EXTENSION OF THE ROMANOVSKY-BARTLETT-SCHEFFÉ TEST

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1. Introduction

We are concerned here with finding a suitable test for the equidependence of the means of two normal populations on respective linear regression variables (which may be identical) when no information is at hand regarding the two variances involved. More precisely, let x be a random variable, normally distributed with mean $h_1 + k_1 \xi$ and variance σ_1^2. Here h_1 and k_1 are constants, and the mean thus depends linearly on the single sure variable ξ. The variance σ_1^2 is independent of ξ. Similarly, let y be a normal random variable, with mean $h_2 + k_2\eta$ and variance σ_2^2. Here, likewise, h_2 and k_2 are constants, η is a sure variable and σ_2^2 is independent of η. Under the set of alternatives

$$\begin{cases} -\infty < h_1, h_2, k_1, k_2 < \infty \\ \quad 0 < \sigma_1^2, \sigma_2^2 \end{cases}$$

we seek an exact, unbiased test for the hypothesis

$$H_0: \quad k_1 = k_2.$$

The test will have reference to a sample $(x_1, x_2, \cdots, x_m) \sim (\xi_1, \xi_2, \cdots, \xi_m)$ out of the first population, and a sample $(y_1, y_2, \cdots, y_n) \sim (\eta_1, \eta_2, \cdots, \eta_n)$ out of the second. The notation here is meant to imply that the random values (or variables) x_i, y_j are observed when ξ and η have the values ξ_i, η_j, respectively. With no loss of generality, we may assume $n \geqq m$.

In this problem [(P), for brevity], as in that of the comparison of constant means of two normal populations with unknown variances, the question of a *best* exact test must for the present go unanswered, for want of sufficiently powerful methods of determining all similar regions. V. Romanovsky, M. S. Bartlett, and H. Scheffé, in obtaining solutions of the latter problem, have brought to bear a specialized procedure based on "Student's" t-test, and with it have produced exact, unbiased tests in that case. The procedure can be applied to (P) as well (and to a large class of problems, in fact), to yield a test of the same character; and one which, like those of Romanovsky, Bartlett, and Scheffé for constant means, has the advantages of having a simple criterion and requiring only the use of t-tables.

The first steps in fashioning and applying the procedure were taken by Romanovsky [3][1] in 1928. It appears, however, that Romanovsky's paper was entirely overlooked and the method was later rediscovered by Bartlett in

[1] Boldface numbers in brackets refer to references at the end of the paper (p. 449).

a less general form. In his paper Bartlett [1] deals with two pairs of independent observations, whereas Romanovsky considers an arbitrary number of pairs of independent or correlated observations. A more general treatment of Bartlett's problem is mentioned by Welch [5] and by Neyman [2], the latter of whom cites a definite test. Scheffé [4] completed the scheme, and accordingly determined a test (referred to in the title of this paper as the Romanovsky-Bartlett-Scheffé test), by pointing the way to those tests, available under Bartlett's prescription, having the greatest power.[2]

The procedure will be laid out in section 2 in a form adapted to (P). It will be found to entail the definition of several linear functions of the sample variables,

$$z_i = \sum_{j=1}^{m} \alpha_{ij} x_j + \sum_{j=1}^{n} \beta_{ij} y_j,$$

having certain properties in common. And the test criterion will be a function of the z_i alone. (P) then becomes embodied in a number of conditions on the α_{ij} and β_{ij}, and in sections 3 and 4 a solution is developed using the methods of vector spaces. We do not find the α_{ij} and β_{ij} themselves, but rather go directly to the specification of the criterion. The solution will be given in sufficient detail to indicate clearly how this circumvention is accomplished.

A full description of the test we have determined, depending only on the notation so far introduced, is contained in the summary, section 5.

2. The Romanovsky-Bartlett-Scheffé procedure for (P)

We define linear functions of the random variables x_j, y_j:

$$z_i = \sum_{j=1}^{m} \alpha_{ij} x_j + \sum_{j=1}^{n} \beta_{ij} y_j, \qquad i = 1, 2, \cdots, f, \tag{1}$$

the coefficients α_{ij}, β_{ij} being subject to the conditions

(i) $z_1, z_2, \cdots, z_f$ *are independently distributed,*

(ii) $\mathcal{E}(z_i) = k_1 - k_2$ *for each* i,

(iii) *the variances of all the* z_i *are equal; say,* σ^2.

Referent to the generic normal random variable z, with mean $k_1 - k_2$ and unknown variance σ^2, the best unbiased test for the hypothesis $k_1 = k_2$ is "Student's" t-test. The critical region in an f-dimensional sample space is, for the level of significance ϵ,

$$W_0: \frac{|\bar{z}|}{\sqrt{\frac{1}{f(f-1)} \sum_{i=1}^{f} (z_i - \bar{z})^2}} > t_\epsilon,$$

[2] In the two papers by Neyman and Scheffé the method is that of confidence intervals. We have interpreted their results in the language of significance tests, which we retain throughout.

the limit of significance, t_ϵ, drawing from t-tables for $f - 1$ degrees of freedom. Then W'_0, the region in the (x,y) sample space defined by W_0, by means of (1), is an unbiased similar critical region for the hypothesis H_0.

The power function of W_0, and consequently of W'_0, is

$$B\left(k_1 - k_2; f, \frac{\sigma}{\sqrt{f}}\right) =$$

$$1 - \frac{2}{\sqrt{\pi}\,\Gamma\left(\frac{f-1}{2}\right)} \int_{v=0}^{\infty} v^{f-2} e^{-v^2} \int_{u=-\frac{t_\epsilon}{\sqrt{f-1}}v}^{+\frac{t_\epsilon}{\sqrt{f-1}}v} e^{-\left[u - \sqrt{\frac{f}{2\sigma^2}}(k_1 - k_2)\right]^2} du dv;$$

and it is known that B is monotonic increasing, uniformly at all non-zero values of $k_1 - k_2$, with increasing f or decreasing σ. It is therefore immediately evident that we shall want to ascribe to σ the minimum value that can be associated with the f concerned. The choice of f will then follow a study of the functions $B(k_1 - k_2)$ for all possible values of f. Accordingly, we frame the following further condition on the α_{ij} and β_{ij}:

