

CONTENTS

Preface	xi
Chapter 0 Historical Strands of Geometry	1
Art/Pattern Strand	2
Building Structures Strand	7
Navigation/Stargazing Strand	9
Motion/Machines Strand	14
Chapter 1 What is Straight?	21
History: How Can We Draw a Straight Line?	22
Problem 1.1 When Do You Call a Line Straight?	26
The Symmetries of a Line	29
Local (and Infinitesimal) Straightness	33
Chapter 2 Straightness on Sphere	35
Early History of Spherical Geometry	35
Problem 2.1 What is Straight on a Sphere?	38
Symmetries of Great Circles	41
Every Geodesic is a Great Circle	43
Intrinsic Curvature	44
Chapter 3 What is an Angle?	46
Problem 3.1 What is an Angle?	46
Problem 3.2 Vertical Angle Theorem (VAT)	48
Hints for Three Different Proofs	49
Chapter 4 Straightness on Cylinders and Cones	51
Problem 4.1 Straightness on Cylinders and Cones	52
Cones with Varying Cone Angles	53
Geodesics on Cylinders	56
Geodesics on Cones	57
Problem 4.2 Global Properties of Geodesics	58
n -Sheeted Coverings of a Cylinder	58
n -Sheeted (Branched) Coverings of a Cone	60
Locally Isometric	62
Is "Shortest" Always "Straight"?	62
Relations to Differential Geometry	63

Chapter 5 Straightness on Hyperbolic Planes	65
A Short History of Hyperbolic Geometry	65
Description of Annular Hyperbolic Planes	68
Hyperbolic Planes of Different Radii (Curvature)	69
Problem 5.1 What is Straight in a Hyperbolic Plane?	70
Problem 5.2 Coordinate System on Annular Hyperbolic Plane	71
Problem 5.3 The Pseudosphere is Hyperbolic	73
Intrinsic/Extrinsic, Local/Global	75
Problem 5.4 Rotations and Reflections on Surfaces	75
Exploring Curvature	76
Chapter 6 Triangles and Congruencies	83
Geodesics are Locally Unique	84
Problem 6.1 Properties of Geodesics	84
Problem 6.2 Isosceles Triangle Theorem (ITT)	85
Circles	86
Triangle Inequality	88
Problem 6.3 Bisector Constructions	88
Problem 6.4 Side-Angle-Side (SAS)	89
Problem 6.5 Angle-Side-Angle (ASA)	92
Chapter 7 Area and Holonomy	95
Problem 7.1 The Area of a Triangle on a Sphere	96
Problem 7.2 Area of Hyperbolic Triangles	97
Problem 7.3 Sum of the Angles of a Triangle	100
Introducing Parallel Transport	100
Introducing Holonomy	102
Problem 7.4 The Holonomy of a Small Triangle	104
The Gauss-Bonnet Formula for Triangles	105
Problem 7.5 Gauss-Bonnet Formula for Polygons	106
Gauss-Bonnet Formula for Polygons on Surfaces	109
Chapter 8 Parallel Transport	111
Problem 8.1 Euclid's Exterior Angle Theorem (EEAT)	111
Problem 8.2 Symmetries of Parallel Transported Lines	113
Problem 8.3 Transversals through a Midpoint	115
Problem 8.4 What Is "Parallel"?	115

Chapter 9 More Triangle Congruencies	118
Problem 9.1 Side-Side-Side (SSS)	118
Problem 9.2 Angle-Side-Side (ASS)	119
Problem 9.3 Side-Angle-Angle (SAA)	121
Problem 9.4 Angle-Angle-Angle (AAA)	122
Chapter 10 Parallel Postulates	124
Parallel Lines on the Plane Are Special	124
Problem 10.1 Parallel Transport on the Plane	125
Problem 10.2 Parallel Postulates Not Involving (Non-) Intersecting Lines	127
Equidistant Curves on Spheres and Hyperbolic Planes	128
Problem 10.3 Parallel Postulates Involving (Non-) Intersecting Lines	128
Problem 10.4 EFP and HSP on Sphere and Hyperbolic Plane	131
Comparisons of Plane, Spheres, and Hyperbolic Planes	132
Parallel Postulates within the Building Structures Strand	134
Non-Euclidean Geometries within the Historical Strands	136
Chapter 11 Isometries and Patterns	138
Problem 11.1 Isometries	139
Problem 11.2 Three Points Determine an Isometry	143
Problem 11.3 Classification of Isometries	144
Klein's Erlangen Program	147
Symmetries and Patterns	148
Problem 11.4 Examples of Patterns	152
Problem 11.5 Classification of Discrete Strip Patterns	153
Problem 11.6 Classification of Finite Plane Patterns	153
Problem 11.7 Regular Tilings with Polygons	154
Other Periodic (and Non-Periodic) Patterns	156
Geometric Meaning of Abstract Group Terminology	158
Chapter 12 Dissection Theory	160
What is Dissection Theory?	160
A Dissection Puzzle from 250 B.C. Solved in 2003	162
History of Dissections in the Theory of Area	163
Problem 12.1 Dissect Plane Triangle and Parallelogram	164
Dissection Theory on Spheres and Hyperbolic Planes	165
Problem 12.2 Khayyam Quadrilaterals	165
Problem 12.3 Dissect Spherical and Hyperbolic Triangles and Khayyam Parallelograms	166
Problem 12.4 Spherical Polygons Dissect to Lunes	167

