4. INCREASING EVENTS.

This chapter contains the well known FKG inequality and a formula of Russo's for the derivative of $P_p\{E\}$ with respect to p for an increasing event E. No periodicity assumptions are necessary in this chapter, so that we shall take as our probability space the triple $(\Omega_{\mathcal{V}}, \mathcal{B}_{\mathcal{V}}, P_{\mathcal{V}})$ as defined in Sect. 3.1. $E_{\mathcal{V}}$ will denote expectation with respect to $P_{\mathcal{V}}$.

Def. 1 A $\mathcal{B}_{\mathcal{V}}$ -measurable function $f:\Omega_{\mathcal{V}} \to \mathbb{R}$ is called increasing (decreasing) if it is[1)] increasing (decreasing) in each $\omega(v)$, $v \in \mathcal{V}$. An event $E \in \mathcal{B}_{\mathcal{V}}$ is called increasing (decreasing) if its indicator function is increasing (decreasing).

Examples

(i) $\{\#W(v)\}$ is an increasing function, since making more sites occupied can only increase $W(v)$.

(ii) $E_1 = \{\#W(v) = \infty\}$ for fixed v is an increasing event; if E_1 occurs in the configuration ω', and every site which is occupied in ω' is also occupied in ω'' - and possibly more sites are occupied in ω'' - then E_1 also occurs in configuration ω''.

(iii) $E_2 = \{\exists$ an occupied path on $\mathcal{G}$ from v_1 to $v_2\}$ for fixed vertices v_1 and v_2 is increasing for the same reasons as E_1 in ex. (ii).

(iv) The most important example of an increasing event for our purposes is the existence of an occupied crosscut of a certain Jordan domain in $\mathbb{R}^2$. More precisely we shall be interested in pair of matching graphs $(\mathcal{G},\mathcal{G}^*)$ in $\mathbb{R}^2$ based on $(\mathcal{M},\mathcal{F})$, $\mathcal{G}_{p\ell}$, $\mathcal{G}^*_{p\ell}$ and $\mathcal{M}_{p\ell}$ will be the planar modifications of $\mathcal{G}$, $\mathcal{G}^*$ and $\mathcal{M}$ respectively (see Sect. 2.2 and 2.3). Let J be a Jordan curve on $\mathcal{M}_{p\ell}$

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1) We use "increasing" and "strictly increasing" instead of "non-decreasing" and "increasing".

consisting of four closed areas B_1, A_1, B_2 and C with disjoint interiors. Then

$$E_3 := \{ \exists \text{ occupied path } r \text{ on } \mathcal{G}_{p\ell} \text{ with initial (final) point on } B_1(B_2) \text{ and such that } r \text{ minus its endpoints is contained in } \operatorname{int}(J)\} \quad .$$

is an increasing event. For further details see Ex. (iii) in the next section. ///

Before treating the principal results of this chapter we prove a simple lemma, stating that the expectation of an increasing function goes up when the probability that a site is occupied goes up. Inequality (4.2) gives an upper bound for this effect, though. The lemma will be useful later.

<u>Lemma 4.1</u>. <u>If</u> $f:\Omega_{\mathcal{V}} \to [0,\infty)$ <u>is an increasing non-negative function and</u>

$$P_{\mathcal{V}} = \prod_{v \in \mathcal{V}} \mu_v \quad , \quad P'_{\mathcal{V}} = \prod_{v \in \mathcal{V}} \mu'_v$$

<u>are two product measure on</u> $\Omega_{\mathcal{V}}$ <u>which satisfy</u>

$$\mu'_v\{\omega(v) = 1\} \geq \mu_v\{\omega(v) = 1\} \quad , \quad v \in \mathcal{V} \quad ,$$

<u>then (with</u> $E_{\mathcal{V}}$ $(E'_{\mathcal{V}})$ <u>denoting expectation with respect to</u> $P_{\mathcal{V}}$ $(P'_{\mathcal{V}})$)

$$(4.1) \qquad E'_{\mathcal{V}} f \geq E_{\mathcal{V}} f \quad .$$

<u>If</u> $f \geq 0$ <u>depends only on the</u> $\omega(v)$ <u>for</u> v <u>in a subset</u> $\mathcal{W}$ <u>of</u> $\mathcal{V}$ <u>with cardinality</u> $m = \# \mathcal{W}$, <u>then</u> .