(iv) *Let $\sigma_{(f)}$ denote the smallest value of σ for the case of f functions z_i. Then the value f_0 is to be chosen for f, such that, if possible,* $B_0 = B\left(k_1 - k_2; f_0, \frac{\sigma_{(f_0)}}{\sqrt{f_0}}\right)$ *is the uniformly greatest of all the above power functions* B; *and otherwise, such that* B_0 *has some optimum property to be specified.*

The necessity for laying down an optimum requirement in conjunction with this last condition may arise only if $\frac{\sigma_{(f)}}{\sqrt{f}}$ is not a non-increasing function of f. If this ratio is non-increasing, then f_0 is the largest value of f admitted by the first three conditions; and the corresponding B_0 is the uniformly greatest power unction that can be achieved by this procedure. We anticipate our results to the extent of noting that the quantity $\frac{\sigma_{(f)}}{\sqrt{f}}$ turns out to be *independent* of f. and f_0 is therefore indicated to be the largest value of f otherwise permitted; this will be $m - 1$. These facts will be established in the next section, where the general solution is found for the problem posed by conditions (i) to (iv).

3. The general solution

Conditions (i), (ii), and (iii) are translated into constraints on the α_{ij} and β_{ij} by considering expectations, variances, and covariances of the expressions (1). The result is, for (i).

$$\begin{cases} \sum_{j=1}^{m} \alpha_{ij}\alpha_{kj} = 0, \\ \sum_{j=1}^{n} \beta_{ij}\beta_{kj} = 0, \end{cases} \qquad i \neq k; \tag{2}$$

for (ii),

$$\begin{cases} \sum_{j=1}^{m} \alpha_{ij} = 0, & \sum_{j=1}^{m} \alpha_{ij}\xi_j = 1, \\ \sum_{j=1}^{n} \beta_{ij} = 0, & \sum_{j=1}^{n} \beta_{ij}\eta_j = 1, \end{cases} \qquad i = 1, 2, \cdots, f; \tag{3}$$

and for (iii),

$$\begin{cases} \sum_{j=1}^{m} \alpha_{ij}^2 = c_1^2, \\ \sum_{j=1}^{n} \beta_{ij}^2 = c_2^2, \end{cases} \qquad i = 1, 2, \cdots, f; \tag{4}$$

where c_1 and c_2 are two positive numbers, the same for all i. The common variance of the z_i is then

$$\sigma^2 = c_1^2\sigma_1^2 + c_2^2\sigma_2^2. \tag{5}$$

In accordance with condition (iv) we must seek a solution with minimum c_1 and c_2.

It is to be noted, as is indicated by equations (3), that the ξ_i may not all be equal, and similarly for the η_i. Clearly, any test cannot be very discerning if either the ξ_i or the η_i, or both sets simultaneously, are very nearly equal. For, in the case of the first population, for example, estimates of k_1 independent of h_1, arise out of the system of equations $h_1 + k_1\xi_i = x_i$, and depend inversely on the difference $\xi_i - \xi_j$. When all these differences are small, compared to the spread of the x_i, any estimate of k_1 is extremely sensitive to small changes in the sample; and in the limit $\xi_1 = \xi_2 = \cdots = \xi_m$, k_1 is absolutely indistinguishable from $h_1 + k_1\xi_i$. This effect will be seen in the dependence of the minimum c_1 and c_2 on the ξ_i and η_i [cf.(13)]. The former, of course, influence the power function B strongly through σ.

In preparation for the introduction of vector methods, we make a transformation of the system of equations (2), (3), (4). The numbers c_1 and c_2 are restricted, by (4) and (3), to be non-zero; we define

$$a_{ij} = \frac{\alpha_{ij}}{c_1}, \qquad b_{ij} = \frac{\beta_{ij}}{c_2}.$$

Further, set

$$s_1^2 = \frac{1}{m}\sum_{i=1}^{m} (\xi_i - \bar{\xi})^2, \qquad s_2^2 = \frac{1}{n}\sum_{i=1}^{n} (\eta_i - \bar{\eta})^2,$$

$$\left(\bar{\xi} = \frac{1}{m}\sum_{i=1}^{m} \xi_i, \qquad \bar{\eta} = \frac{1}{n}\sum_{i=1}^{n} \eta_i\right),$$

and

$$\rho_i = \frac{\xi_i - \bar{\xi}}{\sqrt{m}s_1}, \qquad \tau_i = \frac{-\eta_i + \bar{\eta}}{\sqrt{n}s_2}.$$

The reader will easily satisfy himself that the following system of equations is equivalent to equations (2), (3), (4):

$$\text{(6a)} \quad \sum_{j=1}^{m} a_{ij}a_{kj} = \delta_{ik}, \qquad \text{(6b)} \quad \sum_{j=1}^{n} b_{ij}b_{kj} = \delta_{ik},$$

$$\text{(7a)} \quad \sum_{j=1}^{m} a_{ij} = 0, \qquad \text{(7b)} \quad \sum_{j=1}^{n} b_{ij} = 0,$$

$$\text{(8a)} \quad \sum_{j=1}^{m} a_{ij}\rho_j = \frac{1}{\sqrt{m}s_1c_1}, \qquad \text{(8b)} \quad \sum_{j=1}^{n} b_{ij}\tau_j = \frac{1}{\sqrt{n}s_2c_2},$$

$$i,k = 1, 2, \cdots, f.$$

A result of our normalizations is that

$$\text{(9a)} \quad \sum_{i=1}^{m} \rho_i^2 = 1, \qquad \text{(9b)} \quad \sum_{i=1}^{n} \tau_i^2 = 1,$$

$$\text{(10a)} \quad \sum_{i=1}^{m} \rho_i = 0, \qquad \text{(10b)} \quad \sum_{i=1}^{n} \tau_i = 0.$$

Equations (6a) and (6) follow from (2) and (4); (7a) and (7b) from the left column of (3); and (8a) and (8b) are respectively composed of both upper and both lower equations of (3). The problem is seen to lend itself, in many respects, to separate and identical considerations of the a_{ij} and b_{ij}.

