LOCAL TRUTH

> *"a Grothendieck topology appears most naturally as a modal operator, of the nature 'it is locally the case that'"*
>
> F. W. Lawvere

The notion of a topological bundle represents but one side of the coin of sheaf theory. The other involves the conception of a sheaf as a functor defined on the category of open sets in a topological space. Our aim now is to trace the development of ideas that leads from this notion, via Grothendieck's generalisation, to the notion of a "topology" on a category and its attendant sheaf concept, and from there to the first-order concept of a topology on a topos and the resultant axiomatic sheaf theory of Lawvere and Tierney. The chapter is basically a survey, and its intention is to direct the reader to the appropriate literature.

14.1. Stacks and sheaves

Let I be a topological space, with Θ its set of open subsets. Θ becomes a poset category under the set inclusion ordering, in which the arrows are just the inclusions $U \hookrightarrow V$.

A *stack* or *pre-sheaf* over I is a *contravariant* functor from Θ to **Set**. Thus a stack F assigns to each open V a set $F(V)$, and to each inclusion $U \hookrightarrow V$ a function $F^V_U : F(V) \to F(U)$ (note the contravariance – reversal of arrow direction), such that

(i) $F^U_U = \mathrm{id}_U$, and

(ii) if $U \subseteq V \subseteq W$, then

$$\begin{tikzcd} & F(W) \arrow[dl, "F^W_V"'] \arrow[dr, "F^W_U"] & \\ F(V) \arrow[rr, "F^V_U"'] & & F(U) \end{tikzcd}$$

commutes, i.e. $F^W_U = F^V_U \circ F^W_V$.

EXAMPLE. Let $f: A \to I$ be a sheaf of sets of germs over I, as in Chapter 4. Define a stack $F_f : \Theta \to \mathbf{Set}$ as follows.

$$\begin{aligned} F_f(V) &= \text{the set of local sections of } f \text{ defined on } V \\ &= \{V \xrightarrow{s} A : s \text{ is continuous and } f \circ s = V \hookrightarrow I\} \end{aligned}$$

For an inclusion $U \hookrightarrow V$, F^V_{fU} is the "restricting" or "localising" map that assigns to each section $s: V \to A$ over V its restriction $s \restriction U : U \to A$ to U. Identifying sections s with their images $s(I) \subseteq A$ we have the picture

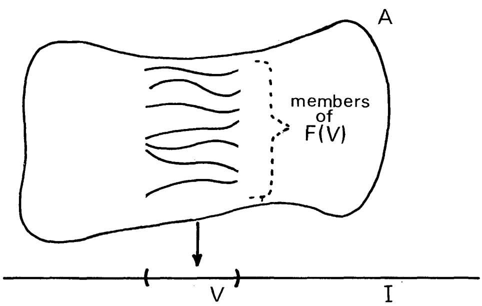


Fig. 14.1.

which indicates the origin of the word "stack". F_f is the *stack of sections over I*. The category $\mathbf{St}(I)$ has as objects the stacks $F : \Theta \to \mathbf{Set}$ and as arrows $\tau : F \dot{\to} G$ the natural transformations, i.e. collections $\{\tau_U : U \in \Theta\}$ of functions $\tau_U : F(U) \to G(U)$ such that

$$\begin{array}{ccc} V & & F(V) \xrightarrow{\tau_V} G(V) \\ \uparrow & & F^V_U \downarrow \qquad\qquad \downarrow G^V_U \\ U & & F(U) \xrightarrow{\tau_U} G(U) \end{array}$$

commutes whenever $U \subseteq V$.

Now a contravariant functor $\Theta \to \mathbf{Set}$ can be construed as a covariant functor from Θ^{op}, the opposite category to Θ, to $\mathbf{Set}$ (cf. §9.1). Thus $\mathbf{St}(I)$ is equivalent to the topos $\mathbf{Set}^{\Theta^{\mathrm{op}}}$

EXERCISE 1. Let $h:(A, f) \to (B, g)$ be an arrow in the spatial topos $\mathbf{Top}(I)$ of sheaves of sets of germs over I. For each open V, define $h_V : F_f(V) \to F_g(V)$ to be the function that maps a section $s \in F_f(V)$ to

$$\begin{tikzcd} A \arrow[rr, "h"] \arrow[dr, "f"'] & & B \arrow[dl, "g"] \\ V \arrow[u, "s"] \arrow[r, hook] & I & \end{tikzcd}$$

$h \circ s$, i.e $h_V(s) = h \circ s$. Verify that $h \circ s$ *is* a section of g, i.e. $h_V(s) \in F_g(V)$, and that the h_V's, for all $V \in \Theta$, form the components of an arrow $\tau_h : F_f \to F_g$ in $\mathbf{St}(I)$. Show that the assignments $f \mapsto F_f$ and $h \mapsto \tau_h$ constitute a functor $\mathcal{S}$ from $\mathbf{Top}(I)$ to $\mathbf{St}(I)$. □

Now given a stack F, the question arises as to when F is (isomorphic to) a stack of sections, i.e. when is there a sheaf of germs f such that in $\mathbf{St}(I)$, $F \cong F_f$. The answer is to be found in the answer to another question about the behavior of local sections of $f: A \to I$. Suppose that $\{V_x \xrightarrow{s_x} A : x \in X\}$ is a collection of local sections of f, indexed by some set X, and that each of their domains V_x is a subset of some open set V. Thus, for all x, $s_x \in F_f(V_x)$ and $V_x \subseteq V$. The question is – when can we "paste" together all of the sections s_x to form a single section $s: V \to A \in F_f(V)$. The rule defining the desired s is this: if $i \in V$, choose some V_x that has $i \in V_x$, and put $s(i) = s_x(i)$. In order to have dom $s = V$ we require that each $i \in V$ be a member of at least one V_x. This means that V is the union of the collection of V_x's, i.e. $V = \bigcup\{V_x : x \in X\} = \{i$: for some $x \in X$, $i \in V_x\}$. In general a collection of open sets whose union is V will be called an *open cover* of V.

In order for s to satisfy the "unique output" property of functions, the definition of $s(i)$ should be independent of the choice of V_x containing i. Thus if $i \in V_x$ and $i \in V_y$, we require $s_x(i) = s_y(i)$. So any two of our local sections s_x and s_y must agree on the part $V_x \cap V_y$ of their domains that they have in common. In symbols –

$$\text{for all} \quad x, y \in X, \qquad s_x \restriction V_x \cap V_y = s_y \restriction V_x \cap V_y.$$

Under this "compatibility" condition, s will be a well-defined member of $F_f(V)$, with $s \restriction V_x = s_x$, all x. Moreover s is the only section over V whose restriction to V_x is always s_x. For, if $t: V \to A$ has $t \restriction V_x = s_x$, all $x \in X$, then $t = s$.

Now the compatibility condition on the s_x's can be expressed in terms of the restricting maps F_U^V of a functor F. We let $F_y^x : F(V_x) \to F(V_x \cap V_y)$ and $F_x^y : F(V_y) \to F(V_x \cap V_y)$ be the F-images of the inclusions $V_x \cap V_y \hookrightarrow V_x$ and $V_x \cap V_y \hookrightarrow V_y$, and $F_x : F(V) \to F(V_x)$ the image of $V_x \hookrightarrow V$. Then what we have shown is that the following condition obtains for the case that F is of the form F_f.

COM: *Given any open cover $\{V_x : x \in X\}$ of an open set V, and any selection of elements $s_x \in F(V_x)$, for all $x \in X$, that are pairwise compatible, i.e. $F_y^x(s_x) = F_x^y(s_y)$ all $x, y \in X$, then there is exactly one $s \in F(V)$ such that $F_x(s) = s_x$ all $x \in X$.*

Notice that COM is a statement that can be made about any stack $F : \Theta \to \mathbf{Set}$. Any F satisfying COM will be called a *sheaf of sections over* I, and the full subcategory of $\mathbf{St}(I)$ generated by those objects that satisfy COM will be denoted $\mathbf{Sh}(I)$.

EXERCISE 2. Show that the constant stack $1 : \Theta \to \mathbf{Set}$, where $1(U) = \{0\}$, is a sheaf.

EXERCISE 3. Consider the space $I = \{0, 1\}$, with $\Theta = \mathcal{P}(I)$ (the discrete topology). Let $F(U) = \{0, 1\}$, all $U \in \Theta$ and $F_U^V = f$, all $U \subseteq V$, where $f(0) = f(1) = 0$. By considering the cover $\{\{0\}, \{1\}\}$ of I, show that F is not a sheaf, i.e. COM fails.

EXERCISE 4. Why must $F(\emptyset)$, *for any sheaf* F, be a one-element set?

EXERCISE 5. Show that

$$\begin{tikzcd} & F(V) \arrow[dl, "F_x"'] \arrow[dr, "F_y"] & \\ F(V_x) \arrow[rr, "F_y^x"] & & F(V_y) \end{tikzcd}$$

commutes whenever $V_y \subseteq V_x$, and so $F(V)$, together with the maps F_x, for all $x \in X$, forms a cone for the diagram consisting of the objects $F(V_x)$ and the arrows F_y^x. Show that COM is equivalent to the condition that this cone be universal for that diagram, i.e. that $F(V)$ be the limit of the diagram, denoted $F(V) = \varprojlim_{x \in X} F(V_x)$ (cf. §3.11). □

Now given an arbitrary stack $F : \Theta \to \mathbf{Set}$, a corresponding sheaf of germs $p_F : A_F \to I$ may be defined. For each $i \in I$ the collection

$\{F(V): i \in V\}$ of F-images of neighbourhoods of i, together with their associated restricting maps, forms a diagram in **Set**. The stalk over i in A_F is defined to be the co-limit, denoted $\varinjlim_{i \in V} F(V)$ of this diagram. Explicitly, an equivalence relation $\sim_i$ is defined on $\bigcup\{F(V): i \in V\}$ thus: if $s_x \in F(V_x)$ and $s_y \in F(V_y)$ (where V_x and V_y are i-neighbourhoods), we put

$$s_x \sim_i s_y \quad \text{iff} \quad F^x_z(s_x) = F^y_z(s_y), \quad \text{for some } i\text{-neighbourhood } V_z \subseteq V_x \cap V_y.$$

Intuitively, $F^x_z(s_x)$ is the "localisation" of the element $s_x \in F(V_x)$ to V_z. Thus $s_x \sim_i s_y$ when they are "locally equal", that is when they have the same localisation to some i-neighbourhood. The equivalence class $[s]_i$ of $s \in F(V)$ under $\sim_i$, i.e. the set $[s]_i = \{t: s \sim_i t\}$, is called the *germ of s at i.* The stalk for p_F over i is then the set $F_i = \{\langle i, [s]_i \rangle: s \in \bigcup\{F(V): i \in V\}\}$. The stalk space is the union $A_F = \bigcup\{F_i : i \in I\}$, and p_F is the projection of A_F onto I. For each open $V \in \Theta$ and $s \in F(V)$, let $N(s, V) = \{\langle i, [s]_i \rangle: i \in V\}$. The collection of all $N(s, V)$'s generates a topology on A_F making p_F a local homeomorphism.

EXERCISE 6. Verify that $\sim_i$ is an equivalence relation.

EXERCISE 7. Define $p^i_V : F(V) \to F_i$ by

$$p^i_V(s) = \langle i, [s]_i \rangle, \quad \text{all} \quad s \in F(V).$$

Show that

$$\begin{tikzcd} F(V) \arrow[rr, "F^V_U"] \arrow[dr, "p^i_V"'] & & F(U) \arrow[dl, "p^i_U"] \\ & F_i & \end{tikzcd}$$

commutes when $U \subseteq V$, so that the p^i_V's form a co-cone for the diagram based on $\{F(V): i \in V\}$. Prove that this co-cone is co-universal for the diagram, so that F_i is its co-limit, $F_i = \varinjlim_{i \in V} F(V)$. (cf. §3.11).

EXERCISE 8. If $s \in F(V)$, define $s_V : V \to A_F$ by putting $s_V(i) = \langle i, [s]_i \rangle = p^i_V(s)$, for all $i \in V$. Show that s_V is a section of the sheaf $p_F : A_F \to I$.

EXERCISE 9. Let F_{p_F} be the sheaf (stack) of sections of the sheaf of germs p_F. For each V, define $\sigma_V : F(V) \to F_{p_F}(V)$ by putting, for $s \in F(V)$, $\sigma_V(s) = s_V$, where s_V is the section of p_F defined in Exercise 8. Show that the σ_V's form the components of an arrow $\sigma : F \dot{\to} F_{p_F}$ in $\mathbf{St}(I)$.

EXERCISE 10. Let $\tau : F \rightarrow G$ be an arrow in $\mathbf{St}(I)$. Define $h_\tau : A_F \rightarrow A_G$ as follows: if $\langle i, [s]_i \rangle$ is a germ at i in A_F, with $s \in F(V)$ say, let $h_\tau(i)$ be the germ $\langle i, [\tau_V(s)]_i \rangle$ in A_G, where τ_V is the component $F(V) \rightarrow G(V)$ of τ. Show that

$$\begin{tikzcd} A_F \arrow[rr, "h_\tau"] \arrow[dr, "p_F"'] & & A_G \arrow[dl, "p_G"] \\ & I & \end{tikzcd}$$

commutes, and that h_τ is a $\mathbf{Top}(I)$-arrow from p_F to p_G.

EXERCISE 11. Verify that the constructions $F \mapsto p_F$, $\tau \mapsto h_\tau$ constitute a functor $\mathscr{G}$ from $\mathbf{St}(I)$ to $\mathbf{Top}(I)$.

EXERCISE 12. Let $f : A \rightarrow I$ be any sheaf of germs over I, F_f its stack of sections, and $p_{F_f} : A_{F_f} \rightarrow I$ the associated sheaf of germs. Define a map $k : A \rightarrow A_{F_f}$ as follows. If $a \in A$, use the local homeomorphism property of f to show that f has a local section $s : V \rightarrow A$ through a, i.e. $a \in s(V)$. Let $k(a) = \langle f(a), [s]_{f(a)} \rangle$ be the germ of s at $f(a)$.

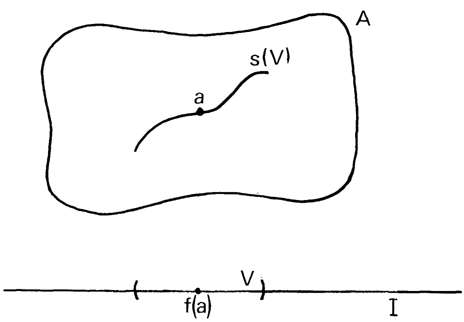


Fig. 14.2.

Check that the definition of $k(a)$ does not depend on which section through a is chosen. Show that

$$\begin{tikzcd} A \arrow[rr, "k"] \arrow[dr, "f"'] & & A_{F_f} \arrow[dl, "p_{F_f}"] \\ & I & \end{tikzcd}$$

commutes, so that k is a $\mathbf{Top}(I)$-arrow from f to p_{F_f}.

EXERCISE 13. Prove that the map k of the last exercise is a bijection, and hence is an iso arrow in **Top**(I), making $f \cong p_{F_f}$.

EXERCISE 14. Let $\sigma_V : F(V) \to F_{p_F}(V)$ be the component of $\sigma : F \dashrightarrow F_{p_F}$ defined in Exercise 9. Show that σ_V is a bijection iff the condition COM holds for open covers of V. Hence show that σ is iso iff the stack F is a sheaf, i.e. that $F \cong F_{p_F}$ iff F belongs to **Sh**(I). □

Exercises 1 and 11 provide us with functors $\mathcal{S} : \mathbf{Top}(I) \to \mathbf{St}(I)$ and $\mathcal{G} : \mathbf{St}(I) \to \mathbf{Top}(I)$, with the image of $\mathcal{S}$ being (contained in) **Sh**(I). By Exercise 13,

$$f \cong \mathcal{G}(\mathcal{S}(f)), \quad \text{all} \quad \mathbf{Top}(I)\text{-objects } f.$$

However by Exercise 14, for $F \in \mathbf{St}(\mathrm{I})$, we have

$$F \cong \mathcal{S}(\mathcal{G}(F)) \quad \text{iff} \quad F \in \mathbf{Sh}(I).$$

Thus $\mathcal{S}$, and the restriction of $\mathcal{G}$ to **Sh**(I) are equivalences of categories (§9.2). They establish that the category of sheaves of sections over I is equivalent to the topos of sheaves of germs over I.