$$(4.2) \qquad E'_{\mathcal{V}} f \leq \left(\max_{v \in \mathcal{W}} \frac{\mu'_v\{\omega(v) = 1\}}{\mu_v\{\omega(v) = 1\}} \right)^m E_{\mathcal{V}} f \; .$$

<u>For a decreasing non-negative function</u> f <u>the inequality in (4.1) is reversed, while (4.2) is to be replaced by</u>

$$E_{\mathcal{V}} f \leq \left(\max_{v \in \mathcal{W}} \frac{\mu_v\{\omega(v) = -1\}}{\mu'_v\{\omega(v) = -1\}} \right)^m E'_{\mathcal{V}} f$$

<u>Proof</u>: The lemma is proved by "coupling". We construct a measure

P on $(\Omega_{\mathcal{V}} \times \Omega_{\mathcal{V}}, \mathcal{B}_{\mathcal{V}} \times \mathcal{B}_{\mathcal{V}})$ such that its marginal distribution on the first (second) factor is $P_{\mathcal{V}}(P'_{\mathcal{V}})$ and with the following properties:

$$(4.3) \qquad P \text{ is a product measure } \prod_{v \in \mathcal{V}} ,$$

where ν_v is a measure on $\{-1,1\} \times \{-1,1\}$. Thus if we write a generic point of $\Omega_{\mathcal{V}} \times \Omega_{\mathcal{V}}$ as $\{(\omega(v),\omega'(v)): v \in \mathcal{V}\}$, then the random variables $(\omega(v),\omega'(v))$, $v \in \mathcal{V}$, are independent under P. Moreover we will have

$$(4.4) \qquad P\{\omega(v) = 1 \mid \omega'(v) = 1\} = \frac{\mu\{\omega(v) = 1\}}{\mu'\{\omega(v) = 1\}}$$

and

$$(4.5) \qquad P\{(\omega,\omega') \in \Omega_{\mathcal{V}} \times \Omega_{\mathcal{V}}: \omega(v) \le \omega'(v) \text{ for all } v\} = 1.$$

To construct such a product measure we merely have to choose the ν_v suitably. We take

$$\nu_v\{\omega(v) = -1, \ \omega'(v) = -1\} = \mu'_v\{\omega(v) = -1\} ,$$

$$\nu_v\{\omega(v) = -1, \ \omega'(v) = 1\} = \mu'_v\{\omega(v) = 1\} - \mu_v\{\omega(v) = 1\} ,$$

$$\nu_v\{\omega(v) = 1, \ \omega'(v) = -1\} = 0 ,$$

$$\nu_v\{\omega(v) = 1, \ \omega'(v) = 1\} = \mu\{\omega(v) = 1\} .$$

(4.4) and (4.5) obviously hold for these ν_v, and one easily checks that P has the prescribed marginal distributions. Now, for any increasing $f \ge 0$, by (4.5)

$$E'_{\mathcal{V}} f = \int_{\Omega_{\mathcal{V}}} f(\omega') \, dP'_{\mathcal{V}}(\omega') = \int_{\Omega_{\mathcal{V}} \times \Omega_{\mathcal{V}}} f(\omega') \, dP(\omega,\omega')$$

$$\ge \int_{\Omega_{\mathcal{V}} \times \Omega_{\mathcal{V}}} f(\omega) \, dP(\omega,\omega') = E_{\mathcal{V}} f .$$

This proves (4.1).

To prove (4.2) note that (4.4) implies

$$P\{\omega(v) \geq \omega'(v) \text{ for all } v \in \mathfrak{W} \mid \omega'\}$$
$$= \prod_{\substack{v \in \mathfrak{W} \\ \omega'(v) = 1}} P\{\omega(v) = 1 \mid \omega'(v) = 1\} \geq \left(\min_{v \in \mathfrak{W}} \frac{\mu\{\omega(v) = 1\}}{\mu'\{\omega(v) = 1\}} \right)^m .$$

