$ (> 0 or < 0); the A_n are Laplacian variables; the joint probability distribution of these variables is a multivariate Laplacian distribution. We write $\sim$ and not $=$, because the series is generally not convergent in the usual sense. But, if $F(t, u)$ is continuous (with respect to t), $X(t)$ is a continuous function and the series is convergent in the sense of Fejer.

The A_n are determined by the Fourier formula

$$A_n = \frac{1}{2\pi} \int_0^{2\pi} X(t)\, e^{-nit} dt, \tag{2.4.8}$$

and the process is determined by the covariance

$$\begin{aligned} E_{p,q} = E\{A_p \bar{A}_q\} &= \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} e^{i(qu-pt)} E\{X(t)\, \bar{X}(u)\, du \\ &= \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} e^{i(qu-pt)} \Gamma(t, u)\, dt du . \end{aligned} \tag{2.4.9}$$

Conversely, if the $E_{p,q}$ are known, $\Gamma(t, u)$ may be calculated by the formula

$$\Gamma(t, u) = \sum{}' \sum{}' E_{p,q} e^{i(pt-qu)} . \tag{2.4.10}$$

In the most general case, $A_1, A_{-1}, A_2, \ldots$, may be expressed as linear functions of independent Laplacian variables $\zeta_1, \zeta_{-1}, \zeta_2, \ldots$. An important particular case occurs if the A_n are themselves independent Laplacian variables.

THEOREM. *The Laplacian variables A_n are independent if, and only if,* $\Gamma(t, u) = g(t - u)$, *that is, if the process is a process with stationary increments.*

PROOF. 1°. If $E_{p,q} = 0$ ($p \neq q$) and $E_{p,p} = E_p$, we deduce from (2.4.9)

$$\Gamma(t, u) = \sum{}' E_n e^{in(t-u)} = g(t-u) .$$

2°. If $\Gamma(t, u) = g(t - u)$, putting in (2.4.8) $t = u + v$, we have

$$E_{p,q} = \frac{1}{4\pi^2} \int_0^{2\pi} e^{i(p-q)u} du \int_0^{2\pi} e^{ipv} g(v)\, dv .$$

Then, if $p \neq q$, $E_{p,q} = 0$. Thus the theorem is proved.

If $p = q = n$, we have

$$E_n = \sigma_n^2 = \frac{1}{2\pi} \int_0^{2\pi} e^{int} g(t)\, dt \tag{2.4.11}$$

and the formula (2.4.7) may be written in the form

$$X(t) \sim \sum{}' \sigma_n \zeta_n e^{nit} . \tag{2.4.12}$$

This is a stationary function. Naturally, if we consider the solution of (2.4.6) which vanishes for $t = 0$ (and for $t = 2\pi$), we have to write

$$X(t) = \sum{}' \sigma_n \zeta_n (e^{nit} - 1) , \tag{2.4.13}$$

and it is no longer a stationary function.

As an important particular case, we may consider the Wiener function $X_1(t)$, defined by (2.4.3) and (2.4.4). Let us observe that $\Gamma(t, u)$, is then a Green's function for the operator $\frac{d^2}{dt^2}$, that is, the integral

$$f_q(t) = \int_0^{2\pi} \Gamma(t, u)\, e^{iqu} du$$

is the solution of $f_q''(t) = -e^{iqt}$, vanishing at $t = 0$ and $t = 2\pi$. Then

$$f_q(t) = \frac{e^{iqt} - 1}{q^2}, \qquad q \neq 0 ,$$

and, from (2.4.8), we deduce

$$E_{p,q} = \frac{1}{4\pi^2} \int_0^{2\pi} \frac{e^{i(q-p)t} - e^{-ipt}}{q^2}\, dt = 0$$

if $pq(p - q) \neq 0$, and

$$E_n = \frac{1}{2\pi n^2}, \qquad n \neq 0 .$$

Then

$$X_1(t) \sim \sum{}' \frac{\zeta_n e^{int}}{n\sqrt{2\pi}},$$

and the complex Wiener function $X(t)$, if $X(0) = 0$, is given by

$$X(t) = \frac{t}{2\pi} X(2\pi) + X_1(t) \sim \frac{t}{\sqrt{2\pi}} \zeta_0 + \sum{}' \frac{\zeta_n e^{nit}}{\sqrt{2\pi}}, \qquad 0 \leqq t \leqq 2\pi ,$$

where ζ_0 is independent of the other ζ_n. In the complex form, this is the well known formula of Wiener.

2.5. *A new particular case.*[1] Let A and B be two points in the Euclidean plane, $r(A, B)$ the distance between these points, and $U(A)$ a complex Laplacian random

[1] The main results of this n^r were stated without proof in the author's earlier papers [7], [8].

function of A, defined, up to an additive constant, by the condition

$$E\{|U(B) - U(A)|^2\} = r(A, B). \tag{2.5.1}$$

The existence of this function was proved in the author's book [6, pp. 277–281].