We conclude this brief introduction to stacks and sheaves with two major illustrations of the behaviour of **Sh**(I)-objects.

I. NNO in Sh(I)

The category **Sh**(I) has a natural numbers object – the sheaf of *locally constant natural-number-valued functions on I*. Specifically $N : \Theta \to \mathbf{Set}$ is the sheaf that has

$$N(V) = \{V \xrightarrow{g} \omega : g \text{ is continuous}\},$$

where ω is presumed to have the discrete topology, and $N_U^V(g) = g \restriction U$ whenever $U \subseteq V$.

The requirement that g be continuous for the discrete topology on ω means precisely that g is locally constant, i.e. that for each $i \in V$ there is a neighbourhood U_i of i, with $i \in U_i \subseteq V$, such that $g \restriction U_i$ is a constant function. Thus there is a number $g_i \in \omega$ such that $g(i) = g_i$ for all $i \in U_i$. This condition on g can be interpreted as saying that the statement "g is constant" is locally true of its domain V, i.e. true of some neighbourhood of each point of V.

The arrow $O:1 \to N$ has component $O_V:\{0\} \to N(V)$ picking out the constantly zero function $V \to \omega$ on V. The V-th component $\mathscr{s}_V: N(V) \to N(V)$ of the successor arrow for $\mathbf{Sh}(I)$ has $\mathscr{s}_V(g) = s \circ g$, where $s:\omega \to \omega$ is the successor function on ω. (Note that $s \circ g$ is locally constant if g is).

EXERCISE 15. Verify the axiom NNO for this construction.

EXERCISE 16. For $n \in \omega$, let $n_V: V \to \omega$ have $n_V(i) = n$, all $i \in V$. Explain how the n_V's provide the components for the ordinal arrow $\mathbf{n}:1 \to N$ in $\mathbf{Sh}(I)$.

EXERCISE 17. If $g \in N(V)$ show that V has an open cover $\{V_x : x \in X\}$ of pairwise disjoint sets, i.e. $V_x \cap V_y = \emptyset$ if $x \neq y$, on each of which g is actually constant. □

Now let $pr_I: I \times \omega \to I$ be the sheaf of germs that is the nno for $\mathbf{Top}(I)$, as described in §12.2. For each continuous $g: V \to \omega$, the product map $\langle \mathrm{id}_V, g\rangle: V \to I \times \omega$ is readily seen to be a section of pr_I, i.e. an element of the stack $F_{pr_I}(V)$ of sections over V. Indeed this construction gives a bijection $N(V) \cong F_{pr_I}(V)$ for each $V \in \Theta$, hence in $\mathbf{Sh}(I)$ we have $N \cong F_{pr_I} = \mathscr{S}(pr_I)$, so that in $\mathbf{Top}(I)$, $\mathscr{G}(N) \cong \mathscr{G}(\mathscr{S}(pr_I)) \cong pr_I$.

EXERCISE 18. Let $p_N: A_N \to I = \mathscr{G}(N)$ be the sheaf of germs of locally constant ω-valued functions. Define $f: I \times \omega \to A_N$ by $f(\langle i, n\rangle) = \langle i, [n_I]_i\rangle$, where $n_I \in N(I)$ is the "constantly n" function defined in Exercise 16. Show directly that f is a bijection, giving $pr_I \cong \mathscr{G}(N)$. □

II. $\mathbf{Set}^{\mathbf{P}}$ and $\mathbf{Sh}(\mathbf{P})$

If $\mathbf{P}$ is a poset then the collection $\mathbf{P}^+$ of $\mathbf{P}$-hereditary sets is a topology on P, in terms of which we have the category (topos) $\mathbf{Sh}(\mathbf{P})$ of sheaves of the form $F:\mathbf{P}^+ \to \mathbf{Set}$. Given such a functor we can define a Kripke-model (variable set) $F^*:\mathbf{P} \to \mathbf{Set}$ as follows. F^* is to be a collection $\{F^*_p : p \in P\}$ of sets, indexed by P, with transitions $F^*_{pq}: F^*_p \to F^*_q$ whenever $p \sqsubseteq q$. We put

$$F^*_p = F([p)) \qquad \text{(note } [p) \in \mathbf{P}^+\text{)}$$

Whenever $p \sqsubseteq q$, we have $[q) \subseteq [p)$, so we take $F^*_{pq}: F([p)) \to F([q))$ to be the image of the inclusion $[q) \hookrightarrow [p)$ under the contra-variant functor F.

Since F is a sheaf, and since for each $V \in \mathbf{P}^+$, $\{[\mathrm{p}) : \mathrm{p} \in \mathrm{V}\}$ covers V (cf. §10.2), by Exercise 5 above we have

$$(*) \qquad F(V) = \varprojlim_{\mathrm{p} \in \mathrm{V}} F([p)) = \varprojlim_{\mathrm{p} \in \mathrm{V}} (F^*_{\mathrm{p}})$$

This shows us how to *define* a sheaf F of sections over the topology $\mathbf{P}^+$ from a variable set $F^* : \mathbf{P} \to \mathbf{Set}$. In **Set**, all diagrams have limits, and so we can define $F(V)$ from $\{F^*_{\mathrm{p}} : p \in V\}$ by the equation $(*)$. Moreover, if $U \subseteq V$, then $\{F^*_{\mathrm{p}} : p \in U\} \subseteq \{F^*_{\mathrm{p}} : p \in V\}$, so the universal cone $F(V)$ for the latter diagram will be a cone for the former, and so F^V_U may be

$$\begin{tikzcd}
 & F(V) \arrow[d, dashed] \arrow[ddl] \arrow[ddr] & \\
 & F(U) \arrow[dl] \arrow[dr] & \\
F^*_p & & F^*_q
\end{tikzcd}$$

defined as the unique factoring arrow as shown.

Thus we obtain an exact correspondence between objects in $\mathbf{Set}^{\mathbf{P}}$ and $\mathbf{Sh}(\mathbf{P})$. If we pass via the functor $\mathcal{G}$ from the sheaf of sections $F : \mathbf{P}^+ \to \mathbf{Set}$ to the sheaf of germs $p_F : A_F \to P$ we find that the stalk in A_F over a point $p \in P$ turns out to be an isomorphic copy of the original set $F^*_{\mathrm{p}} = F([p))$. The bijection $F([p)) \cong F_p$ (F_p = stalk over p) is given by the function $\rho^{\mathrm{p}}_{[\mathrm{p})}$ (defined in Exercise 7) having

$$\rho^{\mathrm{p}}_{[\mathrm{p})}(s) = \langle p, [s]_{\mathrm{p}} \rangle, \quad \text{all} \quad s \in F([p)).$$

The reason why this is so is that the p-neighbourhood $[p)$ lies inside all other p-neighbourhoods (Exercise 10.2.3), so that the germ of any $s' \in F(V)$ at p is the same as the germ at p of its localisation $s = F^V_{[\mathrm{p})}(s')$ to $[\mathrm{p})$.

In view of the description (Exercise 7) of the stalk F_p as a co-limit we then have that if F is related to F^* by the equation $(*)$, then for each $p \in P$,

$$F^*_{\mathrm{p}} \cong \varinjlim_{\mathrm{p} \in \mathrm{V}} F(V)$$

EXERCISE 19. Verify that $\rho^{\mathrm{p}}_{[\mathrm{p})}$ is a bijection

EXERCISE 20. Show that

$$\begin{tikzcd} F(V) \arrow[rr, "F^V_U"] \arrow[dr, "F^V_{[p)}"'] & & F(U) \arrow[dl, "F^U_{[p)}"] \\ & F^*_p & \end{tikzcd}$$

commutes whenever $p \in U \subseteq V$, and so F^*_p is the apex of a co-cone for $\{F(V)\colon p \in V\}$. Verify the co-universal property for this co-cone. □

The reader interested in the origins and history of sheaf theory should consult the paper "What is a Sheaf?" by Seebach et al. [70].

14.2. Classifying stacks and sheaves

The object of truth-values in $\mathbf{St}(I)$ is obtainable by dualising the description of that given for $\mathbf{Set}^{\Theta^{op}}$ in §9.3 (or §10.3, as Θ^{op} is a poset category).

If $V \in \Theta$, let $\Theta_V = \Theta \cap \mathscr{P}(V) = \{U \in \Theta\colon U \subseteq V\}$ be the collection of open subsets of V. (Since V is open, Θ_V is in fact the relative (subspace) topology on V.) A collection $C \subseteq \Theta_V$ of V-open sets is called a *V-crible* when it is closed under the taking of open subsets, i.e. when we have that

if $U \in C$, and $W \subseteq U$ has $W \in \Theta$ (i.e. $W \in \Theta_U$), then $W \in C$.

The stack $\Omega : \Theta \to \mathbf{Set}$ has

$$\Omega(V) = \{C\colon C \text{ is a } V\text{-crible}\}$$

and $\Omega^V_U(C) = C \cap \Theta_U = \{W\colon W \in C \text{ and } W \subseteq U\}$ whenever $U \hookrightarrow V$.

EXERCISE 1. $\Theta_V = \Theta_U$ iff $V = U$.

EXERCISE 2. In the opposite to the inclusion ordering, $V \sqsubseteq U$ iff $U \subseteq V$, of Θ, $\Theta_V = [V)$.

EXERCISE 3. If $U \subseteq V$, then Θ_U is a V-crible, with $\bigcup \Theta_U = U$.

EXERCISE 4. The poset $(\Omega(V), \subseteq)$ of V-cribles under the inclusion relation is a Heyting algebra with the meet and join of V-cribles C, D being their intersection $C \cap D$ and union $C \cup D$. What are $\neg C$ and $C \Rightarrow D$? □

The arrow $true: 1 \to \Omega$ has components $true_V: \{0\} \to \Omega(V)$ given by

$$true_V(0) = \Theta_V, \quad \text{the largest } V\text{-crible.}$$

Given a monic arrow $\tau: F \rightarrowtail G$ of stacks, with each τ_V being the inclusion $F(V) \hookrightarrow G(V)$, the character $\chi_\tau: G \to \Omega$ has V-component $(\chi_\tau)_V: G(V) \to \Omega(V)$, where for $x \in G(V)$,

$$(\chi_\tau)_V(x) = \{U \subseteq V: G_U^V(x) \in F(U)\}$$

EXERCISE 5. Verify that $(\chi_\tau)_V(x)$ is a V-crible. □

In the category $\mathbf{Sh}(I)$ of sheaves of sections over I there is a subobject classifier, which is not the same as that for $\mathbf{St}(I)$. This time the object of truth-values is the contravariant functor $\Omega_j: \Theta \to \mathbf{Set}$ that has

$$\Omega_j(V) = \Theta_V, \quad \text{the collection of open subsets of } V,$$

while Ω_j assigns to each inclusion $U \hookrightarrow V$ the restricting map $\Omega_j(V) \to \Omega_j(U)$ that takes $W \in \Theta_V$ to $W \cap U \in \Theta_U$.

The arrow $true_j: 1 \to \Omega_j$ has V-th component $true_{jV}: \{0\} \to \Theta_V$ given by

$$true_{jV}(0) = V, \quad \text{the largest } V\text{-open set.}$$

If $\tau: F \rightarrowtail G$ is a monic arrow in $\mathbf{Sh}(I)$ its character $\chi_\tau^j: G \to \Omega_j$ has component

$$(\chi_\tau^j)_V: G(V) \to \Omega_j(V),$$

where

$$\begin{aligned}(\chi_\tau^j)_V(x) &= \bigcup\{U \subseteq V: G_U^V(x) \in F(U)\}\\ &= \bigcup(\chi_\tau)_V(x)\end{aligned}$$

(Θ, being a topology, is closed under unions of arbitrary sub-collections).

EXERCISE 6. Show that Ω_j is a sheaf, i.e. satisfies COM.

EXERCISE 7. Verify that the construction just given shows that the Ω-axiom holds in $\mathbf{Sh}(I)$ for $true_j: 1 \to \Omega_j$, identifying the point at which the condition COM is needed. □

Notice that if the $F \mapsto p_F$ construction (the functor $\mathcal{G}$) is applied to Ω_j, the result is the sheaf of germs of open subsets of I, which is precisely the subobject classifier for the spatial topos $\mathbf{Top}(I)$ as described in Chapter 4.

In order now to describe the relationship between Ω_j and Ω in categorial terms we define a function $j_V : \Omega(V) \to \Omega(V)$, by putting for each V-crible $C \subseteq \Theta_V$,

$$j_V(C) = \{U \in \Theta : U \subseteq \cup C\} = \Theta_{\cup C}$$

EXERCISE 8. $j_V(\Theta_U) = \Theta_U$, for $U \in \Theta_V$ (cf. Exercise 3), and so $j_V(true_V(0)) = true_V(0)$.

EXERCISE 9. $C \subseteq j_V(C)$, i.e. $C \cap j_V(C) = C$.

EXERCISE 10. $j_V(j_V(C)) = j_V(C)$.

EXERCISE 11. $j_V(C \cap D) = j_V(C) \cap j_V(D)$, and hence

EXERCISE 12. if $C \subseteq D$ then $j_V(D) \subseteq j_V(D)$, for any C, $D \in \Omega(V)$.

EXERCISE 13. A V-crible of the form Θ_U, for $U \in \Theta_V$, is called a *principal* V-crible. Noting that if $C \subseteq \Theta_V$, then $\cup C \in \Theta_V$, show that

$$j_V(C) = C \quad \text{iff} \quad C \text{ is a principal } V\text{-crible}$$

(cf. Exercise 8).

EXERCISE 14. $j_V(C) = \Theta_V$ iff C covers V (i.e. iff $\cup C = V$).

EXERCISE 15. If C is any V-crible, and $U \subseteq V$, let

$$C_U = \{W \cap U : W \in C\}.$$

Show that $C_U \subseteq \Omega_U^V(C)$.

EXERCISE 16. Prove, in Exercise 15, that

$$\begin{aligned} U \subseteq \cup C \quad &\text{iff} \quad C_U \text{ is an open cover of } U \\ &\text{iff} \quad U = \cup C_U = \cup\{W \cap U : W \in C\}. \end{aligned}$$

EXERCISE 17. Show, using the last two exercises, that

$$\begin{aligned} U \in j_V(C) \quad \text{iff} \quad &U \text{ has an open cover } D \\ &\text{with } D \subseteq \Omega_U^V(C). \end{aligned}$$

EXERCISE 18. If $U \subseteq V$, show that

$$\begin{tikzcd} \Omega(V) \arrow[r, "j_V"] \arrow[d, "\Omega^V_U"'] & \Omega(V) \arrow[d, "\Omega^V_U"] \\ \Omega(U) \arrow[r, "j_U"] & \Omega(U) \end{tikzcd}$$

commutes. □

Now if $E \hookrightarrow \Omega(V)$ is the equaliser of id and $j_V : \Omega(V) \to \Omega(V)$, then by what we know of equalisers in **Set**, and by Exercise 13, we have

$$E = \{C \in \Omega(V) : j_V(C) = C\} = \{\Theta_U : U \in \Theta_V\}$$

But the map $e_V : \Theta_V \to \Omega(V)$ having $e_V(U) = \Theta_U$ is monic, by Exercise 1, and so gives a bijection between $\Theta_V = \Omega_j(V)$ and E. Thus we find that

$$\Omega_j(V) \overset{e_V}{\rightarrowtail} \Omega(V) \underset{j_V}{\overset{id}{\rightrightarrows}} \Omega(V)$$

is an equaliser diagram in **Set**. But the import of Exercise 18 is that the j_V's form the components of an arrow $j_\Theta : \Omega \dashrightarrow \Omega$ in $\mathbf{St}(I)$. The e_V's are also components of a monic $e : \Omega_j \rightarrowtail \Omega$, and we find that

$$\Omega_j \overset{e}{\rightarrowtail} \Omega \underset{j_\Theta}{\overset{1_\Omega}{\rightrightarrows}} \Omega$$

is an equaliser diagram in $\mathbf{St}(I)$. Thus in $\mathbf{St}(I)$, Ω_j arises as that subobject of Ω obtained by equalising j_Θ and 1_Ω. Moreover since by Exercise 8 we have $j_\Theta \circ true = true$, there is a unique arrow τ making

$$\begin{tikzcd} \Omega_j \arrow[r, tail, "e"] & \Omega \\ 1 \arrow[u, dashed, "\tau"] \arrow[ur, "true"'] & \end{tikzcd}$$

commute. Clearly τ is in fact the arrow $true_j$.