For an increasing $f \geq 0$ which depends only on the occupancies in $\mathfrak{W}$ we now have

$$E_{\mathfrak{V}} f = \int_{\Omega_{\mathfrak{V}} \times \Omega_{\mathfrak{V}}} f(\omega) \, dP(\omega,\omega')$$
$$\geq \int_{\substack{\Omega_{\mathfrak{V}} \times \Omega_{\mathfrak{V}} \\ \omega(v) \geq \omega'(v) \text{ on } \mathfrak{W}}} f(\omega') \, dP(\omega,\omega')$$
$$= \int_{\Omega} f(\omega') \, P\{\omega(v) \geq \omega'(v) \text{ for all } v \in \mathfrak{W} \mid \omega'\} \, dP(\omega')$$
$$\geq \left(\min_{v \in \mathfrak{W}} \frac{\mu\{\omega(v) = 1\}}{\mu'\{\omega(v) = 1\}} \right)^m E'_{\mathfrak{V}} f .$$

This is equivalent to (4.2). We leave it to the reader to derive the analogues of (4.1) and (4.2) for decreasing f, by interchanging the roles of $E_{\mathfrak{V}}$ and $E'_{\mathfrak{V}}$. □

4.1. The FKG inequality.

We only discuss the very special case of the FKG inequality which we need in these notes. This special case already appeared in Harris (1960). For more general versions the reader can consult the original article of Fortuin, Kasteleyn and Ginibre (1971) or the recent article by Batty and Bollman (1980) and its references.

Proposition 4.1. *If f and g are two bounded functions on $\Omega_{\mathfrak{V}}$ which depend on finitely many coordinates of ω only and which are both increasing or both decreasing functions, then*

$$(4.6) \qquad E_{\mathfrak{V}}\{f(\omega)\, g(\omega)\} \geq E_{\mathfrak{V}}\{f(\omega)\}\, E_{\mathfrak{V}}\{g(\omega)\} .$$

In particular, if E and F are two increasing events, or two decreasing events, which depend on finitely many coordinates of ω only, then

(4.7) $$P_{\mathcal{V}}\{E \cap F\} \geq P_{\mathcal{V}}\{E\} \cdot P_{\mathcal{V}}\{F\} \quad .$$

Proof: For (4.6) it suffices to take f and g increasing. The decreasing case follows by applying (4.6) to $-f$ and $-g$. Order the elements of $\mathcal{V}$ in some arbitrary way as $v_1, v_2, \ldots$, and write ω_i for $\omega(v_i)$. Without loss of generality assume that $f(\omega)$ and $g(\omega)$ depend on $\omega_1, \ldots, \omega_n$ only. If $n = 1$, then (4.6) follows from the fact that for each ω_1, ω_1',

$$\{f(\omega_1) - f(\omega_1')\}\{g(\omega_1) - g(\omega_1')\} \geq 0$$

(check the cases $\omega_1 \geq \omega_1'$ and $\omega_1 \leq \omega_1'$). Thus

$$0 \leq \iint \{f(\omega_1) - f(\omega_1')\}\{g(\omega_1) - g(\omega_1')\} \; P_{\mathcal{V}}(d\omega) \; P_{\mathcal{V}}(d\omega')$$

$$= 2\, E_{\mathcal{V}}\{fg\} - 2\, E_{\mathcal{V}}\{f\}\, E_{\mathcal{V}}\{g\} \; .$$

The general case of (4.6) follows by induction on n since

$$E_{\mathcal{V}}\{fg\} = E_{\mathcal{V}}\{E_{\mathcal{V}}\{fg|\omega_2, \ldots, {}_n\}\} \quad ,$$

$$\geq E_{\mathcal{V}}\{E_{\mathcal{V}}\{f|\omega_2, \ldots, \omega_n\}\; E_{\mathcal{V}}\{g|\omega_2, \ldots, \omega_n\}\}$$

(since for fixed $\omega_2, \ldots, \omega_n$, $f(\omega)$ and $g(\omega)$ are increasing functions of ω_1 only)

$$\geq E_{\mathcal{V}}\{E_{\mathcal{V}}\{f|\omega_2, \ldots, \omega_n\}\}\; E_{\mathcal{V}}\{E_{\mathcal{V}}\{g|\omega_2, \ldots, \omega_n\}\}$$

(since $E_{\mathcal{V}}\{f|\omega_2, \ldots, \omega_n\}$ is an increasing function of $\omega_2, \ldots, \omega_n$ and similarly for g, plus the induction hypotheses) $= E_{\mathcal{V}}\{f\}\; E_{\mathcal{V}}\{g\}$.