We now bring in the terminology of unitary spaces (vector spaces with an inner product), and introduce a suitable notation. Let $\mathcal{R}_a$ and $\mathcal{R}_b$ be two real unitary spaces of dimensions m and n, outfitted with complete orthonormal sets (coördinate systems) Ω_a and Ω_b, respectively. In $\mathcal{R}_a$ define the vectors $\boldsymbol{a}_i$, $\boldsymbol{\rho}$, and $\boldsymbol{\varepsilon}_a$ (vectors will throughout be denoted by bold face symbols) as those having the components

$$(a_{i1}, a_{i2}, \cdots, a_{im}),$$

$$(\rho_1, \rho_2, \cdots, \rho_m),$$

$$\left(\frac{1}{\sqrt{m}}, \frac{1}{\sqrt{m}}, \cdots, \frac{1}{\sqrt{m}}\right),$$

respectively, in the system Ω_a. In similar fashion define the vectors $\mathbf{b}_i$, $\boldsymbol{\tau}$, and $\boldsymbol{\varepsilon}_b$, in $\mathcal{R}_b$, relative to the coördinate system Ω_b. The inner product of two vectors $\mathbf{g}_1$ and $\mathbf{g}_2$, in $\mathcal{R}_a$ or $\mathcal{R}_b$, will be represented briefly by $(\mathbf{g}_1, \mathbf{g}_2)$. Finally, let $\mathcal{O}_a$ be the orthocomplement of $\boldsymbol{\varepsilon}_a$ in $\mathcal{R}_a$, and $\mathcal{O}_b$ that of $\boldsymbol{\varepsilon}_b$ in $\mathcal{R}_b$.

In the new notation, equations (6a) to (10b) are expressed by

$$(\boldsymbol{a}_i, \boldsymbol{a}_k) = \delta_{ik}, \qquad (\boldsymbol{b}_i, \boldsymbol{b}_k) = \delta_{ik},$$

$$(\boldsymbol{a}_i, \boldsymbol{\varepsilon}_a) = 0, \qquad (\boldsymbol{b}_i, \boldsymbol{\varepsilon}_b) = 0,$$

$$(\boldsymbol{a}_i, \boldsymbol{\varrho}) = \frac{1}{\sqrt{m s_1 c_1}}, \qquad (\boldsymbol{b}_i, \boldsymbol{\tau}) = \frac{1}{\sqrt{n s_2 c_2}},$$

$$(\boldsymbol{\varrho}, \boldsymbol{\varrho}) = 1, \qquad (\boldsymbol{\tau}, \boldsymbol{\tau}) = 1,$$

$$(\boldsymbol{\varrho}, \boldsymbol{\varepsilon}_a) = 0, \qquad (\boldsymbol{\tau}, \boldsymbol{\varepsilon}_b) = 0,$$

$$i,k = 1, 2, \cdots, f.$$

Thus, the $\boldsymbol{a}_i$ are to be f unit vectors, mutually orthogonal, all lying in $\mathcal{O}_a$, and all equally inclined (with the smallest possible inclination, moreover) to the fixed vector $\boldsymbol{\varrho}$, the latter being likewise a unit vector and lying in $\mathcal{O}_a$. The interpretation of the conditions on the $\boldsymbol{b}_i$ is the corresponding statement in $\mathcal{R}_b$.

It is readily seen that a solution, for some values of c_1 and c_2 (putting off for a moment the question of their minima), exists for every $f \leqq m - 1$. The mutual orthogonality of the $\boldsymbol{a}_i$ precludes a solution for $f > m$; and their further property of belonging to $\mathcal{O}_a$ rules out the case $f = m$. It is evident that m must be at least 2.[3] Let $\{\boldsymbol{\varepsilon}_a, \boldsymbol{e}_1, \boldsymbol{e}_2, \cdots, \boldsymbol{e}_{m-1}\}$ and $\{\boldsymbol{\varepsilon}_b, \boldsymbol{e}_1', \boldsymbol{e}_2', \cdots, \boldsymbol{e}'_{n-1}\}$ be coördinate systems in $\mathcal{R}_a$ and $\mathcal{R}_b$; and V_a and V_b be orthogonal transformations in these respective spaces, such that

$$V_a \boldsymbol{\varepsilon}_a = \boldsymbol{\varepsilon}_a, \qquad V_b \boldsymbol{\varepsilon}_b = \boldsymbol{\varepsilon}_b,$$

and

$$V_a\left(\frac{1}{\sqrt{f}}\sum_{i=1}^{f} \boldsymbol{e}_i\right) = \boldsymbol{\varrho}, \qquad V_b\left(\frac{1}{\sqrt{f}}\sum_{i=1}^{f} \boldsymbol{e}'_i\right) = \boldsymbol{\tau}.$$

Then a solution is

$$\boldsymbol{a}_i = V_a \boldsymbol{e}_i, \qquad \boldsymbol{b}_i = V_b \boldsymbol{e}'_i, \qquad i = 1, 2, \cdots, f.$$

This can be verified by direct substitution, but is better seen as follows. The vectors $\boldsymbol{e}_1, \boldsymbol{e}_2, \cdots \boldsymbol{e}_f$, are mutually orthogonal, lie in $\mathcal{O}_a$, and are equally inclined to the unit vector $\frac{1}{\sqrt{f}}\sum_{i=1}^{f} \boldsymbol{e}_i$. An orthogonal transformation preserves inner products, and in particular V_a has $\{\boldsymbol{\varepsilon}_a\}$ and $\mathcal{O}_a$ for invariant manifolds. There-

[3] This condition is fundamental for any exact test. It makes it possible to obtain estimates of k_1 and (since $n \geqq m$) k_2 which are independent of h_1 and h_2.

fore, since V_a further transforms $\frac{1}{\sqrt{f}}\sum_{i=1}^{f} e_i$ into ϱ, it carries $e_1, e_2, \cdots, e_f$ into f mutually orthogonal, unit vectors, in $\mathcal{O}_a$, and equally inclined to ϱ. If the determinant of V_a has the value $+1$ (which it may be chosen to have), the solution may be seen even more vividly: the vectors $e_1, e_2, \cdots, e_f$ lie along generators of a "circular" cone in $\mathcal{O}_a$ with axis defined by $\frac{1}{\sqrt{f}}\sum_{i=1}^{f} e_i$. V_a reorients this cone in $\mathcal{O}_a$ without distorting it, so that its axis coincides with ϱ. The generators originally coincident with $e_1, e_2, \cdots, e_f$ now define f vectors with the properties requisite for a solution.