Not only does the arrow j_Θ give a characterisation of Ω_j, it also characterises, by a property expressible in the first-order language of categories, those stacks over I that are sheaves, i.e. satisfy COM. To see how this works we first observe that j_Θ induces an operator $J : \mathrm{Sub}(G) \to \mathrm{Sub}(G)$ on the **HA** of subobjects of each $\mathbf{St}(I)$-object G.

J assigns to the subobject $\tau: F \rightarrowtail G$ the subobject $J(\tau): J(F) \rightarrowtail G$ obtained by pulling *true* back

$$\begin{tikzcd} J(F) \arrow[r, tail, "J(\tau)"] \arrow[d] & G \arrow[d, "j \circ \chi_\tau"] \\ 1 \arrow[r, "true"] & \Omega \end{tikzcd}$$

along $j \circ \chi_\tau$, so that $\chi_{J(\tau)} = j \circ \chi_\tau$.

EXERCISE 19. In Sub(G) we have
(i) $\tau \subseteq J(\tau)$, i.e. $\tau \cap J(\tau) = \tau$
(ii) $J(J(\tau)) = J(\tau)$
(iii) $J(\tau \cap \sigma) = J(\tau) \cap J(\sigma)$, hence
(iv) if $\tau \subseteq \sigma$, then $J(\tau) \subseteq J(\sigma)$
(cf. Exercises 9–12). □

In general, an operator on a lattice that satisfies (i), (ii), and (iv) (corresponding to Exercises 9, 10, 12) is known as a *closure operator.* An example is the operator on the **BA** $\mathcal{P}(I)$, where I is a topological space, that assigns to each subset $X \subseteq I$ its topological closure (smallest closed superset) $\mathrm{cl}(X)$ in I. If $\mathrm{cl}(X) = I$, then X is said to be *dense* in the space I. By analogy then we say that a monic $\tau: F \rightarrowtail G$ in **St**(I) is *dense* iff $J(\tau) \simeq 1_G$ in Sub(G).

EXERCISE 20. Show that $J(F): \Theta \to Set$ assigns to $V \in \Theta$ the subset

$$\{x: (\chi_\tau)_V(x) \text{ covers } V\}$$

of $G(V)$, and that the components of $J(\tau)$ are the corresponding inclusions.

EXERCISE 21. Show that $\tau: F \rightarrowtail G$ is dense iff for all $V \in \Theta$, if $x \in G(V)$, then

$$(\chi_\tau)_V(x) = \{U: G_U^V(x) \in F(U)\} \text{ covers } V. \qquad \square$$

Now the statement "$G_U^V(x) \in F(U)$" can be construed as the localisation to $U \subseteq V$ of the statement "$x \in F(V)$". Thus if $(\chi_\tau)_V(x)$ covers V, the statement "$x \in F(V)$" is locally true of V, i.e. true at some neighbourhood of each point in V. Hence τ is dense when every element of $G(V)$ is *locally* an element of $F(V)$.

THEOREM (Lawvere). *A stack H is a sheaf (satisfies* COM) *iff for every* **St**(I)*-arrow $\sigma : F \dashrightarrow H$ with codomain H, and every dense arrow $\tau : F \rightarrowtail G$, there is exactly one $\sigma' : G \dashrightarrow H$ such that*

$$\begin{tikzcd} F \arrow[r, tail, "\tau"] \arrow[dr, "\sigma"'] & G \arrow[d, dashed, "\sigma'"] \\ & H \end{tikzcd}$$

commutes. □

Thus H is a sheaf iff every arrow ending at H can be "lifted" in one and only one way from its domain to any object in which that domain is dense.

It can be shown (Tierney) that the proof of this characterisation can be derived entirely from the fact that the diagrams

$$\begin{tikzcd} \Omega \arrow[r, "\langle 1_\Omega, j_\Theta \rangle"] \arrow[dr, "1_\Omega"'] & \Omega \times \Omega \arrow[d, "\cap"] \\ & \Omega \end{tikzcd} \qquad \begin{tikzcd} \Omega \arrow[r, "j_\Theta"] \arrow[dr, "j_\Theta"'] & \Omega \arrow[d, "j_\Theta"] \\ & \Omega \end{tikzcd}$$

$$\begin{tikzcd} \Omega \times \Omega \arrow[r, "j_\Theta \times j_\Theta"] \arrow[d, "\cap"'] & \Omega \times \Omega \arrow[d, "\cap"] \\ \Omega \arrow[r, "j_\Theta"] & \Omega \end{tikzcd}$$

all commute in **St**(I). These diagrams correspond to Exercises 9, 10, 11, and hence to conditions (i)–(iii) of Exercise 19. We shall reserve the name *local operator* for any operator on a lattice that satisfies (i)–(iii) of Exercise 19.

EXERCISE 22. Let **St**(F, H) be the collection of all **St**(I)-arrows from F to H. Given $\tau : F \dashrightarrow G$, let $\tau_0 : \mathbf{St}(G, H) \to \mathbf{St}(F, H)$ be given by $\tau_0(\sigma) = \sigma \circ \tau$

$$\begin{tikzcd} F \arrow[r, "\tau"] \arrow[dr, "\sigma \circ \tau"'] & G \arrow[d, "\sigma"] \\ & H \end{tikzcd}$$

Show that H is a sheaf iff for every dense monic τ, τ_0 is a bijection.

EXERCISE 23. Let $\mathbf{H}=(H, \sqsubseteq)$ be any Heyting algebra. Show that the assignment to each $a \in H$ of its "double pseudo-complement" $\neg\neg a$ is a local operator on $\mathbf{H}$. □

14.3. Grothendieck topoi

Grothendieck's generalisation (cf. Artin et al. [SGA4]) of the functorial notion of a sheaf over a topological space is based on the observation that the axiom COM is expressible in terms of categorial properties of open covers $\{V_x : x \in X\}$, or $\{V_x \hookrightarrow V : x \in X\}$, of objects V in the category Θ. The essential properties of covers needed are

(1) The singleton set $\{V\}$ is a cover of V.

(2) If $\{V_x : x \in X\}$ covers V, and if, for each $x \in X$, $C_x = \{V^x_y : y \in Y_x\}$ is an open cover of V_x, then

$$\bigcup\{C_x : x \in X\} = \{V^x_y : x \in X \text{ and } y \in Y_x\}$$

is an open cover of V.

Thus the union of covers for open sets itself covers the union of those open sets.

(3) If $\{V_x : x \in X\}$ covers V, then for any inclusion $U \hookrightarrow V$, the collection $\{U \cap V_x : x \in X\}$ covers U. Notice that $U \cap V_x \hookrightarrow U$ is the pullback

$$\begin{tikzcd}
U \cap V_x \arrow[r, hook] \arrow[d, hook] & V_x \arrow[d, hook] \\
U \arrow[r, hook] & V
\end{tikzcd}$$

of $V_x \hookrightarrow V$ along $U \hookrightarrow V$.

A *pretopology* on a category $\mathscr{C}$ is an assignment to each $\mathscr{C}$-object a of a collection $Cov(a)$ of sets of $\mathscr{C}$-arrows with codomain a, called *covers* of a, such that

(1) The singleton $\{1_a : a \to a\} \in Cov(a)$.

(2) If $\{a_x \xrightarrow{f_x} a : x \in X\} \in Cov(a)$, and for each $x \in X$, we have an a_x-cover

$$\{a^x_y \xrightarrow{f^x_y} a_x : y \in Y_x\} \in Cov(a_x),$$

then

$$\{a^x_y \xrightarrow{f_x \circ f^x_y} a : x \in X \text{ and } y \in Y_x\} \in Cov(a).$$

(3) If $\{a_x \xrightarrow{f_x} a : x \in X\} \in Cov(a)$, and $g: b \rightarrow a$ is any $\mathscr{C}$-arrow, then for each $x \in X$ the pullback

$$\begin{array}{ccc} b \underset{a}{\times} a_x & \longrightarrow & a_x \\ {\scriptstyle g_x}\big\downarrow & & \big\downarrow{\scriptstyle f_x} \\ b & \xrightarrow{\quad g \quad} & a \end{array}$$

of f_x along g exists, and

$$\{b \underset{a}{\times} a_x \xrightarrow{g_x} b : x \in X\} \in Cov(b).$$

The pair $(\mathscr{C}, Cov)$ of the category $\mathscr{C}$ with the pretopology Cov is called a *site.*

Examples of sites

EXERCISE 1. (Θ, Cov_Θ), where, for open $V \in \Theta$, $Cov_\Theta(V) = \{C: C \subseteq \Theta$ and $\bigcup C = V\}$ is the collection of open covers of V.

EXERCISE 2. $(\mathscr{C}, {}^iCov)$, $\mathscr{C}$ any category, where ${}^iCov(a) = \{\{1_a : a \rightarrow a\}\}$, all $\mathscr{C}$-objects a.

EXERCISE 3. $(\mathscr{C}, {}^dCov)$, $\mathscr{C}$ any category, where ${}^dCov(a) = \mathscr{P}(\{f: \operatorname{cod} f = a\})$ is the collection of all sets of $\mathscr{C}$-arrows with codomain a.

EXERCISE 4. Let **2** be the poset category $(\{0, 1\}, \leqslant)$ with $!: 0 \rightarrow 1$ the only non-identity arrow. Can you find ten different pretopologies on **2**? □

A *stack*, or *presheaf*, of sets over a category $\mathscr{C}$ is by definition a contravariant functor $F: \mathscr{C} \rightarrow \mathbf{Set}$. The category $\mathbf{St}(\mathscr{C})$ of all stacks over $\mathscr{C}$ is thus equivalent to the topos $\mathbf{Set}^{\mathscr{C}^{op}}$.

If Cov is a pretopology on $\mathscr{C}$, and $\{a_x \xrightarrow{f_x} a : x \in X\} \in Cov(a)$, let

$$\begin{array}{ccc} a_x \underset{a}{\times} a_y & \longrightarrow & a_y \\ \big\downarrow & & \big\downarrow{\scriptstyle f_y} \\ a_x & \xrightarrow{\quad f_x \quad} & a \end{array}$$

be the pullback of f_x and f_y, for each $x, y \in X$.

If F is a stack over $\mathscr{C}$ then F gives rise to the functions $F^x_y : F(a_x) \to F(a_x \times_a a_y)$ and $F^y_x : F(a_y) \to F(a_x \times_a a_y)$ as the F-images of the two new arrows obtained by forming this pullback. We denote also by F_x the arrow $F(f_x): F(a) \to F(a_x)$.

A stack F is a *sheaf* over the site $(\mathscr{C}, Cov)$ iff it satisfies

COM: *Given any cover $\{a_x \xrightarrow{f_x} a : x \in X\} \in Cov(a)$ of a $\mathscr{C}$-object a, and any selection of elements $s_x \in F(a_x)$, for all $x \in X$, that are pairwise compatible, i.e. $F^x_y(s_x) = F^y_x(s_y)$ all x, $y \in X$, then there is exactly one $s \in F(a)$ such that $F_x(s) = s_x$ all $x \in X$.*

The full subcategory of $\mathbf{St}(\mathscr{C})$ generated by those objects that are sheaves over the site $(\mathscr{C}, Cov)$ will be denoted $\mathbf{Sh}(Cov)$. A *Grothendieck topos* is, by definition, any category that is equivalent to one of the form $\mathbf{Sh}(Cov)$.

EXERCISE 5. If ${}^iCov$ is the "indiscrete" pretopology on $\mathscr{C}$ of Exercise 2, then $\mathbf{Sh}({}^iCov) = \mathbf{St}(\mathscr{C})$.

EXERCISE 6. Let $F: \mathbf{2} \to \mathbf{Set}$ be a stack over $\mathbf{2}$, and choose $s_0 \in F(0)$, $s_1 \in F(1)$. Assuming that $\{1_1, ! : 0 \to 1\} \in Cov(1)$, show that s_0 and s_1 are compatible iff $F^1_0(s_1) = s_0$, where F^1_0 is the F-image of $! : 0 \to 1$.

EXERCISE 7. Use the last exercise to show that $\mathbf{St}(\mathbf{2}) = \mathbf{Sh}(Cov)$ if $Cov(0) = \{\{1_0\}\}$ and $Cov(1) = \{\{1_1\}, \{1_1, !\}\}$. □

An *a-crible* (dual to a-sieve) is a collection C of arrows with codomain a that is closed under right composition, i.e. if $f: b \to a \in C$ then $f \circ g : c \to a \in C$ for any $\mathscr{C}$-arrow $g: c \to b$. The stack $\Omega : \mathscr{C} \to \mathbf{Set}$ has

$$\Omega(a) = \{C : C \text{ is an } a\text{-crible}\}$$

while for each $\mathscr{C}$-arrow $f: b \to a$, $\Omega_f : \Omega(a) \to \Omega(b)$ has

$$\Omega_f(C) = \{c \xrightarrow{g} b : f \circ g \in C\}.$$

The $\mathbf{St}(\mathscr{C})$-arrow *true* $: 1 \to \Omega$ has component $\top_a : \{0\} \to \Omega(a)$ given by

$$\top_a(0) = C_a = \{f : \operatorname{cod} f = a\}, \text{ the largest } a\text{-crible.}$$

EXERCISE 8. Show that if $C \in \Omega(a)$, then $\Omega_f(C) \in \Omega(b)$, and that if $f: b \to a \in C$ then $\Omega_f(C) = C_b$. □

The sheaves over a site $(\mathscr{C}, Cov)$ can be described by an arrow $j_{Cov}: \Omega \rightarrow \Omega$ exactly as in the classical case $Cov = Cov_\Theta$. The a-th component $j_{Cov\,a}: \Omega(a) \to \Omega(a)$ is defined, for each a-crible $C \in \Omega(a)$, by

$$j_{Cov\,a}(C) = \{ b \xrightarrow{f} a : \text{ there exists a cover } C_f \in Cov(b) \text{ with } C_f \subseteq \Omega_f(C)\}.$$

This is a direct generalisation of the description of $j_V(C)$ for $V \in \Theta$ given in Exercise 17 of the last section.

EXERCISE 9. Verify that j_{Cov} as defined is a $\mathbf{St}(\mathscr{C})$-arrow.

EXERCISE 10. Show that
(1) $\cap \circ \langle \mathbf{1}_\Omega, j_{Cov} \rangle = \mathbf{1}_\Omega$
(2) $j_{Cov} \circ j_{Cov} = j_{Cov}$
(3) $\cap \circ (j_{Cov} \times j_{Cov}) = j_{Cov} \circ \cap$ □

The characterisation theorem of the last section for sheaves in $\mathbf{St}(I)$ holds for Cov-sheaves in $\mathbf{St}(\mathscr{C})$ when j_Θ is replaced by j_{Cov}. The properties of j_{Cov} needed to prove this are precisely (1)–(3) of Exercise 10, as in the classical topological case.

Notice that, by Exercises 5 and 7, it is possible to have different pretopologies Cov_1 and Cov_2 on the same category that lead to the one category of sheaves, i.e.

$$\mathbf{Sh}(Cov_1) = \mathbf{Sh}(Cov_2).$$

However, it can be shown that this last equation holds iff $j_{Cov_1} = j_{Cov_2}$. Thus the arrow j_{Cov} corresponds to a unique Grothendieck topos.

The notion of pretopology has been further refined by Verdier [SGA4] to yield a notion of "topology" on a category such that distinct topologies yield distinct categories of sheaves. The Verdier topologies are subfunctors of Ω–precisely those whose characters satisfy (1)–(3) of Exercise 10. A detailed introductory account of this theory is given by Shlomiuk [74].

An extensive discussion of sites and logical operations on related categories can be found in the article [74] by Gonzalo Reyes.