This proves (4.6) and (4.7) is the special case with $f = I_E$, $g = I_F$. □

Application.

For a simple application of the FKG inequality let v_1, v_2 be two vertices of a connected graph $\mathcal{G}$. Then if there is an occupied path from v_1 to v_2 the occupied clusters of v_1 and v_2 are identical. Therefore, by (4.7) and Ex. 4(i) and 4(iii).

$$(4.8)\qquad P_{\mathcal{V}}\{\#W(v_1) \geq n\} \geq P_{\mathcal{V}}\{\exists \text{ occupied path from } v_1 \text{ to } v_2 \text{ and } \#W(v_2) \geq n\} \geq P_{\mathcal{V}}\{\exists \text{ occupied path from } v_1 \text{ to } v_2\}\, P_{\mathcal{V}}\{\#W(v_2) \geq n\} .$$

If $\mathcal{G}$ is connected and $P_{\mathcal{V}}\{v$ is occupied$\} > 0$ for all v, then also

$$P_{\mathcal{V}}\{\exists \text{ occupied path from } v_1 \text{ to } v_2\} > 0 .$$

Therefore

$$\theta(v_2) > 0 \text{ implies } \theta(v_1) > 0 \text{ and}$$

$$E\{\#W(v_2)\} = \infty \text{ implies } E\{\#W(v_1)\} = \infty .$$

This justifies our statement in Sect. 3.4, that P_H and P_T are independent of the choice of v.

4.2. Pivotal sites and Russo's formula.

Def. 2. Let $E \in \mathcal{B}_{\mathcal{V}}$ be an event and $\omega \in \Omega_{\mathcal{V}}$ an occupancy configuration. A site $v \in \mathcal{V}$ is called pivotal for (E,ω) (or for E for short) iff

$$I_E(\omega) \neq I_E(T_v\omega) ,$$

where $T_v\omega \in \Omega$ is determined by

$$(4.9)\qquad T_v\omega(w) = \begin{cases} \omega(w) & \text{for } w \in \mathcal{V} \text{ but } w \neq v \\ -\omega(v) & \text{for } w = v . \end{cases}$$

In other words, v is pivotal, if changing the occupancy of v only changes the occupancy configuration from one where E occurs to one where E does not occur, or vice versa.

Examples.

(i) Let E_1 be as in Ex. 4(ii) and take

$$F_1 = \{\omega : \#W(w,\omega) = \infty \text{ for some neighbor } w \text{ of } v\}$$

Then v is pivotal for (E_1,ω) iff $\omega \varepsilon F_1$. Indeed for $\omega \varepsilon F_1$, E_1 occurs iff v itself is occupied (recall that $W(v) = \emptyset$ if v is vacant), and hence $I_{E_1}(\omega)$ will change with $\omega(v)$ for $\omega \varepsilon F_1$. On the other hand, if $\omega \not\varepsilon F_1$, then $\#W(v,\omega) < \infty$, no matter what $\omega(v)$ is.

(ii) Let E_2 be as in Ex. 4(iii) and take

$$F_2 = E_2 \cap \{\omega : \text{ all occupied paths from } v_1 \text{ to } v_2 \text{ contain the vertex } v\}.$$

Then

$$I_{E_2}(\omega) = 1 \quad \text{and} \quad \omega(v) = 1 \quad \text{for} \quad \omega \varepsilon F_2$$

But in $T_v\omega$, v is vacant and there are no longer any occupied paths from v_1 to v_2, since on F_2 all these paths had to go through v, and v has now been made vacant. Thus v is pivotal for (E_2,ω) whenever $\omega \varepsilon F_2$.