Now let $X(t)$ be the value of $U(M)$ when M is the point $x = \cos t$, $y = \sin t$. $X(t)$ is clearly a stationary and periodic random function. As to the additive constant, we shall consider three particular cases, and denote $X(t)$ as

$$\begin{array}{lll} X_0(t) & \text{if} & \int_0^{2\pi} X(t)\,dt = 0, \\ X(t) & \text{if} & X(0) = X(2\pi) = 0, \\ X_1(t) & \text{if} & U(o) = 0 \qquad (o \text{ being the point } x = y = 0). \end{array}$$

From (2.5.1), we deduce

$$E\{|X_1(t)|^2\} = 1, \qquad E\{|X_1(t) - X_1(u)|^2\} = 2\left|\sin\frac{t-u}{2}\right|,$$

and consequently

$$\sigma^2(t) = 1 \qquad \Gamma_1(t, u) = 1 - \left|\sin\frac{t-u}{2}\right|, \tag{2.5.2}$$

$$\gamma(t, u) = -\frac{1}{4}\left|\sin\frac{t-u}{2}\right|, \tag{2.5.3}$$

the values of $\sigma^2(t)$ and $\gamma(t, u)$ being the same for the three functions $X_0(t)$, $X_1(t)$ and $X(t)$.

Since the process is stationary, the Fourier coefficients A_n are independent, and, from (2.4.10) and (2.5.2) we deduce

$$\left\{\begin{aligned} \sigma_n^2 &= \frac{-1}{2\pi}\int_0^{2\pi} e^{int}\sin\frac{t}{2}\,dt \\ &= \frac{2}{\pi(4n^2-1)}, \qquad n = \pm 1, \pm 2, \ldots . \end{aligned}\right. \tag{2.5.4}$$

Consequently,

$$X_0(t) \sim \sum{}' \sqrt{\frac{2}{\pi(4n^2-1)}}\,\zeta_n e^{nit}, \tag{2.5.5}$$

$$X(t) \sim \sum_{-\infty}^{+\infty} \sqrt{\frac{2}{\pi(4n^2-1)}}\,\zeta_n(e^{nit} - 1). \tag{2.5.6}$$

As for $X_1(t)$, we may write it in the form $X_0(t) + \sigma\zeta$ where ζ is independent of the variables ζ_n. To determine σ, we let $\Gamma_0(t, u)$ denote the covariance of $X_0(t)$ and have

$$\Gamma_1(t, 0) = \Gamma_0(t, 0) + \sigma^2 = 1 - \sin\frac{t}{2}, \qquad 0 \leqq t \leqq 2\pi,$$

and

$$\int_0^{2\pi} \Gamma_0(t, 0)\,dt = E\left\{\bar{X}_0(0)\int_0^{2\pi} X_0(t)\,dt\right\} = 0.$$

From these equations, we deduce

$$\sigma^2 = 1 - \frac{1}{2\pi}\int_0^{2\pi} \sin\frac{t}{2}\,dt = 1 - \frac{2}{\pi} \tag{2.5.7}$$

and finally

$$X_1(t) \sim \sqrt{1-\frac{2}{\pi}}\,\zeta + \sum{}' \sqrt{\frac{2}{\pi(4n^2-1)}}\,\zeta_n e^{nit}. \tag{2.5.8}$$

Now, in order to obtain the infinitesimal equation satisfied by $X(t)$, we deduce from (2.3.4), (2.5.2) and (2.5.3)

$$\sin\frac{t-x}{2} = -4F(t,x) + \int_0^t F(t,u)\left|\sin\frac{u-x}{2}\right| du, \quad 0 \leqq x \leqq t \leqq 2\pi. \tag{2.5.9}$$

By taking the derivative twice with respect to x, we have easily

$$\sin\frac{t-x}{2} = 16\frac{\partial^2 F(t,x)}{\partial x^2} - 4F(t,x) + \int_0^t F(t,u)\left|\sin\frac{u-x}{2}\right| du.$$

Then, by subtracting the second of these formulas from the first

$$\frac{\partial^2 F(t,x)}{\partial x^2} = 0, \qquad F(t,x) = f_0(t) + x f_1(t),$$

and, using again (2.5.9), we find easily

$$F(t,u) = \frac{\frac{u}{2}\left(1+\cos\frac{t}{2}\right) - \left(\frac{t}{2}+\sin\frac{t}{2}\right)}{4\left(1+\cos\frac{t}{2}\right) + t\sin\frac{t}{2}}, \qquad 0 \leqq u \leqq t \leqq 2\pi. \tag{2.5.10}$$

Finally, the solution of

$$\delta X(t) \sim dt\int_0^t F(t,u)\,dX(u) + \zeta\sqrt{dt}, \qquad 0 \leqq t < 2\pi,\ dt > 0, \tag{2.5.11}$$

$F(t, u)$ being defined by (2.5.10), which vanishes for $t = 0$, is $X(t)$. If $t = 2\pi - \epsilon \to 2\pi$, we have

$$F(t,u) = \frac{-1}{\epsilon} + O\left(\frac{1}{\epsilon}\right).$$

Then we have a verification that the theorem concerning the periodic random functions which was proved in 2.4 is applicable. More precisely, if $2\pi - t$ is small enough, the equation (2.5.11) is asymptotically the same as (2.4.5).

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