14.4. Elementary sites

A *topology* on an elementary topos $\mathscr{E}$ is by definition any arrow $j:\Omega \to \Omega$ that satisfies

(1) $j \circ true = true$

(2) $j \circ j = j$

(3) $\cap \circ (j \times j) = j \circ \cap$.

The pair $\mathscr{E}_j = (\mathscr{E}, j)$ is called an *elementary site*. Notice that condition (1) of Exercise 14.3.10 has been replaced by the simpler $j \circ \top = \top$. This is justified by the following result, for which we need

EXERCISE 1. In any category with 1, a square of the form

$$\begin{tikzcd} 1 \arrow[r, "f"] \arrow[d] & b \arrow[d, "\langle 1_b, g\rangle"] \\ 1 \arrow[r, "\langle f, h\rangle"] & b \times b \end{tikzcd}$$

commutes, i.e. $g \circ f = h$, only if it is a pullback. □

THEOREM 1. *For any arrow $j:\Omega \to \Omega$ in a topos,*

$$\cap \circ \langle 1_\Omega, j\rangle = 1_\Omega \quad \textit{iff} \quad j \circ true = true.$$

PROOF. Consider the diagram

$$\begin{tikzcd} 1 \arrow[r, "\top"] \arrow[d] & \Omega \arrow[d, "\langle 1_\Omega, j\rangle"] \\ 1 \arrow[r, "\langle \top, \top\rangle"] \arrow[d] & \Omega \times \Omega \arrow[d, "\cap"] \\ 1 \arrow[r, "\top"] & \Omega \end{tikzcd}$$

If $\cap \circ \langle 1_\Omega, j\rangle = 1_\Omega$ then the boundary commutes. But the bottom square is the pullback defining $\cap$, so its universal property implies that $!:1 \to 1$ is the unique arrow making the top square commute. But then $\langle \top, \top\rangle = \langle 1_\Omega, j\rangle \circ \top = \langle \top, j \circ \top\rangle$, and so $\top = j \circ \top$.

Conversely if $j \circ \top = \top$, then the top square commutes and is (Exercise 1) a pullback. The PBL then gives the boundary as a pullback and so by the Ω-axiom, $\cap \circ \langle 1_\Omega, j\rangle = \chi_\top = 1_\Omega$. □

Examples of elementary sites

EXERCISE 2. For any site $(\mathscr{C}, Cov)$, $(\mathbf{St}(\mathscr{C}), j_{Cov})$ is an elementary site.

EXERCISE 3. $1_\Omega : \Omega \to \Omega$ is a topology for any $\mathscr{E}$.

EXERCISE 4. $true_\Omega : \Omega \to \Omega$ is a topology.

EXERCISE 5. $\neg\circ\neg : \Omega \to \Omega$ is a topology, the *double negation* topology, on any topos $\mathscr{E}$ (cf. Exercise 14.2.23).

A topology $j : \Omega \to \Omega$ induces a local operator J on the **HA** Sub(d) for each $\mathscr{E}$-object d, exactly as in the case $j = j_\Theta$. J assigns to $f : a \rightarrowtail d$ the subobject $J(f) : J(a) \rightarrowtail d$ having

$$\chi_{J(f)} = j \circ \chi_f.$$

An $\mathscr{E}$-monic $f : a \rightarrowtail d$ is *j-dense* iff $J(f) \simeq 1_d$ in Sub(d).

EXERCISE 6. f is j-dense iff $j \circ \chi_f = true_d$.

EXERCISE 7. In any $\mathscr{E}$, $[\top, \bot] : 1 + 1 \rightarrowtail \Omega$ is $\neg\circ\neg$-dense. (Hint: show that $\chi_{[\top, \bot]} = 1_\Omega \cup \neg$ and use Exercise 8.3.27).

EXERCISE 8. For $j = 1_\Omega$, $J(f) \simeq f$ and f is j-dense iff $f \simeq 1_d$.

EXERCISE 9. In the site $(\mathscr{E}, true_\Omega)$, $\chi_{J(f)} = true_d$, and every monic is dense.

EXERCISE 10. In the elementary site $\mathscr{E}_{\neg\neg} = (\mathscr{E}, \neg\circ\neg)$, f is dense iff $-(-f) \simeq 1_d$ in Sub(d). Use this to give a different proof of Exercise 7.

EXERCISE 11. Show that for any monic $f : a \rightarrowtail d$, $f \cup -f$ is $\neg\circ\neg$-dense. $\square$

A *j-sheaf* is, by definition, an $\mathscr{E}$-object b with the property that for any $\mathscr{E}$-arrow $g : a \to b$ and any j-dense $f : a \rightarrowtail d$ there is exactly one $\mathscr{E}$-arrow $g' : d \to b$ such that

$$\begin{tikzcd} a \arrow[r, tail, "f"] \arrow[dr, "g"'] & d \arrow[d, dashed, "g'"] \\ & b \end{tikzcd}$$

$g' \circ f = g$.

EXERCISE 12. 1 is a j-sheaf.

EXERCISE 13. If

$$a \rightarrowtail b \rightrightarrows c$$

is an equaliser diagram in $\mathscr{E}$ and b and c are j-sheaves, then so is a.

EXERCISE 14. If a and b are j-sheaves, so is $a \times b$.

EXERCISE 15. If

$$\begin{tikzcd} a \arrow[r] \arrow[d] & b \arrow[d] \\ c \arrow[r] & d \end{tikzcd}$$

is a pullback, and c, d, b are j-sheaves, then so is a.
EXERCISE 16. If $j = 1_\Omega$, *every* $\mathscr{E}$-object is a j-sheaf.

EXERCISE 17. In the site $(\mathscr{E}, true_\Omega)$ the only sheaves are the terminal objects of $\mathscr{E}$. Hint: Consider the diagram

$$\begin{tikzcd} 0 \arrow[r, tail] \arrow[dr] & a \\ & b \end{tikzcd}$$

EXERCISE 18. b is a j-sheaf iff each j-dense $f: a \rightarrowtail d$ induces a bijection $-\circ f: \mathscr{E}(d, b) \cong \mathscr{E}(a, b)$. □

THEOREM (Lawvere–Tierney). *The full subcategory $sh_j(\mathscr{E})$ of an elementary site $\mathscr{E}_j$ generated by the j-sheaves is an elementary topos. Moreover there is a "sheafication" functor $\mathscr{S}\!\mathscr{h}_j : \mathscr{E} \to sh_j(\mathscr{E})$ that has $\mathscr{S}\!\mathscr{h}_j(b) \cong b$ for each j-sheaf b, and that preserves all finite limits.*

From this result it follows that any Grothendieck topos is an elementary topos. In the case of the elementary site $(\mathbf{St}(I), j_\Theta)$, $\mathscr{S}\!\mathscr{h} : \mathbf{St}(I) \to \mathbf{Sh}(I)$ is the composite $\mathscr{S} \circ \mathscr{G}$, taking stack F to sheaf F_{p_F}.

A proof of this theorem may be found in Freyd [72] or Kock and Wraith [72]. That $sh_j(\mathscr{E})$ has all finite limits is indicated by Exercises 12–15. That it has exponentials is proven by showing that b^a is a j-sheaf whenever b is. Its subobject classifier $true_j : 1 \to \Omega_j$ is formed as the equaliser

$$\begin{tikzcd} \Omega_j \arrow[r, tail] & \Omega \arrow[r, shift left, "1_\Omega"] \arrow[r, shift right, "j"'] & \Omega \\ & 1 \arrow[ul, dashed, "true_j"] \arrow[u, "true"'] & \end{tikzcd}$$

of j and 1_Ω.

An important application of the sheaf construction occurs in the case $j = \neg \circ \neg$. *The topos $sh_{\neg\neg}(\mathscr{E})$ of "double-negation sheaves" in $\mathscr{E}$ is always a Boolean topos*! This is established by showing that in $sh_j(\mathscr{E})$, $[\top_j, \bot_j]$ is $\mathscr{S}\!\mathscr{h}_j([\top, \bot])$ and that $\mathscr{S}\!\mathscr{h}_j$ maps a j-dense monic to an iso in $sh_j(\mathscr{E})$. The result then follows by Exercise 7, and can be seen as an analogue of the fact that the *regular* ($\neg\neg a = a$) elements of a Heyting algebra $\mathbf{H}$ form a *Boolean* subalgebra of $\mathbf{H}$.

Thus from any topos $\mathscr{E}$ we can pass via the functor $\mathscr{S}\!\mathscr{h}_{\neg\neg}$ to a classical subtopos $sh_{\neg\neg}(\mathscr{E})$. This process is used by Tierney [72] to develop a categorial proof of the independence of the Continuum Hypothesis that parallels Cohen's proof for classical set theory. This work reveals that Cohen's "weak-forcing" technique is a version of the technique of passing from a pre-sheaf to its associated sheaf. More recently the method has been used by Marta Bunge [74] to give a topos-theoretic proof of the independence of Souslin's hypothesis.

EXERCISE 19. Let $\mathscr{E} = \mathbf{Set}^{\mathbf{P}}$ and $j = \neg \circ \neg$. Show that $\Omega_{\neg\neg} : \mathbf{P} \to \mathbf{Set}$, the classifier for $\neg \circ \neg$-sheaves, has

$$\Omega_{\neg\neg}(p) = \text{the set of regular members of } [p)^+,$$

where $S \in [p)^+$ is regular iff $\neg_p(\neg_p S) = S$. Show that $\Omega_{\neg\neg}(p)$ is a Boolean subalgebra of the **HA** of hereditary subsets of $[p)$.

EXERCISE 20. Show that in $\mathbf{Top}(I)$, the stalk of $\Omega_{\neg\neg}$ over i is a *Boolean* subalgebra of the **HA** of germs of open sets at i.

EXERCISE 21. Show that in $\mathbf{M_2}$-$\mathbf{Set}$, $\Omega_{\neg\neg} = \{M, \emptyset\}$. □

14.5. Geometric modality

Modal logic is concerned with the study of a one-place connective on sentences that has a variety of meanings, including "it is necessarily true that" (alethic modality), "it is known that" (epistemic modality), "it is believed that" (doxastic), and "it ought to be the case that" (deontic). The quotation that heads this chapter invites us to consider what we might call *geometric* modality. Semantically the modal connective corresponds to an arrow of the form $\Omega \to \Omega$, just as the one-place negation connective corresponds to the arrow $\neg$ of this form. Lawvere suggests that when the arrow is a topology $j:\Omega \to \Omega$ on a topos then the modal connective has the "natural" reading "it is locally the case that."

Let us now extend the sentential language PL of Chapter 6 by the inclusion of a new connective ∇ and the formation rule

if α *is a sentence, then so is* $\nabla\alpha$

($\nabla\alpha$ is to be read "it is locally the case that α").

Let Ψ be the class of all sentences generated from propositional letters π_i by the connectives $\wedge$, $\vee$, $\sim$, $\supset$, ∇. If $\mathscr{E}_j = (\mathscr{E}, j)$ is any elementary site, then an $\mathscr{E}_j$-valuation $V:\Phi_0 \to \mathscr{E}(1,\Omega)$ extends uniquely to the whole of Ψ, using the semantic rules of §6.7, together with

$$V(\nabla\alpha) = j \circ V(\alpha)$$

$$\begin{tikzcd} 1 \arrow[r, "V(\alpha)"] \arrow[dr, "V(\nabla\alpha)"'] & \Omega \arrow[d, "j"] \\ & \Omega \end{tikzcd}$$

We may then define the validity of any $\alpha \in \Psi$ on the site $\mathscr{E}_j$, denoted $\mathscr{E}_j \models \alpha$, to mean that $V(\alpha) = true$ for all $\mathscr{E}_j$-valuations.

Let $\mathscr{J}$ be the axiom system that has Detachment as its sole inference rule, and as axioms the forms I–XI of IL together with the schemata

$$\nabla(\alpha \supset \beta) \supset (\nabla\alpha \supset \nabla\beta)$$

$$\alpha \supset \nabla\alpha$$

$$\nabla\nabla\alpha \supset \nabla\alpha.$$

(Alternatively $\mathscr{J}$ can be defined by replacing the first two of these schemata by

$$(\alpha \supset \beta) \supset (\nabla\alpha \supset \nabla\beta)$$

and

$$\nabla(\alpha \supset \alpha).)$$

Then we have the following characterisation of validity on elementary sites: for any $\alpha \in \Psi$

$$\vdash_{\mathscr{J}} \alpha \quad \textit{iff} \quad \textit{for all sites } \mathscr{E}_j,\ \mathscr{E}_j \models \alpha.$$

The proof of this (described in Goldblatt [77]) uses a Kripke-style model theory for the language Ψ, developed from an analysis of the notion of "local truth". There are in fact two senses in which we have used this idea, one relating to sheaves of germs, the other to sheaves of sections.

(I) Recall the definition of the equivalence relation $\sim_i$ that defines the germ $[U]_i$ of an open set $U \in \Theta$ at i in the sheaf Ω for **Top**(I) (Chapter 4). We have $U \sim_i V$ iff U and V have the same intersection with some i-neighbourhood. We interpret this to mean that the statement "$U = V$" or "$x \in U$ iff $x \in V$" is locally true at i, i.e. true throughout some neighbourhood of i. This in turn represents the intuitive notion that the statement holds for all points "close" to i. The same interpretation was given to the description of germs of sections $s \in F(V)$ in the stalk space A_F of §14.1.

Thus the statement "α is locally true at p" may be rendered as

(i) "α is true at all points close to p",

(ii) "α is true through some neighbourhood of p".

Intuitively (i) and (ii) are equivalent. A p-neighbourhood is any set containing all points that are close to p, while a point is close to p when it belongs to all p-neighbourhoods. Of course in most significant classical topological spaces (any that is at least T_1) there are no points close to p in this sense – other than p itself. The notion can however be given substance by Abraham Robinson's theory of non-standard topology, wherein a space is enlarged to include points "infinitely close" to the original ones. Indeed in his article [69], the germ of U at p is literally a subset of U, namely the intersection of U with the *monad* of p (the set of points infinitely close to p).

Given now a poset **P** we introduce a binary relation $p \prec q$ on P, with the reading "q is close to p". Then given a model $\mathcal{M} = (\mathbf{P}, V)$ based on **P** the connective ∇ can be semantically interpreted as

$$\mathcal{M} \models_p \nabla\alpha \quad \textit{iff} \quad p \prec q \quad \textit{implies} \quad \mathcal{M} \models_q \alpha$$

thereby formalising condition (i).

Writing $\mu(p) = \{q : p \prec q\}$ for the "monad" of p, this clause becomes

$$\mathcal{M} \models_p \nabla\alpha \quad \textit{iff} \quad \mu(p) \subseteq \mathcal{M}(\alpha),$$

where, as in §8.4,

$$\mathcal{M}(\alpha) = \{q : \mathcal{M} \models_q \alpha\}$$

In order for the structure $(P, \sqsubseteq, <)$ to validate the logic $\mathcal{J}$ it suffices that it satisfy

(a) $p < q$ implies $p \sqsubseteq q$, i.e. $\mu(p) \subseteq [p)$ all p,

(b) $<$ is dense, i.e. if $p < q$, then $p < r < q$ for some r, and

(c) $p \sqsubseteq q$ implies $\mu(q) \subseteq \mu(p)$ (this is needed to ensure that $\mathcal{M}(\alpha)$ is $\sqsubseteq$-hereditary).

Notice that we do not require that $p < p$, i.e. $p \in \mu(p)$. Indeed were this to hold for all p, we would have $p < q$ iff $p \sqsubseteq q$. Thus "q is close to p" really means "q is close to but not the same as p", which is akin to the topological notion of "p approximates to q" as formalised by "p is a limit point of $\{q\}$".

To formalise the condition (ii) we could introduce a collection N_p of subsets of P (the p-neighbourhoods) and put

$$\mathcal{M} \models_p \nabla\alpha \quad \textit{iff} \quad \textit{for some} \quad C \in \mathrm{N}_p, \qquad C \subseteq \mathcal{M}(\alpha).$$

One possible construction of an N_p, would be to take a relation $<$ and put

$$\mathrm{N}_p^< = \{C\colon \mu(p) \subseteq C\}, \quad \text{for each } p$$

EXERCISE 1. Show that the structures $(\mathbf{P}, <)$ and $(\mathbf{P}, \mathrm{N}^<)$ validate the same sentences.