The values of c_1 and c_2 associated with this solution are

$$c_1 = \frac{\sqrt{f}}{\sqrt{m}\, s_1}, \qquad c_2 = \frac{\sqrt{f}}{\sqrt{n}\, s_2} \tag{11}$$

They obtain by simple calculations,

$$\frac{1}{\sqrt{m}\, s_1 c_1} = (a_i, \varrho) = \left(e_i, \frac{1}{\sqrt{f}}\sum_{i=1}^{f} e_i\right) = \frac{1}{\sqrt{f}},$$

and similarly for c_2. A brief argument will now show that these are the smallest values of c_1 and c_2 that can attend a solution, for the given f. Let a_i, $(i = 1, 2, \cdots, f)$, be the first half of a solution. Define $m - 1 - f$ vectors $g_{f+1}, g_{f+2}, \cdots, g_{m-1}$ so that $\{a_1, \cdots, a_f, g_{f+1}, \cdots, g_{m-1}\}$ is a coördinate system in $\mathcal{O}_a$. Then

$$\varrho = \frac{1}{\sqrt{m}\, s_1 c_1}\sum_{i=1}^{f} a_i + \sum_{i=f+1}^{m-1} \varphi_i g_i,$$

and

$$(\varrho, \varrho) = 1 = \frac{f}{m\, s_1^2 c_1^2} + \sum_{i=f+1}^{m-1} \varphi_i^2.$$

From the last equation it follows that c_1 is least when $\varphi_{f+1} = \cdots = \varphi_{m-1} = 0$, and the value in this case is $\frac{\sqrt{f}}{\sqrt{m}\, s_1}$. The same argument applied in $\mathcal{R}_b$ to the second half of a solution establishes our claim for the minimum value of c_2.

In the last two paragraphs we have established, in addition to the existence of a minimal solution for given f, the following necessary and sufficient prescription for one: *The a_i are any f mutually orthogonal, unit vectors in $\mathcal{O}_a$, such that ϱ is their normalized sum; and the b_i are any f mutually orthogonal unit vectors in $\mathcal{O}_b$, such that τ is their normalized sum.*

The minimum variance $\sigma_{(f)}^2$ is now calculable. Putting the values (11) into (5), we obtain

$$\sigma_{(f)}^2 = f\left(\frac{\sigma_1^2}{m\, s_1^2} + \frac{\sigma_2^2}{n\, s_2^2}\right).$$

As was announced earlier, $\frac{\sigma_{(f)}}{\sqrt{f}}$ is independent of f. Consequently, $f_0 = m - 1$, the largest f for which a solution exists, as has been shown.

We have now fully characterized the most general solution for the z_i under conditions (i) to (iv) of section 2. The $\boldsymbol{a}_i$ are any $m - 1$ mutually orthogonal, unit vectors in $\mathcal{O}_a$, the $\boldsymbol{b}_i$ and $m - 1$ mutually orthogonal, unit vectors in $\mathcal{O}_b$, such that

$$\frac{1}{\sqrt{m-1}} \sum_{i=1}^{m-1} \boldsymbol{a}_i = \boldsymbol{\varrho}, \qquad \frac{1}{\sqrt{m-1}} \sum_{i=1}^{m-1} \boldsymbol{b}_i = \boldsymbol{\tau}. \tag{12}$$

For such vectors,

$$\begin{cases} c_1 = \sqrt{\dfrac{m-1}{m}}\,\dfrac{1}{s_1}, \qquad c_2 = \sqrt{\dfrac{m-1}{n}}\,\dfrac{1}{s_2}, \\[2ex] \dfrac{\sigma^2_{(m-1)}}{m-1} = \dfrac{\sigma_1^2}{m\, s_1^2} + \dfrac{\sigma_2^2}{n\, s_2^2}. \end{cases} \tag{13}$$

In the next section we shall go on to specialize the $\boldsymbol{a}_i$ and $\boldsymbol{b}_i$ in a particular way, to an extent which will enable us to state a criterion. But we emphasize here that the power function B, of the test to be found, is completely defined at this stage; for f_0 and $\sigma_{(f_0)}$ are specified by the general solution.

4. Determination of a criterion

Let $\mathcal{O}_{a,\rho}$ denote the orthocomplement in $\mathcal{R}_a$ of the manifold spanned by $\boldsymbol{\varepsilon}_a$ and $\boldsymbol{\varrho}$. Define $\mathcal{O}_{b,\tau}$ correspondingly in $\mathcal{R}_b$. For definiteness, let Ω_b be represented by $\{\omega_1, \omega_2, \cdots, \omega_n\}$, so that $\boldsymbol{b}_i = \sum_{j=1}^{n} b_{ij}\omega_j$, etc. We shall henceforth view $\mathcal{R}_a$ as a subspace of $\mathcal{R}_b$, and identify it with the manifold spanned by $\omega_1, \omega_2, \cdots, \omega_m$. In fact, the latter set will be precisely the coördinate system Ω_a; thus, $\boldsymbol{a}_i = \sum_{j=1}^{m} a_{ij}\,\omega_j$, etc. We proceed from here on the assumption that the vector $\sum_{i=1}^{m} \tau_i\,\omega_i$, in $\mathcal{R}_a$, is neither the null vector nor a linear combination of $\boldsymbol{\varepsilon}_a$ and $\boldsymbol{\varrho}$; and later treat these singular cases. Define, in $\mathcal{R}_a$, the vector $\hat{\boldsymbol{\tau}}$, whose coördinates relative to Ω_a are

$$\hat{\tau}_i = \frac{\tau_i - \dfrac{1}{m}\left(\sum_{j=1}^{m} \tau_j\right)}{\sqrt{\sum_{k=1}^{m}\left[\tau_k - \dfrac{1}{m}\left(\sum_{j=1}^{m}\tau_j\right)\right]^2}}, \qquad i = 1, 2, \cdots, m. \tag{14}$$

It will be noted that $\hat{\boldsymbol{\tau}}$ (which is identical with $\boldsymbol{\tau}$ when $m = n$) is a unit vector and is orthogonal to $\boldsymbol{\varepsilon}_a$. The latter vector is the normalized projection of $\boldsymbol{\varepsilon}_b$ into $\mathcal{R}_a$, and $\hat{\boldsymbol{\tau}}$ has property that it and $\boldsymbol{\varepsilon}_a$ span the manifold in $\mathcal{R}_a$ which is the projection into $\mathcal{R}_a$ of the manifold in $\mathcal{R}_b$ spanned by $\boldsymbol{\tau}$ and $\boldsymbol{\varepsilon}_b$. We denote, finally, by $\mathcal{O}_{a,\hat{\tau}}$ the orthocomplement in $\mathcal{R}_a$ of the manifold spanned by $\boldsymbol{\varepsilon}_a$ and $\hat{\boldsymbol{\tau}}$. Then $\mathcal{O}_{a,\hat{\tau}} \subseteqq \mathcal{O}_{b,\tau}$.

