SOME COMMENTS ON LARGE SAMPLE TESTS

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LET $p_{X|\Theta}$ denote the elementary probability law of a random variable X depending on a parameter Θ, and let $X_1, \ldots, X_n$ be a sample of X. By a test of the hypothesis $H: \Theta = \Theta_0$, we mean a region of rejection W_n in the n-dimensional space of $X_1, \ldots, X_n$. We denote by $P(W_n | \Theta)$ the power of the test region W_n, that is,

$$P(W_n \mid \Theta) = \Pr\left\{\begin{matrix}\text{rejecting } H \text{ when } \Theta \text{ is the} \\ \text{true value of the parameter.}\end{matrix}\right\}$$

It is then desired that

$$P(W_n \mid \Theta_0) = \alpha,$$

where α $(0 < \alpha < 1)$ is the preassigned level of significance, and that $P(W_n | \Theta)$ be as close to one as possible for all $\Theta \neq \Theta_0$. If for every other test Z_n satisfying $P(Z_n | \Theta_0) = \alpha$ we have $P(W_n | \Theta) \geqq P(Z_n | \Theta)$ for all Θ, W_n is said to be uniformly most powerful. In the theory of large-sample tests, sequences of tests $\{W_n\}$, $\{Z_n\}$, $(n = 1, 2, \ldots)$, are compared on the basis of asymptotic properties of the power functions $P(W_n | \Theta)$, $P(Z_n | \Theta)$.

Various classes of asymptotically "good" or "best" tests have been defined. Among them: Tests "unbiased in the limit" **[1]**,[1] "asymptotically most powerful tests" **[2]**, and "asymptotically most stringent tests **[3]**. As was pointed out by Wald (**[4]**, p. 32) concerning his asymptotically most powerful (AMP) tests, these definitions specify that the tests are powerful for sufficiently large n, but say little or nothing about the speed with which the limiting power is approached.

The purpose of the present note is to illustrate this remark by two examples. In example 1 an AMP test sequence W_n is constructed and a sequence of tests Z_n, corresponding to the same level of significance, such that for all Θ

$$\frac{1 - P(Z_n \mid \Theta)}{1 - P(W_n \mid \Theta)} \to 0 \text{ as } n \to \infty .$$

Example 2 shows that, given any sequence of numbers $r_n \to 1$, there exists an AMP test W_n such that $P(W_n \mid \Theta) \leqq r_n$ for all Θ.

The tests W_n of examples 1 and 2 are also asymptotically most stringent, and similar tests exist which are unbiased in the limit. Actually it seems doubtful that any definition of optimum tests, based only on asymptotic

[1] Boldface numbers in brackets refer to references at the end of the paper (p. 457).

properties of power functions, can be very satisfactory, since in practice the sample size is always limited and since obviously an asymptotic property implies nothing about the behavior of any finite segment of a sequence of power functions.

Denoting, for any regions U_n, T_n, the l.u.b. (with respect to Θ) of $P(U_n \mid \Theta) - P(T_n \mid \Theta)$ by $L(U_n, T_n)$, Wald defines a sequence $\{W_n\}$ of regions to be an AMP test of the hypothesis $H\colon \Theta = \Theta_0$ at the level of significance α if, for any sequence $\{Z_n\}$ of regions for which $P(Z_n \mid \Theta_0) = \alpha$, the inequality

$$\limsup_{n \to \infty} L(Z_n, W_n) \leqq 0$$

holds. Hence a necessary and sufficient condition for $\{W_n\}$ to be AMP is that for any $\{Z_n\}$ of the same level of significance, given ϵ, there exists $N = N(\epsilon)$ such that $n \geqq N$ implies

$$P(W_n \mid \Theta) \geqq P(Z_n \mid \Theta) - \epsilon \text{ for all } \Theta.$$

If, in particular, a uniformly most powerful test Z_n exists for every n (for testing $H\colon \Theta = \Theta_0$ at the level of significance α), W_n is AMP if and only if, as $n \to \infty$,

$$P(Z_n \mid \Theta) - P(W_n \mid \Theta) \to 0 \text{ uniformly in } \Theta. \text{ [2]}$$

Example 1. Let

$$p_{X|\theta}(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-\theta)^2}. \tag{1}$$

For testing the hypothesis $H\colon \Theta = 0$ against the set of alternatives $\Theta > 0$, a uniformly most powerful test Z_n exists,

$$Z_n\colon \quad \overline{X} \geqq \frac{c}{\sqrt{n}} \,. \tag{2}$$

Define W_n by the inequalities,

$$W_n\colon \quad \overline{X} \geqq \frac{b_n}{\sqrt{n}}, \qquad \overline{X} \leqq \frac{-a_n}{\sqrt{n}}; \qquad a_n, b_n > 0, \tag{3}$$

where a_n, b_n are constants, to be defined later.