EXERCISE 2. Given any poset $(P, \sqsubseteq)$ define

> $p < q$ iff p is a limit point of $\{q\}$ in the topology $\mathbf{P}^+$ (in which "open" = "hereditary").

Show that

$$p < q \quad \text{iff} \quad p \sqsubset q \quad (\text{i.e. } p \sqsubseteq q \text{ and } p \neq q). \qquad \square$$

(II) The sense of "local truth" that applies to stacks of sections refers to a property holding locally of an open set, or an object of a site, rather than at a point. Thus for example a classical topological space is said to be locally connected if each open set is covered by connected open sets.

In this Chapter a function has been described as "locally constant" on its open domain when that domain is covered by open sets, on each of which the function is constant (Exercise 14.1.17).

In the context of a stack F, if $s, t \in F(V)$ have

$$F_x(s) = F_x(t), \quad \text{all} \quad x \in X,$$

given some cover $\{V_x : x \in X\}$ of V, we can take this to mean that "$s = t$" is locally true of V, i.e. true at all members of some cover of V. It follows from COM then that if F is a sheaf, locally equal sections of F are actually equal.

This same sense of a statement being locally true of an open set V when true of all members of a cover of V appears in the interpretation of j_Θ-density of monics given in §14.2.

If we think now of a poset $\mathbf{P} = (P, \sqsubseteq)$ as being the category of open sets of a topology, with $\sqsubseteq$ the opposite to the inclusion ordering, then we may formalise the foregoing discussion by contemplating structures $(\mathbf{P}, Cov)$, where Cov assigns to each $p \in P$ a collection $Cov(p) \subseteq \mathcal{P}(P)$, the "covers" of p. We define, for $\mathcal{M} = (\mathbf{P}, V)$

$$(\overset{*}{*}) \qquad \mathcal{M} \models_p \nabla\alpha \quad \textit{iff} \quad \textit{for some} \quad C \in Cov(p), \qquad C \subseteq \mathcal{M}(\alpha)$$

(Note that, formally, this is the same as the "neighbourhood system" approach described above.)

In order to guarantee that $\mathcal{M}(\alpha)$ be hereditary the operator Cov must satisfy

$$p \sqsubseteq q \quad \textit{only if} \quad Cov(p) \subseteq Cov(q),$$

i.e., every p-cover is a q-cover.

EXAMPLE 1. *Grothendieck topology*: Let $(\mathbf{P}, Cov)$ be the site (Θ, Cov_Θ) as defined in Exercise 14.3.1.

If j_V is the V-th component of j_Θ, then $j_V(C) = \Theta_V = true_V(0)$ iff C covers V, for C a V-crible (Exercise 14.2.14). If $\mathcal{M}$ is any model based on Θ, then $\mathcal{M}(\alpha)_V = \mathcal{M}(\alpha) \cap \Theta_V$ is always a V-crible, and we find that, using the above definition $(\overset{*}{*})$,

$$\mathcal{M} \models_V \nabla\alpha \quad \textit{iff} \quad j_V(\mathcal{M}(\alpha)_V) = true_V(0).$$

Identifying the element $\mathcal{M}(\alpha)_V$ of $\Omega(V)$ with an arrow $\{0\} \to \Omega(V)$ we obtain a $\mathbf{St}(Cov_\Theta)$-arrow $\mathcal{M}(\alpha)\colon 1 \to \Omega$. The role of j_Θ as a modal operator is then given explicitly, as

$$\mathcal{M} \models \nabla\alpha$$

iff

$$\begin{tikzcd} 1 \arrow[r, "\mathcal{M}(\alpha)"] \arrow[dr, "true"'] & \Omega \arrow[d, "j_\Theta"] \\ & \Omega \end{tikzcd}$$

commutes.

EXAMPLE 2. *Cofinality:* Lawvere suggests [70] that the double negation topology $\neg\circ\neg$ is "more appropriately put into words as 'it is cofinally the case that' ".

In general, if S and T are subsets of a poset $\mathbf{P}$, then S is said to be *cofinal* in T if

$$\textit{for all } p \in T \textit{ there is some } q \in S \textit{ such that } p \sqsubseteq q,$$

i.e. every member of T has a member of S "coming after" it.

If we define

$$Cov(p) = \{S \subseteq P\colon S \text{ is cofinal in } [p)\}$$

then for any model $\mathcal{M}$ on $\mathbf{P}$ we find that

$$\mathcal{M} \models_p \nabla\alpha \quad \text{iff} \quad \mathcal{M} \models_p \sim\sim\alpha.$$

This is based on the fact that

$$\mathcal{M} \models_p \sim\sim\alpha \quad \text{iff} \quad \mathcal{M}(\alpha) \text{ is cofinal in } [p). \qquad \square$$

By adapting the techniques described in §8.4, a canonical structure $\mathbf{P}_{\mathcal{J}} = (P, \sqsubseteq, \prec)$ is definable for which

$$\mathbf{P}_{\mathcal{J}} \models \alpha \quad \textit{iff} \quad \vdash_{\mathcal{J}} \alpha.$$

On the topos $\mathbf{Set}^{\mathbf{P}_{\mathcal{J}}}$ a topology $j : \Omega \to \Omega$ is then obtained by defining the component $j_p : \Omega_p \to \Omega_p$ to satisfy

$$j_p(S) = \{q\colon p \sqsubseteq q \text{ and } \mu(q) \subseteq S\},$$

for each $S \in [p)^+$.

We thus obtain the *canonical site* $\mathscr{C}_{\mathcal{J}}$ for $\mathcal{J}$, for which it may be shown that

$$\mathscr{C}_{\mathcal{J}} \models \alpha \quad \textit{iff} \quad \mathbf{P}_{\mathcal{J}} \models \alpha$$

and from this follows the completeness theorem for $\mathcal{J}$ mentioned earlier.

14.6. Kripke–Joyal semantics

The "local character" of properties of sheaves gives rise to a semantical theory, due to André Joyal, that incorporates aspects of Kripke's IL-semantics, together with the principle that the truth-value of a sentence is determined by its local truth-values.

We have already noted that an equality "$s=t$" of sections of a sheaf is true on some open set V iff it is locally true of V, i.e. true throughout some cover of V. Indeed the very essence of the sheaf concept is that an arrow $s: V \to A$ is a section of $f: A \to I$ iff it is locally a section over V. In other words, $s \in F_f(V)$ iff there is a cover $\{V_x : x \in X\}$ with $F_x(s) \in F_f(V_x)$ – "$s \in F_f(V)$" is true when localised to V_x – for all $x \in X$.

To take an example from Lawvere [76] involving existential quantification, suppose that

$$\begin{tikzcd} A \arrow[rr, "h"] \arrow[dr, "f"'] & & B \arrow[dl, "g"] \\ & I & \end{tikzcd}$$

is a map of sheaves of germs and $t \in F_g(V)$ is a section of g over V. We ask – when does there exist a section $s \in F_f(V)$ of f over V with $h \circ s = t$? Answer – precisely when there is a cover $\{V_x : x \in X\}$ of V with for each x a section $s_x \in F_f(V_x)$ such that $h \circ s_x = t \restriction V_x$. Thus the statement $\exists s(h \circ s = t)$ is true of V precisely when it is locally true of V.

Briefly, the basis of Joyal's semantics is this. We consider interpretations of formulae $\varphi(v_1, v_2)$ in a site $(\mathscr{C}, Cov)$. Given arrows $f: a \to b$, $g: a \to c$, suppose we know what it means for $\langle f, g \rangle$ to satisfy φ at a, denoted $a \models \varphi[f, g]$. Then for a particular $f: a \to b$ we put

$a \models \exists v_2 \varphi[f]$ iff there is an a-cover $\{a_x \xrightarrow{f_x} a : x \in X\}$ and arrows $\{a_x \xrightarrow{g_x} c : x \in X\}$ such that $a_x \models \varphi[f \circ f_x, g_x]$, all $x \in X$.

The disjunction connective gets a similar interpretation:

$a \models \varphi \vee \psi[f]$ iff for some $\{a_x \xrightarrow{f_x} a : x \in X\} \in Cov(a)$, we have for each $x \in X$ that $a_x \models \varphi[f \circ f_x]$ or $a_x \models \psi[f \circ f_x]$.

The other connectives, and the universal quantifier, are interpreted by analogues of Kripke's rules, e.g.

$a \models \varphi \supset \psi[f]$ iff for any $a_x \xrightarrow{f_x} a$, if $a_x \models \varphi[f \circ f_x]$ then $a_x \models \psi[f \circ f_x]$

$a \models \forall v_2 \varphi[f]$ iff for all $a_x \xrightarrow{f_x} a$ and all $a_x \xrightarrow{g} c$, $a_x \models \varphi[f \circ f_x, g]$.

The "local character of truth" is then embodied in the consequence that for any formula $\varphi(v)$,

$a \models \varphi[f]$ iff for some $\{a_x \xrightarrow{f_x} a : x \in X\} \in Cov(a)$, $a_x \models \varphi[f \circ f_x]$, all $x \in X$.

The details of the Kripke–Joyal semantics are given by Reyes [76] for sites, Boileau [75] for general topoi and Osius [75(i)] for categorial set theory. For applications of it cf. Kock [76]. In as much as it gives a non-classical interpretation to $\vee$ and $\exists$ it is more analogous to Beth models, and "Beth–Joyal semantics" would perhaps be a more appropriate name. A Beth model for first-order logic has a single set A of individuals, rather than one for each state $p \in P$ as in the structures of §11.6. The universal quantifier has the standard interpretation

$$\mathcal{M} \models_p \forall v\, \varphi \quad \text{iff} \quad \text{for all} \quad a \in A, \qquad \mathcal{M} \models_p \varphi[a]$$

while the clause for $\exists$ reads

$$\mathcal{M} \models_p \exists v\, \varphi \quad \text{iff} \quad \text{there is a bar } B \text{ for } p \text{ such that for each } q \in B$$
$$\text{there is some } a \in A \text{ with } \mathcal{M} \models_q \varphi[a].$$

An application of this modelling to intuitionistic metamathematics and an indication of its relation to topological interpretations may be found in van Dalen [78].

14.7. Sheaves as complete Ω-sets

Let $\mathbf{A}$ be an Ω-set, where Ω is a **CHA**. Then, as defined in §11.9, a singleton for $\mathbf{A}$ is a function $s : A \to \Omega$ that satisfies

(i) $s(x) \sqcap [\![x \approx y]\!] \sqsubseteq s(y)$

(ii) $s(x) \sqsubseteq [\![Ex]\!]$

(iii) $s(x) \sqcap s(y) \sqsubseteq [\![x \approx y]\!]$

for all $x, y \in A$. (These are conditions (viii)–(x) of §11.9. Note that (ii) is a consequence of (iii) by putting $x = y$.) Each element $a \in A$ yields the singleton $\{\mathbf{a}\}$ that assigns $[\![x \approx a]\!]$ to each $x \in A$. $\mathbf{A}$ is called a *complete* Ω-set if each of its singletons is of the form $\{\mathbf{a}\}$ for a *unique* $a \in A$.

Example 1. Let $\Omega = \Theta$, the **CHA** of open subsets of a space I. Then for any topological space X we have a corresponding Θ-set $\mathbf{C}_X$, which is the set of continuous X-valued partial functions on I. C_X is the set of all continuous functions of the form $f : V \to X$, for all $V \in \Theta$, with degrees of equality measured as

$$[\![f \approx g]\!] = \{i : f(i) = g(i)\}^{\mathrm{o}}.$$

Now suppose that $s: C_X \to \Theta$ is a singleton. For each $f \in C_X$, let $f_s = f \restriction s(f)$ be the restriction of f to the open set $s(f)$. By condition (ii) above, $s(f)$ is a subset of $[\![f \approx f]\!]$, i.e. of the domain of f, and so $\operatorname{dom} f_s$ is just $s(f)$ itself. But then by (iii) we see that the f_s's form a compatible family of functions, since if i belongs to $\operatorname{dom} f_s$ and $\operatorname{dom} g_s$ it must belong to $[\![f \approx g]\!]$ and so $f_s(i) = f(i) = g(i) = g_s(i)$.

Thus we may "patch" together the f_s's to obtain a single element a_s of C_X whose restriction to each $s(f)$ is just f_s. In other words, a_s agrees with f on the set $s(f)$, giving

$$s(f) \subseteq [\![f \approx a_s]\!].$$

For the converse inclusion, if $f(i) = a_s(i)$, then $f(i) = g_s(i) = g(i)$ for some g with $i \in \operatorname{dom} g_s = s(g)$. But then the extensionality condition (i) implies that $i \in s(f)$.

Thus we see that our original singleton s is the function $\{\mathbf{a}_s\}$. To see that a_s is unique with this property, observe that whenever $\{\mathbf{f}\} = \{\mathbf{g}\}$, i.e. f and g agree with all members of $\mathbf{C}_X$ to the same extent, then f and g agree with each other to the same extent that they agree with themselves, and so

$$[\![f \approx g]\!] = [\![f \approx f]\!] = [\![g \approx g]\!]$$

(cf. Ex. 16 of §11.9). Thus f and g have the same domain, and they agree on that domain, which means that $f = g$.

This establishes the completeness of $\mathbf{C}_X$.

EXAMPLE 2. Analogously, given a continuous function $k: A \to I$, we obtain the Θ-set $\mathbf{C}_k$ whose elements are the local (partial) continuous sections $I \rightsquigarrow A$ of k. The completeness of $\mathbf{C}_k$ is established exactly as above. In particular, this assigns a complete Θ-set to each object of $\mathbf{Top}(I)$. $\mathbf{C}_X$ itself can be identified with the set of local sections of the projection function $X \times I \to X$, by identifying $f: V \to X$ with $\langle f, V \hookrightarrow I \rangle : V \to X \times I$. (Note that the projection need not be a local homeomorphism, and hence not a $\mathbf{Top}(I)$-object.) □

The completeness property for an Ω-set allows a very elegant abstract treatment of the idea of the restriction of a function to an open set. The development of this theory is due to Dana Scott and Michael Fourman. Given $a \in A$ and $p \in \Omega$, the function $\{\mathbf{a}\} \restriction p$ that assigns $[\![x \approx a]\!] \sqcap p$ to x is a singleton (§11.9, Exercise 17). If $\mathbf{A}$ is complete, then there is exactly one $b \in A$ with $\{\mathbf{b}\} = \{\mathbf{a}\} \restriction p$. We call b the *restriction of a to p*, and denote it $a \restriction p$. (From now on we will often abbreviate the extent $[\![Ea]\!] = [\![a \approx a]\!]$ of a to Ea).

EXERCISE 1. $(a \restriction p) \restriction q = a \restriction (p \sqcap q)$

EXERCISE 2. $a \restriction Ea = a$

EXERCISE 3. $E(a \restriction p) = Ea \sqcap p$

EXERCISE 4. $a \restriction [\![a \approx b]\!] = b \restriction [\![a \approx b]\!]$

EXERCISE 5. $a \restriction (Ea \sqcap Eb) = a \restriction Eb$

EXERCISE 6. Write $a \between b$ to mean that

$$a \restriction Eb = b \restriction Ea$$

i.e. that a and b are *compatible.* Prove that

$$a \between b \quad \text{iff} \quad Ea \sqcap Eb \sqsubseteq [\![a \approx b]\!].$$

EXERCISE 7. Show that $a \between b$, as defined in the last exercise, iff

$$[\![x \in \{\mathbf{a}\}]\!] \sqcap [\![y \in \{\mathbf{b}\}]\!] \sqsubseteq [\![x \approx y]\!]$$

holds for all $x, y \in A$.

EXERCISE 8. Show that the relation $\leqslant$, where

$$a \leqslant b \quad \text{iff} \quad a = b \restriction Ea$$

is a partial ordering on A that satisfies

(i) $a \restriction p \leqslant a$,
(ii) $a \leqslant b$ implies $Ea \sqsubseteq Eb$ and $a \restriction p \leqslant b \restriction p$,
(iii) if $a \leqslant c$ and $b \leqslant c$, for some c, then $a \between b$,
(iv) $a \leqslant b$ iff $Ea \sqsubseteq [\![a \approx b]\!]$,
(v) $a \leqslant b \restriction p$ iff $a \leqslant b$ and $Ea \sqsubseteq p$,
(vi) $a \restriction p = a$ iff $Ea \sqsubseteq p$,
(vii) $a \leqslant b$ iff $a \between b$ and $Ea \sqsubseteq Eb$.