Chapter 13 Square Roots, Pythagoras, and Similar Triangles	169
Square Roots	170
Problem 13.1 A Rectangle Dissects into a Square	171
Baudhayana’s Sulbasutram	174
Problem 13.2 Equivalence of Squares	177
Any Polygon Can be Dissected into a Square	178
History of Dissections	179
Problem 13.3 More Dissection-Related Problems	180
Three-Dimensional Dissections and Hilbert’s Third Problem	181
Problem 13.4 Similar Triangles	182
Chapter 14 Projections of a Sphere onto a Plane	184
Problem 14.1 Charts Must Distort	185
Problem 14.2 Gnomonic Projection	185
Problem 14.3 Cylindrical Projection	186
Problem 14.4 Stereographic Projection	187
History of Stereographic Projection and Astrolabe	188
Chapter 15 Circles	191
Problem 15.1 Angles and Power of Points for Circles in the Plane	192
Problem 15.2 Power of Points for Circles on Spheres	193
Problem 15.3 Applications of Power of a Point	196
Problem 15.4 Trisecting Angles and Other Constructions	197
Chapter 16 Inversions in Circles	201
Early History of Inversions	201
Problem 16.1 Inversions in Circles	202
Problem 16.2 Inversions Preserve Angles and Preserve Circles (and Lines)	204
Problem 16.3 Using Inversions to Draw Straight Lines	206
Problem 16.4 Apollonius’ Problem	207
Expansions of the Notion of Inversions	211
Chapter 17 Projections (Models) of Hyperbolic Planes	213
Distortion of Coordinate Systems	214
Problem 17.1 A Conformal Coordinate System	215
Problem 17.2 Upper Half-Plane Is Model of Annular Hyperbolic Plane	216
Problem 17.3 Properties of Hyperbolic Geodesics	218
Problem 17.4 Hyperbolic Ideal Triangles	219
Problem 17.5 Poincaré Disk Model	221
Problem 17.6 Projective Disk Model	222

Chapter 18 Geometric 2-Manifolds	223
Problem 18.1 Flat Torus and Flat Klein Bottle	224
Problem 18.2 Universal Covering of Flat 2-Manifolds	229
Problem 18.3 Spherical 2-Manifolds	231
Coverings of a Sphere	233
Problem 18.4 Hyperbolic Manifolds	235
Problem 18.5 Area and Euler Number	238
Triangles on Geometric Manifolds	239
Problem 18.6 Can the Bug Tell Which Manifold?	241
Chapter 19 Geometric Solutions of Quadratic and Cubic Equations	242
Problem 19.1 Quadratic Equations	243
Problem 19.2 Conic Sections and Cube Roots	246
Problem 19.3 Solving Cubic Equations Geometrically	249
Problem 19.4 Algebraic Solution of Cubics	251
What Does This All Point To?	253
Chapter 20 Trigonometry and Duality	255
Problem 20.1 Circumference of a Circle	255
Problem 20.2 Law of Cosines	257
Problem 20.3 Law of Sines	259
Duality on a Sphere	260
Problem 20.4 The Dual of a Small Triangle on a Sphere	262
Problem 20.5 Trigonometry with Congruences	262
Duality on the Projective Plane	262
Problem 20.6 Properties on the Projective Plane	263
Perspective Drawing and History	264
Chapter 21 Mechanisms	267
Interactions of Mechanisms with Mathematics	267
Problem 21.1 Four-Bar Linkages	271
Problem 21.2 Universal Joint	275
Problem 21.3 Reauleaux Triangle and Constant Width Curves	277
Involutives	280
Linkages Interact with Mathematics	282
Chapter 22 3-Spheres and Hyperbolic 3-Spaces	284
Problem 22.1 Explain 2-Sphere in 3-Space to a 2-Dimensional Bug	285
What is 4-Space? Vector Spaces and Bases	286

Problem 22.2 A 3-Sphere in 4-Space	288
Problem 22.3 Hyperbolic 3-Space, Upper Half Space	290
Problem 22.4 Disjoint Equidistant Great Circles	292
Problem 22.5 Hyperbolic and Spherical Symmetries	294
Problem 22.6 Triangles in 3-Dimensional Spaces	295
Chapter 23 Polyhedra	296
Definitions and Terminology	296
Problem 23.1 Measure of a Solid Angle	297
Problem 23.2 Edges and Face Angles	298
Problem 23.3 Edges and Dihedral Angles	299
Problem 23.4 Other Tetrahedral Congruence Theorems	299
Problem 23.5 The Five Regular Polyhedra	300
Chapter 24 3-Manifolds – The Shape of Space	303
Space as an Oriented Geometric 3-Manifold	304
Problem 24.1 Could Our Universe Be Non-Euclidean?	305
Problem 24.2 Euclidean 3-Manifolds	307
Problem 24.3 Dodecahedral 3-Manifolds	309
Problem 24.4 Some Other Geometric 3-Manifolds	311
In Search of Meaning	313
Appendix A Making Models	325
The Hyperbolic Plane from Paper Annuli	325
Polyhedral Constructions of the Hyperbolic Plane	326
Hyperbolic Soccer Ball	328
Crocheting a Hyperbolic Plane	329
Crocheting Symmetric Hyperbolic Plane	331
Crocheting Hyperbolic Pair of Pants	333
Appendix B Ideas for More Geometric Explorations	340
Some resources	340
Curvahedra	341
Exploring Polyhedra	343
For Further Explorations of Tilings and Polyhedra	347
3-D Symmetry	348
Bibliography	353