$\{W_n\}$ is AMP provided

$$P(Z_n \mid \Theta) - P(W_n \mid \Theta) \to 0 \text{ uniformly in } \Theta.$$

Setting

$$G(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\, dt \,,$$

[2] Cf. [4], p. 457.

we have

$$P(Z_n \mid \Theta) = G(\Theta\sqrt{n} - c) \tag{4}$$

and

$$P(W_n \mid \Theta) = G(\Theta\sqrt{n} - b_n) + G(-\Theta\sqrt{n} - a_n). \tag{5}$$

Since Z_n is uniformly most powerful, we thus obtain

$$\begin{aligned} 0 \leqq P(Z_n \mid \Theta) - P(W_n \mid \Theta) &< G(b_n - \Theta\sqrt{n}) - G(c - \Theta\sqrt{n}) \\ &\leqq G\left(\frac{b_n - c}{2}\right) - G\left(-\frac{b_n - c}{2}\right). \end{aligned} \tag{6}$$

Therefore $P(Z_n \mid \Theta) - P(W_n \mid \Theta) \to 0$ uniformly in Θ provided

$$b_n \to c, \tag{7}$$

and hence (7) is a sufficient condition for $\{W_n\}$ to be AMP.

We next consider the ratio

$$\frac{1 - P(W_n \mid \Theta)}{1 - P(Z_n \mid \Theta)} = \frac{G(b_n - \Theta\sqrt{n}) - G(-a_n - \Theta\sqrt{n})}{G(c - \Theta\sqrt{n})}. \tag{8}$$

Substituting a continuous variable for $\sqrt{n}$, and applying the rule of de l'Hospital, we find that the ratio (8) $\to \infty$ for all values of $\Theta > 0$ provided

$$\sqrt{n}(b_n - c) \to \infty \text{ as } n \to \infty. \tag{9}$$

Thus, taking for example

$$b_n = c + n^{-1/4} \tag{10}$$

and determining a_n from the equation

$$P(W_n \mid 0) = \alpha, \tag{11}$$

we see that $\{W_n\}$ is AMP by (7), and that

$$[1 - P(Z_n \mid \Theta)] = o\,[1 - P(W_n \mid \Theta)] \text{ for all } \Theta > 0.$$

Example 2. Let

$$p_{X|\Theta}(x) = \begin{cases} 1/\Theta & \text{if } 0 \leqq x \leqq \Theta \\ 0 & \text{elsewhere.} \end{cases} \tag{12}$$

For testing the hypothesis H: $\Theta = 1$ against the set of alternatives $\Theta > 0$, a uniformly most powerful test Z_n exists,

$$Z_n\colon \quad 0 \leqq y \leqq \sqrt[n]{\alpha}; \qquad y > 1, \tag{13}$$

where α is the level of significance, and $y = \max(x_1, \ldots, x_n)$.

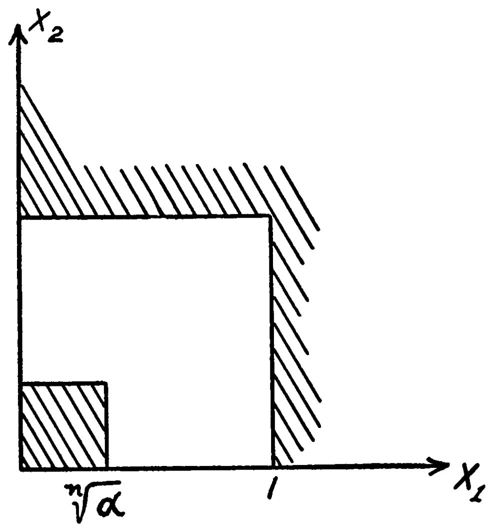

Fig. 1

Figure 1 shows a two-dimensional cut of the n-dimensional sample space, the shaded region representing the region Z_n.

Given a sequence of numbers $r_n \to 1$ (without loss of generality we assume $1 > r_n > \max(\alpha, 1/2)$ for all n), we shall construct an AMP test W_n such that

$$P(W_n \mid \Theta) \leqq r_n \text{ for all } \Theta.$$

Consider a pyramid in n-dimensional space, vertex at the origin, and with all n faces lying in the positive "quadrant" and extending to infinity, and such that the volume of the pyramid, cut off by the cube $0 \leqq x_i \leqq \sqrt[n]{\alpha}$, $(i = 1, \ldots, n)$, equals $\left(2 - \dfrac{1}{r_n}\right)\alpha$.

Let W_n consist

(i) of the part of the pyramid common to the cube $0 \leqq x_i \leqq \sqrt[n]{\alpha}$, $(i = 1, \cdots, n)$

(ii) of the part of the pyramid lying outside the unit cube

(iii) of the cubical shell

$$\sqrt[n]{\alpha} \leqq x_i \leqq \sqrt[n]{\frac{\alpha}{r_n}}, \qquad i = 1, \cdots, n.$$

Figure 2 shows a two-dimensional cut of the n-dimensional sample space, the shaded region representing the region W_n.