Consider the following vectors:

in $\mathcal{R}_a$,

$$\boldsymbol{u}_i = \boldsymbol{a}_i - (\boldsymbol{a}_i, \boldsymbol{\varrho})\, \boldsymbol{\varrho}, \qquad i = 1, 2, \cdots, m-1; \tag{15}$$

in $\mathcal{R}_b$,

$$\boldsymbol{\nu}_i = \boldsymbol{b}_i - (\boldsymbol{b}_i, \boldsymbol{\tau})\, \boldsymbol{\tau}, \qquad i = 1, 2, \cdots, m-1. \tag{16}$$

The $\boldsymbol{u}_i$ are the components of the $\boldsymbol{a}_i$ orthogonal to $\boldsymbol{\varrho}$, and consequently lie in $\mathcal{O}_{a,\rho}$. They have equal lengths, equal mutual inclinations, and have a null sum. This is immediately evident from the symmetric nature of the $\boldsymbol{a}_i$, but may be derived from (15), with the use of the fact that $(\boldsymbol{a}_i, \boldsymbol{\varrho}) = \dfrac{1}{\sqrt{m-1}}$ [cf.(12)].

We shall want the actual values:

$$(\boldsymbol{u}_i, \boldsymbol{u}_j) = \delta_{ij} - \frac{1}{m-1}.$$

$m - 1$ vectors $\boldsymbol{u}_i$, with the properties noted, lying in an $(m-2)$-dimensional space $\mathcal{O}_{a,\rho}$, define the vertices of a polyhedron which is the analogue of the regular tetrahedron in 3-space. The $\boldsymbol{u}_i$ may be expressed in terms of a coördinate system $\{\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \cdots, \boldsymbol{\lambda}_{m-2}\}$ in $\mathcal{O}_{a,\rho}$,

$$\boldsymbol{u}_i = \sum_{j=1}^{m-2} \gamma_{ij} \boldsymbol{\lambda}_j, \qquad i = 1, 2, \cdots, m-1; \tag{17}$$

the γ_{ij} being subject to the conditions,

$$\begin{cases} \displaystyle\sum_{i=1}^{m-1} \gamma_{ij} = 0, & j = 1, 2, \cdots, m-2, \\ \displaystyle\sum_{j=1}^{m-2} \gamma_{ij}\gamma_{kj} = \delta_{ik} - \frac{1}{m-1}, & i, k = 1, 2, \cdots, m-1. \end{cases} \tag{18}$$

The arbitrariness of the $\boldsymbol{a}_i$ now implies the following converse: *for any $m - 2$ mutually orthogonal, unit vectors $\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \cdots, \boldsymbol{\lambda}_{m-2}$, that span $\mathcal{O}_{a,\rho}$, and any system of numbers γ_{ij} satisfying* (18), *the vectors*

$$\boldsymbol{a}_i = \sum_{j=1}^{m-2} \gamma_{ij} \boldsymbol{\lambda}_j + \frac{1}{\sqrt{m-1}}\, \boldsymbol{\varrho}, \qquad i = 1, 2, \cdots, m-1, \tag{19}$$

constitute the first half of a solution.

The vectors ν_i are similarly subject only to the condition that they be $m - 1$ vectors in $\mathcal{O}_{b,\tau}$, with null sum, and such that

$$(\nu_i, \nu_j) = \delta_{ij} - \frac{1}{m-1} .$$

We may choose the ν_i *to lie in* $\mathcal{O}_{a,\hat{\tau}}$. This is our first specialization. We make another immediately. *With the same numbers* γ_{ij} *as in* (19), *the vectors*

$$b_i = \sum_{j=1}^{m-2} \gamma_{ij} \mathfrak{u}_j + \frac{1}{\sqrt{m-1}} \tau, \qquad i = 1, 2, \cdots, m-1, \tag{20}$$

form the second half of a solution, when $\mathfrak{u}_1, \mathfrak{u}_2, \cdots, \mathfrak{u}_{m-2}$ *are any* $m - 2$ *mutually orthogonal, unit vectors that span* $\mathcal{O}_{a,\hat{\tau}}$. Here, of course,

$$\nu_i = \sum_{j=1}^{m-2} \gamma_{ij} \mathfrak{u}_j, \qquad i = 1, 2, \cdots, m-1. \tag{21}$$

Let us turn now to an examination of the formal criterion (cf. sec. 2):

$$t = \frac{\bar{z}}{\sqrt{\dfrac{1}{(m-1)(m-2)} \displaystyle\sum_{i=1}^{m-1} (z_i - \bar{z})^2}} . \tag{22}$$

We have replaced f by $m - 1$. Up to this point it has been tacitly assumed that $m - 2 > 0$. It should now be observed that this is a necessary condition on the size of the x-sample. And again, this condition is not peculiar to the test we are devising (see footnote 3). Any unbiased test must take account of the standard deviation of estimates of $k_1 - k_2$, and if these estimates are independent, that standard deviation is based on $m - 2$ degrees of freedom. It is, of course, a consequence of this that the t of (22) has $m - 2$ degrees of freedom.