EXERCISE 9. Define $a \in A$ to be the *join* of $B \subseteq A$, written $a = \bigvee B$, iff
(i) $b \leqslant a$ for all $b \in B$, and
(ii) $Ea = \bigsqcup \{Eb : b \in B\}$.

Show that a complete Ω-set $\mathbf{A}$ satisfies the following abstract version of COM:

> *Every subset $B \subseteq A$ whose elements are pairwise compatible has a* unique *join.*

Prove in fact that

$$s(x) = \bigsqcup_{b \in B} [\![x \approx b]\!]$$

defines a singleton when B has pairwise compatible elements (use Ex. 6) and that the corresponding element of $\mathbf{A}$ to s is $\bigvee B$.

N.B.: To do this exercise, and many of those to follow, you will need to know that a **CHA** satisfies the following law of distribution of $\sqcap$ over $\bigsqcup$:

$$x \sqcap (\bigsqcup C) = \bigsqcup_{c \in C} (x \sqcap c), \quad \text{all} \quad C \subseteq \Omega$$

EXERCISE 10. (i) Prove that $\bigvee B$, when it exists, is the l.u.b. of B for the ordering $\leqslant$, and that in general a set B has a join iff it has a l.u.b. for this ordering.

(ii) $(\bigvee B) \restriction p = \bigvee \{b \restriction p : b \in B\}$. □

A presheaf $F_{\mathbf{A}} : \Omega \to \mathbf{Set}$ over the poset category Ω is defined for *complete* $\mathbf{A}$ by putting

$$F_{\mathbf{A}}(p) = \{x \in A : Ex = p\}$$

for each $p \in \Omega$. Whenever $p \sqsubseteq q$, the assignment of $x \restriction p$ to x is a function from $F_{\mathbf{A}}(q)$ to $F_{\mathbf{A}}(p)$ (Exercise 3). We take this function as the $F_{\mathbf{A}}$-image of the Ω-arrow $p \to q$.

In order to discuss sheaves over the category Ω we define $Cov_\Omega(p)$ to be the collection of all subsets C of Ω that have $\bigsqcup C = p$. This is an obvious generalisation of the definition of Cov_Θ given in §14.3 and (Ω, Cov_Ω) can be shown to be a site. The corresponding category of *sheaves over Ω* is denoted $\mathbf{Sh}(\Omega)$.

EXERCISE 11. Let $C \in Cov_\Omega(p)$ and consider a selection of elements $x_q \in F_{\mathbf{A}}(q)$, all $q \in C$, that are pairwise compatible (in the sense given in COM, or in this section – they mean the same thing). Use the definition of join given in Exercise 9 to construct a unique $x \in F_{\mathbf{A}}(p)$ with $x \restriction q = x_q$, all $q \in C$. Hence verify that $F_{\mathbf{A}}$ is a sheaf (satisfies COM). □

In the converse direction, given a sheaf F over Ω we construct a corresponding Ω-set $\mathbf{A}_F$. We let

$$A_F = \{\langle x, q\rangle : x \in F(q)\}$$

be the disjoint union of the sets $F(q)$ for all $q \in \Omega$. For $a = \langle x, q\rangle$, put $E(a) = q$. Then for any p, we define

$$a \restriction p = \langle F^q_{p\sqcap q}(x), p \sqcap q\rangle,$$

and this allows us to put

$$[\![a \approxeq b]\!]_{\mathbf{A}_F} = \bigsqcup\{p \in \Omega : a \restriction p = b \restriction p\}.$$

Equality can now be given by

$$[\![a \approx b]\!]_{\mathbf{A}_F} = [\![a \approxeq b]\!]_{\mathbf{A}_F} \sqcap E(a) \sqcap E(b).$$

EXERCISE 12. $[\![a \approx a]\!]_{\mathbf{A}_F} = E(a)$

EXERCISE 13. $[\![a \approxeq b]\!]_{\mathbf{A}_F} = (E(a) \sqcup E(b)) \Rightarrow [\![a \approx b]\!]_{\mathbf{A}_F}$

EXERCISE 14. Verify that $\mathbf{A}_F$ is an Ω-set.

EXERCISE 15. Let $s : A_F \to \Omega$ be a singleton. Generalise the argument of Example 1 by showing that the elements $a \restriction s(a)$, for all $a \in A_F$ are pairwise compatible, and use the property COM as it applies to F to show that $s = \{\mathbf{a}\}$, for a unique a (which will in fact be the join of the elements $a \restriction s(a)$). Hence show that $\mathbf{A}_F$ is complete. □

EXAMPLE 3. Let $\Omega = \Theta$ and X be a topological space as in Example 1. The sheaf $F_X : \Theta \to \mathbf{Set}$ of continuous X-valued (partial) functions on I has

$$F_X(V) = \{V \xrightarrow{f} X : f \text{ is continuous}\}$$

with each inclusion $V \hookrightarrow W$ being assigned the usual restriction operator by F_X. In this case $F_X(V)$ and $F_X(W)$ are already disjoint when $V \neq W$, as they consist of sets of functions with distinct domains, and so in forming the associated Θ-set we may simply take the union of the $F_X(V)$'s. We see then that $\mathbf{A}_{F_X}$ is none other than the Θ-set $\mathbf{C}_X$ of Example 1. □

EXERCISE 16. Develop a truly axiomatic theory of "restrictions of elements over a **CHA**" by *defining* a presheaf over Ω to be a set A together with a pair of functions

$$\restriction : A \times \Omega \to A$$
$$E : A \to \Omega$$

that satisfy the laws of Exercises 1–3. Define *compatibility*, the *restriction ordering* $\leqslant$, and *join* for such a structure and call it a *sheaf* if it satisfies the version of COM given in Exercise 9. Use the definition of equality for $\mathbf{A}_F$ to show that such a sheaf carries a complete-Ω-set structure whose $a \restriction p$ operation (defined via singletons) and extent function $[\![Ea]\!]$ are the original $\restriction$ and E you started with.

EXERCISE 17. Let $\mathbf{A} \restriction p$ be the Ω-set based on

$$A \restriction p = \{a \in A : Ea \sqsubseteq p\}$$

with equality as for $\mathbf{A}$. Show that if $B \subseteq A \restriction p$ has a join in $\mathbf{A}$ then this join belongs to $A \restriction p$. Hence show that $\mathbf{A} \restriction p$ is complete if $\mathbf{A}$ is. □

The constructions $\mathbf{A} \mapsto F_{\mathbf{A}}$ and $F \mapsto \mathbf{A}_F$ can be extended to arrows to give an equivalence between $\mathbf{Sh}(\Omega)$ and the sub-category of Ω-**Set** generated by the complete objects. In fact $\mathbf{Sh}(\Omega)$ is equivalent to the larger category Ω-**Set** itself, a result due originally to D. Higgs [73]. This is because each Ω-set $\mathbf{A}$ is isomorphic in Ω-**Set** to a complete Ω-set $\mathbf{A}^*$. We take A^* as the set of all singletons $s : A \to \Omega$ of $\mathbf{A}$, with

$$[\![s \approx t]\!]_{\mathbf{A}^*} = \bigsqcup_{x \in A} (s(x) \sqcap t(x))$$

("there is an x belonging to both s and t". In **Set**, overlapping singletons are identical).

EXERCISE 18. $[\![Es]\!]_{\mathbf{A}^*} = \bigsqcup_{x \in A} [\![x \in s]\!] = \bigsqcup_{a \in A} [\![s \approx \{\mathbf{a}\}]\!]_{\mathbf{A}^*}$

EXERCISE 19. $[\![\{\mathbf{a}\} \approx s]\!]_{\mathbf{A}^*} = s(a)$

EXERCISE 20. $[\![\{\mathbf{a}\} \approx \{\mathbf{b}\}]\!]_{\mathbf{A}^*} = [\![a \approx b]\!]_{\mathbf{A}}$

EXERCISE 21. $[\![E\{\mathbf{a}\}]\!]_{\mathbf{A}^*} = [\![Ea]\!]_{\mathbf{A}}$

EXERCISE 22. (cf. Example 1). Let $s : A^* \to \Omega$ be a singleton of $\mathbf{A}^*$. For each $f \in A^*$, let f_s be the singleton $f \restriction s(f)$ as defined in Exercise 17 of §11.9, so that $f_s(x) = f(x) \sqcap s(f)$.

Define

$$a_s(x) = \bigsqcup_{f \in A^*} f_s(x).$$

(i) Prove $[\![Ef_s]\!]_{\mathbf{A}^*} = s(f)$.

(ii) Show that the f_s's are pairwise compatible in the sense (of Exercise 7) that they satisfy

$$f_s(x) \sqcap g_s(y) \sqsubseteq [\![x \approx y]\!]_{\mathbf{A}}.$$

(iii) Show that a_s is a singleton of $\mathbf{A}$, with $[\![f \approx a_s]\!]_{\mathbf{A}^*} = s(f)$, all $f \in A^*$.

(iv) Suppose that $h \in A^*$ has $[\![f \approx h]\!] = s(f)$, all f. Show that $h \restriction s(f) = f_s$ for all f (i.e. $h(x) \sqcap s(f) = f_s(x)$), and hence that $h = a_s$.
Thus prove that $\mathbf{A}^*$ is complete.

EXERCISE 23. Since $\mathbf{A}^*$ is complete, each element $s \in A^*$ has, for each $p \in \Omega$, a restriction $s \restriction p$ defined as the unique element $t \in A^*$ corresponding to the singleton $\{\mathbf{s}\} \restriction p$ of $\mathbf{A}^*$ (i.e. t is defined by the equation

$$[\![x \approx t]\!]_{\mathbf{A}^*} = [\![x \approx s]\!]_{\mathbf{A}^*} \sqcap p).$$

Show that this t is precisely the singleton $s \restriction p$ of Exercise 17 of §11.9 (i.e. that $t(x) = s(x) \sqcap p$).

EXERCISE 24. Show that in $\mathbf{A}^*$,

$$s \restriction s(a) = \{\mathbf{a}\} \restriction s(a)$$

all $s \in A^*$, $a \in A$.

EXERCISE 25. In view of Exercise 23, use the ideas of Exercise 16 to develop an alternative proof that $\mathbf{A}^*$ is complete.

EXERCISE 26. Prove that in Ω-**Set**, an arrow $f : A \times B \to \Omega$ from $\mathbf{A}$ to $\mathbf{B}$ is

(i) monic iff it satisfies

$$f(\langle x, y \rangle) \sqcap f(\langle z, y \rangle) \sqsubseteq [\![x \approx z]\!]$$

(ii) epic iff it satisfies

$$[\![Ey]\!]_{\mathbf{B}} = \bigsqcup_{x \in A} f(\langle x, y \rangle)$$

("y exists in $\mathbf{B}$ to the extent that it is the f-image of some x in $\mathbf{A}$").

EXERCISE 27. Define $i_{\mathbf{A}} : A \times A^* \to \Omega$ by $i_{\mathbf{A}}(\langle x, s \rangle) = s(x)$. Use the last exercise to show that $i_{\mathbf{A}} : \mathbf{A} \to \mathbf{A}^*$ is iso in Ω-**Set**. □

The topos CΩ-Set

The last exercise implies that as far as categorial constructions are concerned, we may confine our attention to complete Ω-sets (also called *Ω-sheaves*). In this context we can take a different approach to arrows, by taking the $[\![f(x)\approx y]\!]$ notation for $f(\langle x, y\rangle)$ literally.

Let $\mathbf{A}$ and $\mathbf{B}$ be Ω-sets, and $g:A\to B$ a function from set A to set B satisfying

(i) $[\![x\approx y]\!]_{\mathbf{A}}\sqsubseteq[\![g(x)\approx g(y)]\!]_{\mathbf{B}}$

(ii) $[\![Eg(x)]\!]_{\mathbf{B}}\sqsubseteq[\![Ex]\!]_{\mathbf{A}}$.

Define $\bar{g}:A\times B\to\Omega$ by

$$\bar{g}(\langle x, y\rangle)=[\![g(x)\approx y]\!]_{\mathbf{B}}$$

EXERCISE 28. Prove $[\![Eg(x)]\!]=[\![Ex]\!]$.

EXERCISE 29. Show that $\bar{g}$ is an arrow from $\mathbf{A}$ to $\mathbf{B}$ in **Ω-Set**, i.e. an extensional, functional, total Ω-valued relation from A to B (conditions (iv)–(vii) of §11.9). □

To avoid confusion, a function g satisfying (i) and (ii) will be called a *strong* arrow, while the **Ω-Set** arrows will be referred to as *weak*. The two notions are equivalent in the case of a complete codomain. If $f:A\times B\to\Omega$ is a weak arrow, for given $a\in A$ define $s_a:B\to\Omega$ by

$$s_a(y)=f(\langle a, y\rangle).$$

EXERCISE 30. Use the weak arrow properties of f to show that s_a is a singleton of $\mathbf{B}$. □

If $\mathbf{B}$ is complete, there will then be a unique $b\in B$ that has $\{\mathbf{b}\}=s_a$. Put $g_f(a)=b$.

EXERCISE 31. Show that $g_f:A\to B$ is a strong arrow from $\mathbf{A}$ to $\mathbf{B}$, with $\bar{g}_f=f$.

EXERCISE 32. If g is strong, with cod g complete, prove that $g_{\bar{g}}=g$.

EXERCISE 33. Show that for complete $\mathbf{A}$, $g_{1_{\mathbf{A}}}=\mathrm{id}_A$.

EXERCISE 34. If $f:\mathbf{A}\to\mathbf{B}$ and $h:\mathbf{B}\to\mathbf{C}$ are weak arrows, with $\mathbf{B}$ and $\mathbf{C}$ complete, then $g_{h\circ f}$ is the functional composition $g_h\circ g_f$.

EXERCISE 35. Suppose every weak arrow with codomain **B** is of the form $\bar{g}$ for exactly one strong arrow g. Show that **B** is complete (Hint: consider the one-element Ω-set $\{0\}$ with $[\![0 \approx 0]\!] = [\![Es]\!]_{\mathbf{B}^*}$, and the weak arrow $f: \{0\} \times B \to \Omega$ with $f(\langle 0, y \rangle) = s(y)$).

EXERCISE 36. If **A** and **B** are complete, show that $g: A \to B$ is strong iff it preserves extents and restrictions, i.e. has

$$Eg(a) = Ea$$

$$g(a \upharpoonright p) = g(a) \upharpoonright p$$

all $a \in A$, $p \in \Omega$. □

The category **CΩ-Set** is defined to be that which consists of the complete Ω-sets with strong arrows between them, identities and composites being as in **Set**.

EXERCISE 37. Let $f: \mathbf{A} \to \mathbf{B}$ be a weak arrow, and $s \in A^*$. Define $f_s: B \to \Omega$ by

$$f_s(y) = \bigsqcup_{x \in A} (f(x, y) \sqcap s(x)).$$

Show that f_s is a singleton of **B** ("y belongs to f_s to the extent that it is the f-image of some member of s"). Show that putting $f^*(s) = f_s$ defines a strong arrow $f^*: \mathbf{A}^* \to \mathbf{B}^*$ for which

$$[\![f^*(\{\mathbf{a}\}) \approx \{\mathbf{b}\}]\!]_{\mathbf{B}^*} = f(a, b)$$

all $a \in A$, $b \in B$.

EXERCISE 38. Let the functor $F: \mathbf{C\Omega\text{-}Set} \to \mathbf{\Omega\text{-}Set}$ be the identity on objects, and have $F(g) = \bar{g}$. Let $F^*: \mathbf{\Omega\text{-}Set} \to \mathbf{C\Omega\text{-}Set}$ have $F^*(\mathbf{A}) = \mathbf{A}^*$ and $F^*(f) = f^*$.

Show that F and F^* establish the equivalence of the two categories.

EXERCISE 39. Let $g: \mathbf{A} \to \mathbf{B}$ be a strong arrow, with **A** and **B** complete. Show that in addition to preserving E and $\upharpoonright$, g preserves $\leqslant$ and $\bigvee$, i.e.