(iii) This example plays a fundamental role in the later development. Let $(\mathcal{G},\mathcal{G}^*)$ be a periodic matching pair of graphs in $\mathbb{R}^2$, based on $(\mathcal{M},\mathcal{F})$ and let $\mathcal{G}_{p\ell}$, $\mathcal{G}^*_{p\ell}$ and $\mathcal{M}_{p\ell}$ be the planar modifications defined in Sect. 2.3. This time we take $\mathcal{V}$ = vertex set of $\mathcal{G}_{p\ell}$ and define $\Omega_{\mathcal{V}}$, $\mathcal{B}_{\mathcal{V}}$ accordingly. We are interested in the existence of "occupied crosscuts of Jordan domains". More precisely, let J be a Jordan curve on $\mathcal{M}_{p\ell}$, consisting of four closed arcs, B_1, A, B_2 and C, with disjoint interiors and occuring in this order as J is traversed in one direction. $\bar{J} = \text{int}(J) \cup J$. We consider paths $r = (v_0,e_1,\dots,e_\nu, v_\nu)$ on $\mathcal{G}_{p\ell}$ which satisfy

$$(4.10) \qquad (e_1\setminus \{v_0\}, v_1, e_2,\dots, e_{\nu-1}, v_{\nu-1}, e_\nu \setminus \{v_\nu\}) \subset \text{int}(J),$$

and

$$(4.11) \qquad v_0 \varepsilon B_1 , v_\nu \varepsilon B_2 .$$

(4.10) and (4.11) are just the conditions (2.23) - (2.25) in the present setup, since an edge of $\mathcal{G}_{p\ell} \subset \mathcal{M}_{p\ell}$ can intersect the curve J on $\mathcal{M}_{p\ell}$ in a vertex of $\mathcal{M}_{p\ell}$ only, by virtue of the planarity of $\mathcal{M}_{p\ell}$. We call any path r on $\mathcal{G}_{p\ell}$ which satisfies (4.10) and (4.11) a <u>crosscut</u> of int(J). We can now define $J^-(r)$ and $J^+(r)$ as in Def. 2.11 and order r_1 and r_2 as in Def. 2.12, whenever r, r_1, r_2 satisfy (4.10) and (4.11). We take

(4.12) $$E_3 = \{\omega: \exists \text{ at least one occupied crosscut of } \text{int}(J)\},$$

and we want to find the pivotal sites for (E_3,ω) when $\omega \varepsilon E_3$. By Prop. 2.3, if E_3 occurs, then there exists a unique lowest crosscut of int(J) on $\mathcal{G}_{p\ell}$, which we denote by $R(\omega)$. Now let $\omega \varepsilon E_3$, so $R(\omega)$ exists and v a vertex which is not on R. Then changing the occupancy of v leaves the crosscut R intact and such a site v is therefore not pivotal for (E_3,ω). Next consider a v on $R \cap \text{int}(J)$ which has a <u>vacant connection to $\mathring{C}$</u>. By this we mean that there exists a path $s^* = (v_0^*, e_1^*, \ldots, e_\rho^*, v_\rho^*)$ on $\mathcal{G}_{p\ell}^*$

satisfying the following conditions (4.13) - (4.16):

(4.13) there exists an edge e of $\mathcal{M}_{p\ell}$ between v and v_0 such that $\mathring{e} \subset J^+(R)$ (in particular $v \; \mathcal{M}_{p\ell} \; v_0$),

(4.14) $$v_\rho^* \varepsilon \mathring{C} \quad ,$$

(4.15) $$(v_0^*, e_1^*, \ldots, v_{\rho-1}^*, e_\rho^* \setminus \{v_\rho^*\}) \subset J^+(R) \;,$$

(4.16) all vertices of s^* are vacant.

We allow here the possibility $\rho = 0$ in which case s^* reduces to the single vertex $v_0^* = v_\rho^*$, and we make the convention that (4.15) is automatically fulfilled in this case. We claim that any $v \varepsilon R \cap \text{int}(J)$ with such a vacant connection to $\mathring{C}$ is pivotal for (E_3,ω) whenever $\omega \varepsilon E_3$. To prove this claim note that v is on $R(\omega)$, hence is occupied in ω, and therefore vacant in $T_v \omega$. If there would exist an occupied crosscut r of int(J) in $T_v\omega$, then r could not contain v, which is vacant in $T_v\omega$. Thus r would also be occupied in ω and by Prop. 2.3 (see (2.27)) we would have

(4.17) $$r \varepsilon \bar{J}^+(R) \;.$$

Now, if $R = (v_0, e_1, \ldots, e_\nu, v_\nu)$, then the boundary of $J^+(R)$ consists of R, the segment of B_2 from v_ν to the intersection of B_2

with C(call this segment B_2^+), C, and the segment of B_1 from the intersection of B_1 with C to v_0(call this segment B_1^+) ; see Fig. 4.1. This boundary is, in fact, a Jordan curve.