Let

$$A_j = \sum_{i=1}^{m-1} a_{ij}, \qquad B_j = \sum_{i=1}^{m-1} b_{ij}.$$

Using the values of c_1 and c_2 from (13), we obtain, after substituting expressions (1) for the z_i,

$$\bar{z} = \frac{1}{\sqrt{m-1}} \left[\frac{1}{\sqrt{m}\, s_1} \sum_{i=1}^{m} A_i x_i + \frac{1}{\sqrt{n}\, s_2} \sum_{i=1}^{n} B_i y_i \right] , \tag{23}$$

and

$$\frac{1}{m-1} \sum_{i=1}^{m-1} (z_i - \bar{z})^2 = \frac{1}{m s_1^2} \sum_{j,k=1}^{m} P_{jk} x_j x_k \tag{24}$$
$$+ \frac{2}{\sqrt{mn}\, s_1 s_2} \sum_{\substack{j=1,\dots,m \\ k=1,\dots,n}} R_{jk} x_j y_k + \frac{1}{n s_2^2} \sum_{j,k=1}^{n} Q_{jk} y_j y_k,$$

where

$$\begin{cases} P_{jk} = \sum_{i=1}^{m-1}\left(a_{ij} - \frac{A_j}{m-1}\right)\left(a_{ik} - \frac{A_k}{m-1}\right), & j, k = 1, 2, \cdots, m, \\ Q_{jk} = \sum_{i=1}^{m-1}\left(b_{ij} - \frac{B_j}{m-1}\right)\left(b_{ik} - \frac{B_k}{m-1}\right), & j, k = 1, 2, \cdots, n, \\ R_{jk} = \sum_{i=1}^{m-1}\left(a_{ij} - \frac{A_j}{m-1}\right)\left(b_{ik} - \frac{B_k}{m-1}\right), & j = 1, 2, \cdots, m, \; k = 1, 2, \cdots, n. \end{cases} \tag{25}$$

It is thus necessary only to specify the A_i, B_i, P_{jk}, Q_{jk}, and R_{jk}. The first two are given immediately by (12). The first of (12) is, in component form,

$$\frac{1}{\sqrt{m-1}} \sum_{i=1}^{m-1} a_{ij} = \rho_j, \qquad j = 1, 2, \cdots, m;$$

that is,

$$A_j = \sqrt{m-1}\,\rho_j, \qquad j = 1, 2, \cdots, m. \tag{26}$$

In the same way, from the second of (12),

$$B_j = \sqrt{m-1}\,\tau_j, \qquad j = 1, 2, \cdots, n. \tag{27}$$

With the aid of (26) we find

$$a_{ij} - \frac{A_j}{m-1} = a_{ij} - \frac{1}{\sqrt{m-1}}\,\rho_j = a_{ij} - (\boldsymbol{a}_i, \boldsymbol{\varrho})\rho_j;$$

that is, the expressions on the left are the components of the vectors $\boldsymbol{u}_i$ with respect to Ω_a. For brevity, denote these by u_{ij}. Also, let v_{ij} denote the components of $\boldsymbol{v}_i$ with respect to Ω_b. Equation (27) reveals that

$$b_{ij} - \frac{B_j}{m-1} = v_{ij}.$$

We have then,

$$\begin{cases} P_{jk} = \sum_{j=1}^{m-1} u_{ij} u_{ik}, \\ Q_{jk} = \sum_{i=1}^{m-1} v_{ij} v_{ik}, \\ R_{jk} = \sum_{i=1}^{m-1} u_{ij} v_{ik}. \end{cases} \tag{28}$$

Multiply the second equation of (18) by γ_{ir}, and sum over i; bringing the first of (18) to bear, we get the result

$$\sum_{j=1}^{m-2}\gamma_{kj}\left(\sum_{i=1}^{m-1}\gamma_{ij}\gamma_{ir}\right)=\gamma_{kr}, \qquad k=1,2,\cdots,m-1; \quad r=1,2,\cdots,m-2.$$

These equations have the unique solution

$$\sum_{i=1}^{m-1}\gamma_{ij}\gamma_{ik}=\delta_{jk}, \qquad j,k=1,2,\cdots,m-2. \tag{29}$$

Let λ_{ij} and μ_{ij} be the coördinates of $\boldsymbol{\lambda}_i$ and $\boldsymbol{\mu}_i$ with respect to Ω_a. From (17) and (21),

$$u_{ij}=\sum_{k=1}^{m-2}\gamma_{ik}\lambda_{kj},$$

$$v_{ij}=\begin{cases}\displaystyle\sum_{k=1}^{m-2}\gamma_{ik}\mu_{kj}, & j=1,2,\cdots,m,\\ 0, & j=m+1,\cdots,n.\end{cases}$$

With the help of (29) we calculate

$$\begin{aligned}P_{jk}&=\sum_{i=1}^{m-1}u_{ij}u_{ik}=\sum_{i=1}^{m-1}\left(\sum_{r=1}^{m-2}\gamma_{ir}\lambda_{rj}\right)\left(\sum_{s=1}^{m-2}\gamma_{is}\lambda_{sk}\right)\\ &=\sum_{r,s=1}^{m-2}\left(\sum_{i=1}^{m-1}\gamma_{ir}\gamma_{is}\right)\lambda_{rj}\lambda_{sk}\\ &=\sum_{r=1}^{m-2}\lambda_{rj}\lambda_{rk}.\end{aligned}$$

The corresponding result obtains for Q_{jk}; and (here is the advantage gained by defining the $\boldsymbol{u}_i$ and $\boldsymbol{v}_i$ with the same numbers γ_{ij}) for R_{jk} we get a simple bilinear form. We therefore have, in the place of (28),

$$\left\{\begin{aligned}&P_{jk}=\sum_{i=1}^{m-2}\lambda_{ij}\lambda_{ik}, && j,k=1,2,\cdots,m,\\ &Q_{jk}=\begin{cases}\displaystyle\sum_{i=1}^{m-2}\mu_{ij}\mu_{ik},\\ 0,\end{cases} && \begin{aligned}&j,k=1,2,\cdots,m,\\ &j>m,\quad k>m,\end{aligned}\\ &R_{jk}=\begin{cases}\displaystyle\sum_{i=1}^{m-2}\lambda_{ij}\mu_{ik},\\ 0,\end{cases} && \begin{aligned}&j,k=1,2,\cdots,m,\\ &k>m.\end{aligned}\end{aligned}\right. \tag{30}$$

The next step is to make a judicious choice of the vectors $\boldsymbol{\lambda}_i$ and $\boldsymbol{\mu}_i$. We start by taking

$$\boldsymbol{\lambda}_1 = \frac{1}{\sqrt{1 - (\boldsymbol{\rho},\hat{\boldsymbol{\tau}})^2}} [\hat{\boldsymbol{\tau}} - (\boldsymbol{\rho},\hat{\boldsymbol{\tau}})\boldsymbol{\rho}], \tag{31}$$

and

$$\boldsymbol{\mu}_1 = \frac{1}{\sqrt{1 - (\boldsymbol{\rho},\hat{\boldsymbol{\tau}})^2}} [\boldsymbol{\rho} - (\boldsymbol{\rho},\hat{\boldsymbol{\tau}})\hat{\boldsymbol{\tau}}]. \tag{32}$$