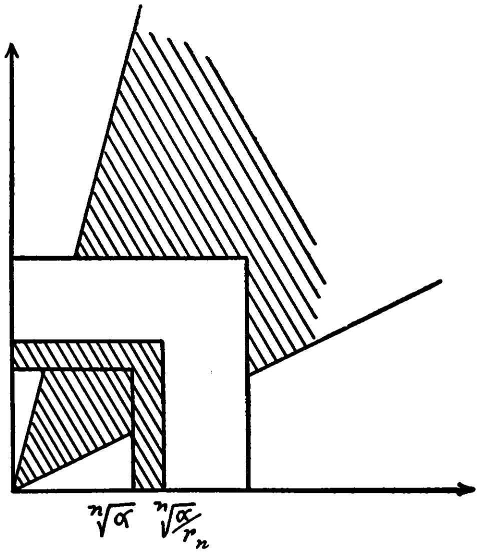

Fig. 2

We compute

$$P(Z_n \mid \Theta) = \begin{cases} 1 & \text{if } \Theta \leqq \sqrt[n]{\alpha}, \\ \dfrac{\alpha}{\Theta^n} & \text{if } \sqrt[n]{\alpha} \leqq \Theta \leqq 1, \\ \dfrac{\Theta^n - 1 + \alpha}{\Theta^n} & \text{if } 1 \leqq \Theta \end{cases} \tag{14}$$

$$P(W_n \mid \Theta) = \begin{cases} 2 - \dfrac{1}{r_n} & \text{if } \Theta \leqq \sqrt[n]{\alpha}, \\ \dfrac{\Theta^n - \alpha + \alpha\left(2 - \dfrac{1}{r}\right)}{\Theta^n} & \text{if } \sqrt[n]{\alpha} \leqq \Theta \leqq \sqrt[n]{\dfrac{\alpha}{r_n}} \\ \dfrac{\alpha}{\Theta^n} & \text{if } \sqrt[n]{\dfrac{\alpha}{r_n}} \leqq \Theta \leqq 1, \\ \dfrac{\alpha + (2 - \frac{1}{r_n})(\Theta^n - 1)}{\Theta^n} & \text{if } 1 \leqq \Theta. \end{cases} \tag{15}$$

Using the fact that $0 < r_n$ implies $2 - \dfrac{1}{r_n} < r_n$ we shall now show that

a) $P(Z_n \mid \Theta) - P(W_n \mid \Theta) \to 0$ uniformly in Θ,

b) $P(W_n \mid \Theta) \leqq r_n$ for all Θ.

For $\Theta \leqq 1$ we have

$$\begin{cases} 2 - \dfrac{1}{r_n} \leqq P(W_n \mid \Theta) \leqq r_n & \text{if } \Theta \leqq \sqrt[n]{\dfrac{\alpha}{r_n}}, \\ P(W_n \mid \Theta) = P(Z_n \mid \Theta) = \dfrac{\alpha}{\Theta^n} \leqq r_n & \text{if } \sqrt[n]{\dfrac{\alpha}{r_n}} \leqq \Theta \leqq 1. \end{cases} \tag{16}$$

Therefore

$$\begin{aligned} &P(Z_n \mid \Theta) - P(W_n \mid \Theta) \leqq 1 - \left(2 - \frac{1}{r_n}\right) && \text{if } \Theta \leqq \sqrt[n]{\frac{\alpha}{r_n}}, \\ &P(Z_n \mid \Theta) - P(W_n \mid \Theta) = 0 && \text{if } \sqrt[n]{\frac{\alpha}{r_n}} \leqq \Theta \leqq 1. \end{aligned} \tag{17}$$

For $\Theta > 1$ we have

$$\begin{aligned} P(W_n \mid \Theta) &= 2 - \frac{1}{r_n} - \frac{2 - \frac{1}{r_n} - \alpha}{\Theta^n} \\ &\leqq \begin{cases} 2 - \dfrac{1}{r_n} < r_n & \text{if } 2 - \dfrac{1}{r_n} - \alpha \geqq 0, \\ \alpha \leqq r_n & \text{if } 2 - \dfrac{1}{r_n} - \alpha \leqq 0. \end{cases} \end{aligned} \tag{18}$$

$$P(Z_n \mid \Theta) - P(W_n \mid \Theta) = \frac{\left(\frac{1}{r_n} - 1\right)\left(\Theta^n - 1\right)}{\Theta^n} \leqq \frac{1}{r_n} - 1. \tag{19}$$

From (17) and (19) it follows that for all Θ

$$P(Z_n \mid \Theta) - P(W_n \mid \Theta) \leqq \frac{1}{r_n} - 1 \to 0 \text{ as } n \to \infty,$$

which establishes (a), while (b) follows from (16) and (18).

[I should like to express my gratitude to Professor J. Neyman for suggesting this problem to me.]

REFERENCES

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