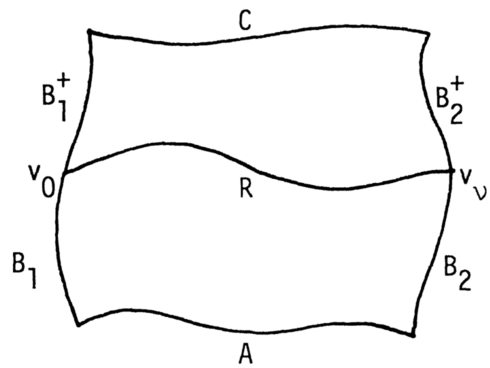

Figure 4.1

Since r would begin on B_1 and end on B_2 and satisfy (4.17) it would in fact connect a point on B_1^+ with a point on B_2^+ inside $\bar{J}^+(R)$. Next consider, the path $\tilde{s}: = (v, e, v_0^*, e_1^*, \ldots, e_\nu^*, v_\nu^*)$, where e is as in (4.13). From the requirements $\mathring{e} \subset J^+(R)$, (4.14) and (4.15) it follows that $\tilde{s}$ is a crosscut of $J^+(R)$. Moreover, its endpoints - v on $R \cap \text{int}(J)$ and v_ρ^* on $\mathring{C}$ - separate the endpoints of r on B_1^+ and B_2^+. Thus r would have to intersect $\tilde{s}$. This, however, is impossible. Indeed, the paths r and $\tilde{s}$ on $\mathcal{M}_{p\ell}$ would have to intersect in a vertex of $\mathcal{M}_{p\ell}$ (recall that $\mathcal{M}_{p\ell}$ is planar) which would have to be occupied - being a vertex of r - as well as vacant - being also a vertex on $\tilde{s}$. (Note that the v_i^* are vacant in ω, hence in $T_v\omega$, and v became vacant in $T_v\,\omega$). Thus no occupied crosscut r of $\text{int}(J)$ can exist in $T_v\omega$, i.e., $T_v\omega \notin E_3$, which proves our claim.

We remark (without proof) that a certain converse of the above holds. Assume that $A \cap B_i$ as well as $C \cap B_i$ is a vertex of $\mathcal{M}_{p\ell}$, $i = 1,2$. Then under the convention (2.15), (2.16) the only pivotal sites on $R \cap \text{int}(J)$ for (E_3, ω) are vertices which have a vacant connection to C. (We call s^* a vacant connection to C if (4.13) (4.15) and (4.16) hold but (4.14) is replaced by $v_\rho^* \in C$). This can be derived from a variant of Prop. 2.2. We shall, however, not need this fact.

<u>Proposition 4.2</u> (<u>Russo's formula</u>) <u>Let</u> $E \in \mathcal{B}_\mathfrak{t}$ <u>be an increasing event and</u> $P_\mathfrak{t}$ <u>as in</u> (3.3), (3.4) <u>with</u> $\mathcal{E}$ <u>replaced by</u> $\mathfrak{t}$.

Then

(4.18) $\quad \frac{\partial}{\partial p(v)} P_{\mathcal{V}}\{E\} = P_{\mathcal{V}}\{v \text{ is pivotal for } (E,\omega)\}$.

Let p' and p'' be any two functions from $\mathcal{V}$ into $[0,1]$ and set

(4.19) $\quad \mu_{vt}\{\omega(v) = 1\} = 1-\mu_{vt}\{\omega(v) = -1\}$

$$= (1-t)p'(v) + tp''(v), \; v \in \mathcal{V}, \; 0 \leq t \leq 1,$$

(4.20) $\quad P_{\mathcal{V}t} = \prod_{v \in \mathcal{V}} \mu_{vt}$.