The effect of these definitions is the following: $\boldsymbol{\lambda}_1$ lies in $\mathcal{O}_{a,\rho}$ and, together with $\boldsymbol{\varepsilon}_a$ and $\boldsymbol{\rho}$, spans the same manifold, $\mathcal{M}$, in $\mathcal{R}_a$ as do $\boldsymbol{\varepsilon}_a$, $\boldsymbol{\rho}$ and $\hat{\boldsymbol{\tau}}$. Therefore $\boldsymbol{\lambda}_2, \boldsymbol{\lambda}_3, \cdots, \boldsymbol{\lambda}_{m-2}$ may be any orthonormal set that spans the orthocomplement of $\mathcal{M}$ in $\mathcal{R}_a$. The vector $\boldsymbol{\mu}_1$ lies in $\mathcal{O}_{a,\hat{\tau}}$ and, together with $\boldsymbol{\varepsilon}_a$ and $\hat{\boldsymbol{\tau}}$, likewise spans $\mathcal{M}$. Therefore also the vectors $\boldsymbol{\mu}_2, \boldsymbol{\mu}_3, \cdots, \boldsymbol{\mu}_{m-2}$ may be any orthonormal set that spans the orthocomplement of $\mathcal{M}$. *We choose*

$$\boldsymbol{\mu}_i = \boldsymbol{\lambda}_i, \qquad i = 2, 3, \cdots, m-2. \tag{33}$$

This cinches the criterion, for we can now evaluate the P_{jk}, Q_{jk}, and R_{jk}. The matrix

$$\begin{pmatrix} \frac{1}{\sqrt{m}} & \frac{1}{\sqrt{m}} & \cdots & \frac{1}{\sqrt{m}} \\ \rho_1 & \rho_2 & \cdots & \rho_m \\ \frac{\hat{\tau}_1 - (\boldsymbol{\rho},\hat{\boldsymbol{\tau}})\rho_1}{\sqrt{1-(\boldsymbol{\rho},\hat{\boldsymbol{\tau}})^2}} & \frac{\hat{\tau}_2 - (\boldsymbol{\rho},\hat{\boldsymbol{\tau}})\rho_2}{\sqrt{1-(\boldsymbol{\rho},\hat{\boldsymbol{\tau}})^2}} & \cdots & \frac{\hat{\tau}_m - (\boldsymbol{\rho},\hat{\boldsymbol{\tau}})\rho_m}{\sqrt{1-(\boldsymbol{\rho},\hat{\boldsymbol{\tau}})^2}} \\ \lambda_{21} & \lambda_{22} & \cdots & \lambda_{2m} \\ \lambda_{31} & \lambda_{32} & \cdots & \lambda_{3m} \\ \vdots & \vdots & \vdots & \vdots \\ \lambda_{m-2,1} & \lambda_{m-2,2} & \cdots & \lambda_{m-2,m} \end{pmatrix}$$

is orthogonal, and so gives us

$$\frac{1}{m} + \rho_j\rho_k + \frac{[\hat{\tau}_j - (\boldsymbol{\rho},\hat{\boldsymbol{\tau}})\rho_j]\cdot[\hat{\tau}_k - (\boldsymbol{\rho},\hat{\boldsymbol{\tau}})\rho_k]}{1 - (\boldsymbol{\rho},\hat{\boldsymbol{\tau}})^2} + \sum_{i=2}^{m-2} \lambda_{ij}\lambda_{ik} = \delta_{jk}. \tag{34}$$

The values of the non-zero expressions in (30) [cf. (33)],

$$\begin{cases} P_{jk} = \lambda_{1j}\,\lambda_{1k} + \sum_{i=2}^{m-2} \lambda_{ij}\,\lambda_{ik}, \\ Q_{jk} = \mu_{1j}\,\mu_{1k} + \sum_{i=2}^{m-2} \lambda_{ij}\,\lambda_{ik}, \\ R_{jk} = \lambda_{1j}\,\mu_{1k} + \sum_{i=1}^{m-2} \lambda_{ij}\,\lambda_{ik}, \end{cases}$$

fall out with the application of (34) and components of (31) and (32); the complete result is

$$\begin{cases} P_{jk} = \delta_{jk} - \dfrac{1}{m} - \rho_j\rho_k, & j,k = 1, 2, \cdots, m, \\ Q_{jk} = \begin{cases} \delta_{jk} - \dfrac{1}{m} - \hat{\tau}_j\hat{\tau}_k, & j,k = 1, 2, \cdots, m, \\ 0, & j > m, k > m, \end{cases} \\ R_{jk} = \begin{cases} \delta_{jk} - \dfrac{1}{m} - \rho_j\hat{\tau}_k - \dfrac{(\rho_j - \hat{\tau}_j)(\rho_k - \hat{\tau}_k)}{1 - \sum_{i=1}^{m} \rho_i\hat{\tau}_i}, & j,k = 1, 2, \cdots, m, \\ 0, & k > m. \end{cases} \end{cases} \tag{35}$$

It remains only to substitute into (23), (24), and then into (22) to get the criterion explicitly. First, however, it will be well to set

$$\begin{cases} X_i = \dfrac{x_i - \bar{x}}{\sqrt{m}\, s_1}, & i = 1, 2, \cdots, m, \\ Y_i = \dfrac{y_i - \dfrac{1}{m}\left(\sum_{i=1}^{m} y_i\right)}{\sqrt{n}\, s_2}, & i = 1, 2, \cdots, n. \end{cases} \tag{36}$$

We direct attention to the second of these expressions to point out that the second term in the numerator is the mean of only the first m values y_i, *not* the mean of the entire y-sample. With this, the criterion is

$$t = \frac{\sqrt{m-2}\left(\sum_{i=1}^{m} \rho_i X_i + \sum_{i=1}^{n} \tau_i Y_i\right)}{\sqrt{\sum_{i=1}^{m} (X_i + Y_i)^2 + [\sum_{i=1}^{m} (\rho_i X_i + \hat{\tau}_i Y_i)]^2 + \dfrac{2[\sum_{i=1}^{m} (\rho_i - \hat{\tau}_i) X_i][\sum_{i=1}^{m} (\rho_i - \hat{\tau}_i) Y_i]}{1 - \sum_{i=1}^{m} \rho_i \hat{\tau}_i}}} \tag{37}$$