(i) $x \leqslant y$ only if $g(x) \leqslant g(y)$
(ii) $g(\bigvee C) = \bigvee\{g(c): c \in C\}$, all $C \subseteq A$.

EXERCISE 40. Let s be a singleton of **A**. Show that in $\mathbf{A}^*$, the elements $\{\mathbf{a}\} \upharpoonright s(a)$ are pairwise compatible, and their join is s, i.e.

$$s = \bigvee\{\{\mathbf{a}\} \upharpoonright s(a): a \in A\}.$$

EXERCISE 41. (i) Let $f: A \to \Omega$ be a subset (extensional and strict) of $\mathbf{A}$ in the sense of §11.9. Define $f^*: A^* \to \Omega$ by putting

$$f^*(s) = \bigsqcup_{a \in A} (f(a) \sqcap s(a)).$$

Show that f^* is a subset of $\mathbf{A}^*$ that has $f^*(\{\mathbf{a}\}) = f(a)$, all $a \in A$.

(ii) Given a subset $g: A^* \to \Omega$ of $\mathbf{A}^*$ show that

$$g_*(a) = g(\{\mathbf{a}\})$$

defines a subset g_* of $\mathbf{A}$, and that

$$g(s) = \bigsqcup_{a \in A} (g_*(a) \sqcap s(a)).$$

Thus show that subsets of $\mathbf{A}^*$ correspond uniquely to subsets of $\mathbf{A}$.

EXERCISE 42. Let $i_{\mathbf{A}}: A \times A^* \rightarrow \Omega$ be the weak iso arrow of Exercise 27. Show that the corresponding strong arrow (which we also denote $i_{\mathbf{A}}$) assigns $\{\mathbf{a}\}$ to a.

EXERCISE 43. Let $g: \mathbf{A} \to \mathbf{C}$ be a strong arrow, with $\mathbf{C}$ complete. Show that there exists exactly one strong arrow $h: \mathbf{A}^* \to \mathbf{C}$ for which the diagram

$$\begin{tikzcd} \mathbf{A} \arrow[r, "i_{\mathbf{A}}"] \arrow[dr, "g"'] & \mathbf{A}^* \arrow[d, dashed, "h"] \\ & \mathbf{C} \end{tikzcd}$$

commutes. (Hint: Consider the elements $g(a) \restriction s(a)$, all $a \in A$, for $s \in A^*$. Use Exercises 39, 40).

EXERCISE 44. Show either directly or via the last Exercise that the function f^* of Exercise 37 is uniquely determined by the fact that it has $[\![f^*(\{\mathbf{a}\}) \approx \{\mathbf{b}\}]\!] = f(a, b)$, all $a \in A$ and $b \in B$. □

The topos-structure of $\mathbf{C}\Omega$-**Set** could be obtained by applying the completing functor F^* to Ω-**Set**. The relevant constructions admit however of simplified descriptions, which we now outline.

Terminal object

1 is the set Ω, with $[\![p \approx q]\!]_{\mathbf{1}} = p \sqcap q$.

We have $Ep = p$, $p \upharpoonright q = p \sqcap q$, and $[\![p \approx q]\!] = (p \Leftrightarrow q)$.

Notice that in this case $\leqslant$ is the lattice ordering $\sqsubseteq$ on the **CHA** Ω, and $\bigvee$ is the lattice join (l.u.b.) $\bigsqcup$.

The unique arrow $\mathbf{A} \to \mathbf{1}$ is the extent function $a \mapsto [\![a \approx a]\!]_{\mathbf{A}}$.

EXERCISE 45. Let $\Omega = \Theta$, and X be a topological space. If $f : I \to X$ is continuous, let $f_o : \Theta \to C_X$ assign to each V the restriction $f \upharpoonright V$ of f to V. Interpret the two conditions that define strong arrows to show that in $\mathbf{C\Theta}$**-Set** we have $f_o : \mathbf{1} \to \mathbf{C}_X$. Conversely, given an "element" $g : \mathbf{1} \to \mathbf{C}_X$ of $\mathbf{C}_X$, show that $g(V)$ has domain V, and that the $g(V)$'s are pairwise compatible. Hence show that there is a unique $f \in C_X$ that has $f_o = g$.

Thus establish that there is a bijective correspondence between elements $\mathbf{1} \to \mathbf{C}_X$ of $\mathbf{C}_X$ in $\mathbf{C\Theta}$**-Set** and *globally defined* continuous functions $I \to X$.

EXERCISE 46. In view of the last exercise, we say that a is a *global element* of $\mathbf{A}$ in $\mathbf{C\Omega}$**-Set** if $Ea = \top$. For such an element, define $f_a : \Omega \to A$ by $f_a(p) = a \upharpoonright p$ and show that $f_a : \mathbf{1} \to \mathbf{A}$.

Conversely, given $h : \mathbf{1} \to \mathbf{A}$, use Exercise 6 to show that the $h(p)$'s are compatible, and hence prove that there is a unique global element a of $\mathbf{A}$ with $f_a = h$.

EXERCISE 47. The complete Ω-set $\mathbf{1} \upharpoonright e$ (Exercise 17) is based on the set

$$\Omega \upharpoonright e = \{q : q \sqsubseteq e\}.$$

Show that this is a **CHA** in its own right with the same $\bigsqcup$ and $\bigsqcap$ operations as Ω, but with pseudo-complement $\neg_e$ and relative pseudo-complement $\Rightarrow_e$ given by

$$\neg_e q = \neg q \upharpoonright e = \neg q \sqcap e$$

and

$$q \Rightarrow_e r = (q \Rightarrow r) \upharpoonright e = (q \Rightarrow r) \sqcap e,$$

all q, $r \in \Omega \upharpoonright e$. □

Initial object

Recall from §11.9 that the function from A to Ω that assigns $\perp$ to every $a \in A$ is a singleton, and so for complete $\mathbf{A}$ corresponds to a unique element $\emptyset_{\mathbf{A}} \in A$. We have $[\![x \approx \emptyset_{\mathbf{A}}]\!] = \perp$, all x.

The initial object $\mathbf{0}$ for $\mathbf{C\Omega}$-**Set** is the one-element set $\{\perp\}$, with $[\![\perp \approx \perp]\!] = \perp$. The unique arrow $\mathbf{0} \to \mathbf{A}$ assigns $\emptyset_{\mathbf{A}}$ to $\perp$.

EXERCISE 48. (i) $\emptyset_{\mathbf{A}}$ is the join of the empty subset of A.
(ii) If $Ea = \perp$, then $a = \emptyset_{\mathbf{A}}$. □

Products

$\mathbf{A} \times \mathbf{B}$ is the set

$A \cdot B = \{\langle a, b\rangle \in A \times B : Ea = Eb\}$, with

$[\![\langle a, b\rangle \approx \langle c, d\rangle]\!] = [\![a \approx c]\!] \sqcap [\![b \approx d]\!]$.

We have

$$E\langle a, b\rangle = Ea \sqcap Eb$$

and

$$\langle a, b\rangle \restriction p = \langle a \restriction p, b \restriction p\rangle$$

Projection arrows, and products of arrows are defined just as in **Set**.

Coproducts

$\mathbf{A} + \mathbf{B}$ is the set

$A + B = \{\langle a, b\rangle \in A \times B : Ea \sqcap Eb = \perp\}$, with

$[\![\langle a, b\rangle \approx \langle c, d\rangle]\!] = [\![a \approx c]\!] \sqcup [\![b \approx d]\!]$,

giving

$$E\langle a, b\rangle = Ea \sqcup Eb$$

and

$$\langle a, b\rangle \restriction p = \langle a \restriction p, b \restriction p\rangle.$$

The injection $i_{\mathbf{A}} : \mathbf{A} \to \mathbf{A} + \mathbf{B}$ takes $a \in A$ to $\langle a, \emptyset_{\mathbf{B}}\rangle$, while $i_{\mathbf{B}}(b) = \langle \emptyset_{\mathbf{A}}, b\rangle$, all $b \in B$.

The coproduct $[f, g] : \mathbf{A} + \mathbf{B} \to \mathbf{C}$ of two strong arrows f and g assigns to $\langle a, b\rangle$ the join in $\mathbf{C}$ of $f(a)$ and $g(b)$.

EXERCISE 49. Verify that $f(a)$ and $g(b)$ are compatible in $\mathbf{C}$ when $\langle a, b\rangle \in A + B$ and f and g are strong. □

Pullback

The domain of the pullback

$$\begin{tikzcd} \mathbf{D} \arrow[r, "g'"] \arrow[d, "f'"'] & \mathbf{B} \arrow[d, "g"] \\ \mathbf{A} \arrow[r, "f"'] & \mathbf{C} \end{tikzcd}$$

of f and g as shown has $D = \{\langle x, y\rangle \in A \cdot B : f(x) = g(y)\}$ with its Ω-equality inherited from $\mathbf{A} \times \mathbf{B}$. f' and g' are the evident projections.

EXERCISE 50. Show that $a \in D$ only if $a \restriction p \in D$ all $p \in \Omega$. Prove that if a subset of D has a join in $A \cdot B$ then this join belongs to D. Hence verify that $\mathbf{D}$ is complete. □

Subobject classifier

The object of truth-values is $\dot{\boldsymbol{\Omega}}$, where

$$\dot{\Omega} = \{\langle p, e\rangle \in \Omega \times \Omega : p \sqsubseteq e\}$$

and

$$[\![\langle p, e\rangle \approx \langle q, e'\rangle]\!] = (p \Leftrightarrow q) \sqcap e \sqcap e'$$

giving

$$E\langle p, e\rangle = e$$

and

$$\langle p, e\rangle \restriction q = \langle p \sqcap q, e \sqcap q\rangle.$$

The arrow $true : \mathbf{1} \to \dot{\boldsymbol{\Omega}}$ has

$$true(p) = \langle p, p\rangle, \quad \text{all} \quad p \in \Omega.$$

If $f : \mathbf{A} \to \mathbf{B}$ is monic in **CΩ-Set** (which just means that it is injective as a set function-exercise) then for each $b \in B$ we define the truth-value of "$b \in f(A)$" as

$$[\![b \in f(A)]\!] = \bigsqcup_{a \in A} [\![f(a) \approx b]\!]$$

("b belongs to $f(A)$ to the extent that it is equal to the f-image of some $a \in A$").

The character $\chi_f : \mathbf{B} \to \dot{\boldsymbol{\Omega}}$ of f is then given by

$$\chi_f(b) = \langle [\![b \in f(A)]\!], Eb \rangle.$$

EXERCISE 51. Prove that

$$[\![b \in f(A)]\!] = \bigsqcup \{Ea : a \in A \text{ and } f(a) \leqslant b\}$$

("b belongs to $f(A)$ to the extent that there exists a restriction of b in $f(A)$"), and show that the image $f(A)$ of A under f is precisely the set

$$\{b \in B : [\![b \in f(A)]\!] = Eb\}.$$

Hence show how to define the subobject of $\mathbf{B}$ that is classified by a given arrow $\mathbf{B} \to \dot{\boldsymbol{\Omega}}$.

EXERCISE 52. Show that the propositional logic of $\mathbf{C\Omega}$-**Set** is as follows:

(i) $\mathit{false}(p) = \langle \bot, p \rangle$ defines $\mathit{false} : \mathbf{1} \to \dot{\boldsymbol{\Omega}}$

(ii) The negation arrow $\neg : \dot{\boldsymbol{\Omega}} \to \dot{\boldsymbol{\Omega}}$ has

$$\begin{aligned} \neg(\langle p, e \rangle) &= \langle \neg_e p, e \rangle \\ &= \langle \neg p \sqcap e, e \rangle \qquad \text{(cf. Ex. 47)} \end{aligned}$$

(iii) Conjunction, disjunction, and implication as arrows $\dot{\Omega} \cdot \dot{\Omega} \to \dot{\Omega}$ have

$$\langle p, e \rangle \cap \langle q, e \rangle = \langle p \sqcap q, e \rangle$$

$$\langle p, e \rangle \cup \langle q, e \rangle = \langle p \sqcup q, e \rangle$$

$$\langle p, e \rangle \Rightarrow \langle q, e \rangle = \langle p \Rightarrow_e q, e \rangle$$

□

Exponentials

$\mathbf{B}^{\mathbf{A}}$ is the set $[A \to B]$ of all pairs of the form $\langle f, e \rangle$ such that $e \in \Omega$ and $f : A \to B$ is a set function satisfying

$$f(a \restriction p) = f(a) \restriction p$$

and

$$Ef(a) = Ea \sqcap e.$$

Equality is defined by

$$[\![\langle f, e \rangle \approx \langle g, e' \rangle]\!] = \bigsqcap_{x \in A} (Ex \Rightarrow [\![f(x) \approx g(x)]\!]) \sqcap e \sqcap e'$$

giving

$$E\langle f, e\rangle = e$$

and

$$\langle f, e\rangle \restriction p = \langle f \restriction p, e \sqcap p\rangle$$

(where $f \restriction p$ is the function $x \mapsto f(x) \sqcap p$ as usual).

The evaluation arrow $ev: A \cdot [A \to B] \to B$ is given by

$$ev(\langle x, \langle f, e\rangle\rangle) = f(x)$$

EXERCISE 53. Given $g: \mathbf{C} \times \mathbf{A} \to \mathbf{B}$, show that the exponential adjoint to g assigns to $c \in C$ the pair $\langle g_c, Ec\rangle$, where $g_c : A \to B$ has

$$g_c(a) = g(\langle c \restriction Ea, a \restriction Ec\rangle).$$

EXERCISE 54. Show that a global element $\langle f, \top\rangle$ of $\mathbf{B}^{\mathbf{A}}$ is essentially a function f that preserves $\restriction$ and E. Hence establish that the global elements of $\mathbf{B}^{\mathbf{A}}$ are essentially the strong **CΩ-Set** arrows $\mathbf{A} \to \mathbf{B}$. $\square$

Power objects

A simpler description than $\dot{\boldsymbol{\Omega}}^{\mathbf{A}}$ is available. $\mathcal{P}(\mathbf{A})$ is the set of pairs $\langle f, e\rangle$, where $f: A \to \Omega$ has

$$f(a) \sqsubseteq e$$

and

$$(\dagger) \qquad f(a \restriction p) = f(a) \sqcap p$$

all $a \in A$, $p \in \Omega$.

Equality, E, and $\restriction$ are as in the exponential case.

EXERCISE 55. Show that assigning to $\langle f, e\rangle$ the pair $\langle g, e\rangle$, where $g: A \to \dot{\Omega}$ has

$$g(a) = \langle f(a), Ea \sqcap e\rangle$$

establishes the isomorphism of $\mathcal{P}(\mathbf{A})$ and $\dot{\boldsymbol{\Omega}}^{\mathbf{A}}$.

EXERCISE 56. A global element of $\mathcal{P}(\mathbf{A})$ is essentially a function $f: A \to \Omega$ that satisfies ($\dagger$) above. Show that such a function is extensional and strict, i.e. satisfies

$$[\![x \approx y]\!] \sqcap f(x) \sqsubseteq f(y)$$

and

$$f(x) \sqsubseteq Ex.$$

Conversely show that any extensional strict function satisfies (†).

In other words, prove that the global elements of the power object $\mathcal{P}(\mathbf{A})$ in $\mathbf{C}\boldsymbol{\Omega}$**-Set** are essentially the *subsets of* $\mathbf{A}$, *i.e.* the elements of the "weak" power object for $\mathbf{A}$ in $\boldsymbol{\Omega}$**-Set** described in §11.9.

EXERCISE 57. Prove that $\mathcal{P}(\mathbf{A})$ is "flabby", which means that each of its elements can be extended to (i.e. is a restriction of) some global element.