If

(4.21) $\quad p'(v) \leq p''(v)$ for all $v \in \mathcal{V}$,

and E is an increasing event which depends on the occupancy of finitely many vertices only, then for any subset $\mathcal{W}$ of $\mathcal{V}$, then

(4.22) $\quad \frac{d}{dt} P_{\mathcal{V}t}\{E\}$

$$= \sum_{v \in \mathcal{V}} \{p''(v) - p'(v)\} P_{\mathcal{V}t}\{v \text{ is pivotal for } E\}$$

$$\geq \inf_{v \in \mathcal{W}} \{p''(v) - p'(v)\} E_{\mathcal{V}t}\{\# \text{ of pivotal sites for } E \text{ in } \mathcal{W}\}.$$

(Of course $E_{\mathcal{V}t}$ denotes expectation with respect to $P_{\mathcal{V}t}$)

Proof: (Russo (1981)). To prove (4.18) write

(4.23) $\quad P_{\mathcal{V}}\{E\} = E_{\mathcal{V}}\{I_E\} = p(v)\, E_{\mathcal{V}}\{I_E \mid v \text{ is occupied}\}$

$$+ (1-p(v))\, E_{\mathcal{V}}\{I_E \mid v \text{ is vacant}\} .$$

Since $\omega(v)$ is independent of all other sites, the conditional expectations in the right hand side of (4.23) are integrals with respect to $\prod_{w \neq v} \mu_w$ and are independent of $p(v)$. Therefore

$$(4.24) \qquad \frac{\partial}{\partial\, p(v)} P_{\mathcal{U}}\{E\} = E_{\mathcal{U}}\{I_E \mid v \text{ is occupied}\} - E_{\mathcal{U}}\{I_E \mid v \text{ is vacant}\} .$$

Next set

$$J = J(\omega) = J(\omega;E,v) = \begin{cases} 1 & \text{if } v \text{ is pivotal for } (E,\omega) \\ 0 & \text{if } v \text{ is not pivotal for } (E,\omega). \end{cases}$$

Then, from (4.24)

$$(4.25) \quad \frac{\partial}{\partial\, p(v)} P_{\mathcal{U}}\{E\} = E_{\mathcal{U}}\{I_E J \mid v \text{ is occupied}\} + E_{\mathcal{U}}\{I_E(1-J) | v \text{ is occupied}\} - E_{\mathcal{U}}\{I_E J | v \text{ is vacant}\} - E_{\mathcal{U}}\{I_E(1-J) | v \text{ is vacant}\} .$$

Now the function $I_E(\omega)(1-J(\omega))$ can take only the values 0 and 1. $I_E(\omega)(1-J(\omega)) = 1$ only if E occurs and v is not pivotal for (E,ω), i.e., E occurs in ω, and also if $\omega(v)$ is changed to $-\omega(v)$. Clearly $I_E(\omega)(1-J(\omega)) = 1$ is a condition on $\omega(w)$, $w \neq v$, only, so that $I_E(\omega)(1-J(\omega))$ is independent of $\omega(v)$. Therefore the second and fourth term in the right hand side of (4.25) cancel. Also, if v is pivotal for (E,ω) and E is increasing, then E must occur if $\omega(v) = 1$ and cannot occur if $\omega(v) = -1$. Therefore the third term in the right hand side of (4.25) vanishes. This leaves us with

$$(4.26) \qquad \frac{\partial}{\partial\, p(v)} P_{\mathcal{U}}\{E\} = \frac{E_{\mathcal{U}}\{I_E(\omega)J(\omega)I[\omega(v) = +1]\}}{P_{\mathcal{U}}\{\omega(v) = 1\}} .$$

But, by the argument just given, E must occur if $J(\omega)I[\omega(v) = 1] = 1$, so that we can drop the factor I_E in the numerator on the right of (4.26). Finally $J(\omega)$ is again independent of $\omega(v)$, since $J(\omega) = 1$ means $\omega(w)$, $w \neq v$, is such that E occurs when $\omega(v) = 1$ and does not occur when $\omega(v) = -1$. Thus, the right hand side of (4.26) equals

$$E_{\mathcal{U}}\{J(\omega)\} = P_{\mathcal{U}}\{v \text{ is pivotal for } E\} .$$

This proves (4.18). (4.22) follows now from the chain rule and

(4.21). (Note that v can be pivotal for E only if I_E depends on $\omega(v)$; hence the sum in the middle of (4.22) has only finitely many non-zero terms). □