Here, caution is again advised with regard to the ranges of summation; all sums extend from 1 to m, except in the second term of the numerator (the only place the unroofed τ_i occur), where the sum extends from 1 to n. We restate the following important definitions:

$$\begin{cases} s_1^2 = \dfrac{1}{m}\displaystyle\sum_{i=1}^{m}(\xi_i - \bar{\xi})^2, \qquad s_2^2 = \dfrac{1}{n}\displaystyle\sum_{i=1}^{n}(\eta_i - \bar{\eta})^2, \\ \left(\bar{\xi} = \dfrac{1}{m}\displaystyle\sum_{i=1}^{m}\xi_i, \qquad \bar{\eta} = \dfrac{1}{n}\displaystyle\sum_{i=1}^{n}\eta_i\right), \\ \rho_i = \dfrac{\xi_i - \xi,}{\sqrt{\ }\ s_1} \qquad \tau_i = \dfrac{-\eta_i + \bar{\eta}}{\sqrt{n}\, s_2}, \\ \hat{\tau}_i = \dfrac{-\eta_i + \dfrac{1}{m}\left(\displaystyle\sum_{j=1}^{m}\eta_j\right)}{\sqrt{\displaystyle\sum_{k=1}^{m}\left[-\eta_k + \dfrac{1}{m}\left(\displaystyle\sum_{j=1}^{m}\eta_j\right)\right]^2}}. \end{cases} \tag{38}$$

The last of these is obviously equivalent to (14).

At the beginning of this section we noted certain exceptional cases of the vector $\sum_{i=1}^{m}\tau_i\boldsymbol{\omega}_i$ that require further consideration. They may be classified as follows:

Case 1. $\sum_{i=1}^{m}\tau_i\boldsymbol{\omega}_i = 0$. This is equivalent to $\eta_1 = \eta_2 = \cdots = \eta_m = 0$.
In this case, $\hat{\boldsymbol{\tau}}$, as defined by (14) or (38), is indeterminate.

Case 2. $\sum_{i=1}^{m}\tau_i\boldsymbol{\omega}_i \propto \boldsymbol{\varepsilon}_a$. This occurs when $\eta_1 = \eta_2 = \cdots = \eta_m \neq 0$.
Here also $\hat{\boldsymbol{\tau}}$ is indeterminate.

Case 3. $\sum_{i=1}^{m}\tau_i\boldsymbol{\omega}_i = \varphi_1\boldsymbol{\varrho} + \varphi_2\boldsymbol{\varepsilon}_a$, $\varphi_1 \neq 0$. This is the situation when
$\eta_i = p\,\xi_i + q$, $(i = 1, 2, \cdots, m)$, with $p \neq 0$. In this case, $\hat{\boldsymbol{\tau}} = \boldsymbol{\varrho}$.

In the first two cases the manifold $\mathcal{O}_{a,\tau}$ is undefined, and our designation of the ν_i is without meaning. We may correct this by replacing $\hat{\boldsymbol{\tau}}$ by $\boldsymbol{\varrho}$ in all our work. Case 3 starts with this situation. Therefore we obtain the same solution in all three cases with the further step: let $\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \cdots, \boldsymbol{\lambda}_{m-2}$ be any orthonormal set spanning $\mathcal{O}_{a,\rho}$, and take $\boldsymbol{\mu}_i = \boldsymbol{\lambda}_i$, $i = 1, 2, \cdots, m-2$. The need for making a particular choice [(31) and (32) are indeterminate for $\hat{\boldsymbol{\tau}} = \boldsymbol{\varrho}$] of $\boldsymbol{\lambda}_1$ and $\boldsymbol{\mu}_1$ is now gone. Just as the sums $\sum_{i=2}^{m-2}\lambda_{ij}\lambda_{ik}$ were calculable above, so now the sums $\sum_{i=1}^{m-2} \lambda_{ij} \lambda_{ik}$ will be found to be determined. The criterion reduces to

$$t = \frac{\sqrt{m-2}\left(\sum_{i=1}^{m} \rho_i X_i + \sum_{i=1}^{n} \tau_i Y_i\right)}{\sqrt{\sum_{i=1}^{m} (X_i + Y_i)^2 + \left[\sum_{i=1}^{m} \rho_i (X_i + Y_i)\right]^2}} . \tag{39}$$

A closer analysis will reveal that each of the three cases above is a limit of non-singular cases with $\hat{\varepsilon} \to \varrho$, whereby the various constituents of the solution go over into the forms we have assigned in the foregoing independent construction. The expression (39) is, in fact, the limiting form of (37) for $\hat{\varepsilon} \to \varrho$ [in this connection formulas (35) may be studied]. Consequently, (37) and (39) do not belong to different tests; together they fully define a single test.

It should be observed, concerning the numerator of t in (37) and (39), that $\sum_{i=1}^{m} \rho_i X_i$ and $-\sum_{i=1}^{m} \tau_i Y_i$ are respectively just the sample regression coefficients of x on ξ and y on η; that is, the respective least-square estimates of k_1 and k_2.

5. Summary

We have determined an exact, unbiased test for the significance of the difference between two regression coefficients, in the situation described at the beginning of section 1 (q.v. for notation). In the course of the work it has been brought out that, for any test, m must be greater than 2, and neither of the sets $(\xi_1, \xi_2, \cdots, \xi_m)$, $(\eta_1, \eta_2, \cdots, \eta_n)$ may consist of all equal elements. Let the samples $(x_1, x_2, \cdots, x_m) \sim (\xi_1, \xi_2, \cdots, \xi_m)$ and $(y_1, y_2, \cdots, y_n) \sim \eta_1, \eta_2, \cdots, \eta_n)$ be ordered in any manner. Then our test, at the level of significance ϵ, is defined by the critical region (cf. sec. 2) W'_0: $|t| > t_\epsilon$, where the criterion t is given by (37), or by (39) when $\eta_i = p\,\xi_i + q$, $(i = 1, 2, \cdots, m)$, for some numbers p and q. The quantities appearing in (37) and (39) are defined by (36) and (38). The criterion has the t-distribution with $m - 2$ degrees of freedom.

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