EXERCISE 58. Prove that the "singleton arrow"

$$\{\cdot\}_{\mathbf{A}} : \mathbf{A} \to \mathcal{P}(\mathbf{A})$$

(cf. §11.8) assigns $\langle\{\mathbf{a}\}, Ea\rangle$ to $a \in A$. □

Object of partial elements

$\tilde{\mathbf{A}}$ has

$$\tilde{A} = \{\langle a, e\rangle : a \in A, e \in \Omega, \text{ and } Ea \sqsubseteq e\}$$

with

$$[\![\langle a, e\rangle \approx \langle a', e'\rangle]\!] = [\![a \approx a']\!] \sqcap e \sqcap e'.$$

As usual $E\langle a, e\rangle = e$, and

$$\langle a, e\rangle \restriction p = \langle a \restriction p, e \sqcap p\rangle.$$

The imbedding $\eta_{\mathbf{A}} : \mathbf{A} \rightarrowtail \tilde{\mathbf{A}}$ has

$$\eta_{\mathbf{A}}(a) = \langle a, Ea\rangle.$$

Notice that $\tilde{\mathbf{1}} = \dot{\boldsymbol{\Omega}}$ explicitly.

EXERCISE 59. If g is a partial arrow

$$\begin{tikzcd} \mathbf{C} \arrow[r, hook] \arrow[d, "g"'] & \mathbf{A} \arrow[d, "\tilde{g}"] \\ \mathbf{B} \arrow[r, tail, "\eta_{\mathbf{B}}"] & \tilde{\mathbf{B}} \end{tikzcd}$$

from $\mathbf{A}$ to $\mathbf{B}$ with dom $g \subseteq A$ as shown, show that its character $\tilde{g} : \mathbf{A} \to \tilde{\mathbf{B}}$ has

$$\tilde{g}(a) = \langle g_a, Ea \rangle$$

where

$$g_a = \bigvee \{g(x) : x \leqslant a\}$$ □

Formal logic in $\mathbf{C}\Omega$-Set

We shall use the same formal semantics for quantificational languages in $\mathbf{C}\Omega$**-Set** as that developed for Ω**-Set** in §11.9. A model $\mathfrak{A}$ for our sample language $\mathscr{L} = \{\mathbf{R}\}$ should assign to $\mathbf{R}$ a strong arrow $r : \mathbf{A} \times \mathbf{A} \to \dot{\boldsymbol{\Omega}}$ in $\mathbf{C}\Omega$**-Set**. By Exercises 55 and 54 such an r corresponds to a unique global element of $\mathscr{P}(\mathbf{A} \times \mathbf{A})$, and hence by Exercise 56 we can identify it with a subset of $\mathbf{A} \times \mathbf{A}$ (extensional strict function $A \times A \to \Omega$), allowing the theory of §11.9 to proceed unchanged.

There is one notable advantage in working with *complete* Ω-sets as far as formal logic is concerned, and that is that they allow a natural interpretation of definite-description terms of the form $\mathbf{I}v\varphi(v)$ (as described in §11.10). If $\mathfrak{A} = \langle \mathbf{A}, r \rangle$ is an $\mathscr{L}$-model in $\mathbf{C}\Omega$**-Set**, and $\varphi(v)$ is a formula with one free variable, define a function $f_\varphi : A \to \Omega$ by

$$f_\varphi(c) = [\![\mathbf{E}(\mathbf{c}) \wedge \forall v(\varphi(v) \equiv (v \approx \mathbf{c}))]\!]$$

EXERCISE 60. Show that f_φ is a singleton of $\mathbf{A}$, either by a direct calculation, or by expressing this fact in terms of the $\mathfrak{A}$-truth of formulae which you can derive from the $\mathfrak{A}$-true ones of Exercise 20 of §11.9, using $\mathfrak{A}$-truth-preserving rules of inference. □

Since $\mathbf{A}$ is complete, there is a unique $a_\varphi \in A$ that has $\{\mathbf{a}_\varphi\} = f_\varphi$. We take this element as the interpretation of the term $\mathbf{I}v\varphi$.

EXERCISE 61. Verify the $\mathfrak{A}$-truth of

$$v_i \approx \mathbf{I}v_j(v_i \approx v_j)$$

$$\mathbf{I}v_i\varphi \approx \mathbf{I}v_i(\mathbf{E}(v_i) \wedge \varphi)$$

$$\mathbf{E}(\mathbf{I}v_i\varphi) \equiv \exists v_j \forall v_i((v_i \approx v_j) \equiv \varphi)$$ □

Now, if $\mathfrak{A} = \langle \mathbf{A}, r \rangle$ is an $\mathscr{L}$-model in the weaker category Ω**-Set**, we define the associated complete model to be $\mathfrak{A}^* = \langle \mathbf{A}^*, r^* \rangle$, where r^* is the subset of $\mathbf{A}^* \times \mathbf{A}^*$ corresponding to r as in Exercise 41. Now Exercise 18 of this Section states that in $\mathbf{A}^*$

$$[\![Es]\!] = \bigsqcup_{a \in A} [\![s \approx \{\mathbf{a}\}]\!]$$

which means that the set $\{\{\mathbf{a}\}: a \in A\}$ generates $\mathbf{A}^*$ in the sense of Exercise 22 of §11.9. The latter may then be used to carry through the next result.

EXERCISE 62. For any sentence φ whose closed terms denote only elements of A,

$$[\![\varphi]\!]_{\mathfrak{A}} = [\![\varphi]\!]_{\mathfrak{A}^*}.$$

EXERCISE 63. Suppose $\mathbf{A}$ is complete. Prove that for any formula $\varphi(v_1, \ldots, v_n)$,

$$[\![\varphi(\mathbf{c}_1, \ldots, \mathbf{c}_n)]\!]_{\mathfrak{A}} \sqcap p = [\![\varphi(\mathbf{c}_1 \restriction \mathbf{p}, \ldots \mathbf{c}_n \restriction \mathbf{p})]\!]_{\mathfrak{A}} \sqcap p$$

for all $c_1, \ldots, c_n \in A$ and $p \in \Omega$. □

Comprehension

Given a model $\mathfrak{A}$ based on an Ω-set $\mathbf{A}$, a formula $\varphi(v)$ with one free variable determines a subobject $\mathbf{A}_\varphi \hookrightarrow \mathbf{A}$ of $\mathbf{A}$, namely the Ω-set of A-elements having the "property" φ. In the light of the realisation (§4.8) of the Ω-axiom as a form of the Comprehension principle, $\mathbf{A}_\varphi$ should be constructible by pulling *true* back along an arrow

$$[\![\varphi]\!]: \mathbf{A} \to \dot{\boldsymbol{\Omega}}$$

that semantically interprets φ.

In $\mathbf{C\Omega}$-$\mathbf{Set}$ the appropriate definition is to let $[\![\varphi]\!]$ be the function which assigns to each $c \in A$ the pair $\langle [\![\varphi(\mathbf{c})]\!]_{\mathfrak{A}} \sqcap Ec, Ec\rangle$.

EXERCISE 64. Prove that $[\![\varphi]\!]$ is a strong arrow.

EXERCISE 65. (Exhausting). Describe the arrows $true_{\mathbf{A}}$, $\forall_{\mathbf{A}}$, $\exists_{\mathbf{A}}$ in $\mathbf{C\Omega}$-$\mathbf{Set}$, and then verify that $[\![\varphi]\!]$ as just defined is precisely the same interpretation $\mathbf{A} \to \dot{\boldsymbol{\Omega}}$ of φ as that that is produced by the $\mathscr{E}$-semantics of §11.4. □

Since $\mathbf{A}_\varphi$ is to be the pullback of *true* along $[\![\varphi]\!]$, we are lead by Exercise 51 to conclude that

$$c \in A_\varphi \quad \text{iff} \quad [\![\varphi(\mathbf{c})]\!]_{\mathfrak{A}} \sqcap Ec = Ec$$

and so

$$A_\varphi = \{c \in A : Ec \sqsubseteq [\![\varphi(\mathbf{c})]\!]\}$$

with Ω-equality in $\mathbf{A}_\varphi$ being as for $\mathbf{A}$.

In the notation introduced for subobject classifiers, we have

$$[\![d \in A_\varphi]\!] = [\![\mathbf{E}(\mathbf{d}) \wedge \varphi(\mathbf{d})]\!]_{\mathfrak{A}}$$

EXERCISE 66. $[\![d \in A_\varphi]\!] = [\![\exists v(\varphi(v) \wedge v \approx \mathbf{d})]\!]_{\mathfrak{A}}$ □

Simple sheaves

Any set X can be made into an Ω-set by providing it with the *rigid* equality

$$[\![x \approx y]\!] = \begin{cases} \top & \text{if} \quad x = y \\ \bot & \text{if} \quad x \neq y \end{cases},$$

yielding the *rigid Ω-set* $\bar{\mathbf{X}}$. The completion (set of singletons) of $\bar{\mathbf{X}}$ will simply be denoted $\mathbf{X}^*$. A **CΩ-Set**-object obtained in this way is called a *simple sheaf.*

In the case $\Omega = \Theta$, $\mathbf{X}^*$ has a natural representation as the sheaf $\mathbf{C}_X$ of continuous X-valued partial functions on I (Example 1), where here we take the *discrete* topology on X for which singleton subsets $\{x\} \subseteq X$ are open.

Given continuous $f: V \to X$, define

$$s_f(x) = f^{-1}(\{x\}) \in \Theta$$

for all $x \in X$. The $s_f(x)$'s are disjoint for distinct x's, and since $Ex = \top = I$ in $\bar{\mathbf{X}}$, it follows that s_f is a singleton of $\bar{\mathbf{X}}$.

Conversely, if $s: X \to \Theta$ is an element of $\mathbf{X}^*$, for $x \neq y$ we have $s(x) \cap s(y) = \emptyset$, so with $V = \bigcup\{s(x) : x \in X\}$ we may define a function $f_s : V \to X$ which corresponds uniquely to s by the construction just given. The rule is that for input $i \in V$,

$$f_s(i) = \text{the unique } x \in X \text{ such that } i \in s(x).$$

Then $f_s^{-1}(\{x\}) = s(x)$ is open, making f_s continuous for the discrete topology.

EXERCISE 67. Verify that the operations $s \mapsto f_s$ and $f \mapsto s_f$ are mutually inverse. □

We noted in §14.1, in discussing natural-numbers objects for $\mathbf{Sh}(I)$, that continuous functions for a discrete codomain are precisely those that are *locally constant* (constant throughout some neighbourhood of each point of their domain). Thus in the topological case, the simple Θ-sheaf $\mathbf{X}^*$ may be thought of as the sheaf of locally constant X-valued functions on I. Its global ($Ea = \top$) elements are of course just those functions that are globally defined (have domain I). We may identify members of the original set X with such functions by associating with $a \in X$ the function $f_a : I \to X$ that has $f_a(i) = a$ for *all* $i \in I$ (f_a corresponds to the singleton $\{\mathbf{a}\}$ in $\mathbf{X}^*$ by the above construction). Thus X is identified with the set of globally defined totally constant functions. There may however be other global elements of $\mathbf{X}^*$. If I is made up of a number of disjoint open pieces then there may well be globally defined but only locally constant functions which assign different constant values to each of these disjoint pieces.

In general, a rigid Ω-set $\bar{\mathbf{X}}$ is *reduced*, which means that it has

$$\{\mathbf{a}\} = \{\mathbf{b}\} \quad \text{only if} \quad a = b$$

for all $a, b \in X$. This implies that the assignment of $\{\mathbf{a}\}$ to a is an injection of X into X^*, and since (Exercise 20)

$$[\![\{\mathbf{a}\} \approx \{\mathbf{b}\}]\!]_{\mathbf{X}^*} = [\![a \approx b]\!]_{\mathbf{X}},$$

we may simply identify a and $\{\mathbf{a}\}$ and regard X as a subset of X^*, i.e. $X \subseteq X^*$. Then by Exercise 18 we find that

$$Es = \bigsqcup_{a \in X} [\![s \approx a]\!]_{\mathbf{X}^*}$$

for all $s \in X^*$, which means that X generates $\mathbf{X}^*$. This greatly simplifies the computation of formal truth-values for a model $\mathfrak{A}$ based on the simple sheaf $\mathbf{X}^*$, since by Exercise 22 of §11.9 we have

$$[\![\forall v \varphi]\!]_{\mathfrak{A}} = \bigsqcap_{c \in X} [\![\mathbf{E}(\mathbf{c}) \supset \varphi(\mathbf{c})]\!]_{\mathfrak{A}}$$

and

$$[\![\exists v \varphi]\!]_{\mathfrak{A}} = \bigsqcup_{c \in X} [\![\mathbf{E}(\mathbf{c}) \wedge \varphi(\mathbf{c})]\!]_{\mathfrak{A}},$$

so that we can confine the range of quantification to the elements of the original (rigid) set X. But the latter elements are all global in $\mathbf{X}^*$, so these equations reduce (via $\top \Rightarrow \mathrm{p} = \mathrm{p}$) to

$$(\dagger) \quad \begin{aligned} [\![\forall v \varphi]\!]_{\mathfrak{A}} &= \bigsqcap_{c \in X} [\![\varphi(\mathbf{c})]\!]_{\mathfrak{A}} \\ [\![\exists v \varphi]\!]_{\mathfrak{A}} &= \bigsqcup_{c \in X} [\![\varphi(\mathbf{c})]\!]_{\mathfrak{A}} \end{aligned}$$

We shall use these facts later when we come to construct number systems in categories of Ω-sheaves.

Topoi as sheaf-categories

When may an elementary topos $\mathscr{E}$ be construed as the category of sheaves over some **CHA**? To answer that question we shall examine below some of the properties enjoyed by topoi of the form **CΩ-Set**.

The Heyting algebra most naturally associated with $\mathscr{E}$ is the algebra $\Omega_{\mathscr{E}} = \mathrm{Sub}(1)$ of subobjects $u \rightarrowtail 1$ of $\mathscr{E}$'s terminal object ($\Omega_{\mathscr{E}}$ can alternatively be thought of as the **HA** $\mathscr{E}(1, \Omega)$ of global truth-values of $\mathscr{E}$). But in order to develop sheaf theory over Ω we need the latter to be complete as a lattice. For this it suffices that $\mathscr{E}$ have *arbitrary coproducts of subobjects of* 1, i.e. that for any set $\{u_x : x \in X\}$ of $\mathscr{E}$-objects whose unique arrow $u_x \to 1$ is monic there is an associated co-product object, which we denote $\varinjlim_{x\in X} u_x$. The lattice join $\bigsqcup_{x\in X} u_x$ may then be obtained as the epi-monic factorisation

$$\begin{tikzcd}
\varinjlim_{x\in X} u_x \arrow[rr] \arrow[dr, two heads] & & 1 \\
& \bigsqcup_{x\in X} u_x \arrow[ur, tail] &
\end{tikzcd}$$

of the coproduct of the u_x's (cf. the construction of unions in §7.1).

The existence of coproducts of arbitrary sub-collections of $\Omega_{\mathscr{E}}$ is also necessary for $\mathscr{E}$ to be a sheaf-category since **CΩ-Set**, for any **CHA** Ω, has coproducts of *all* sets of objects. Given a set $\{\mathbf{A}_x : x \in X\}$ of Ω-sheaves, the coproduct $\varinjlim_{x\in X} \mathbf{A}_x$ is defined, by generalisation of the above definition of $\mathbf{A}+\mathbf{B}$, to be the Ω-set of all *disjoint selections* of the $\mathbf{A}_x$'s. A member of this coproduct is a selection $a = \{a_x : x \in X\}$ of an element $a_x \in A_x$ for each $x \in X$ such that

$$Ea_x \sqcap Ea_y = \bot \quad \text{whenever} \quad x \neq y.$$

Equality of selections is given by

$$[\![a \approx b]\!] = \bigsqcup_{x\in X} [\![a_x \approx b_x]\!]$$

and the injection $\mathbf{A}_y \to \varinjlim_{x\in X} \mathbf{A}x$ assigns to $a_y \in A_y$ the selection a that has

$$a_x = \begin{cases} a_y & \text{if} \quad x = y \\ \emptyset_{\mathbf{A}_x} & \text{otherwise} \end{cases}$$

EXERCISE 68. Given a collection $\{\mathbf{A}_x \xrightarrow{f_x} \mathbf{C} : x \in X\}$ of arrows in $\mathbf{C\Omega\text{-}Set}$, describe their coproduct arrow $\varinjlim_{x \in X} \mathbf{A}_x \to \mathbf{C}$. □

The other property enjoyed by $\mathbf{C\Omega\text{-}Set}$ that will be used to answer our question is *weak extensionality*. In general an $\mathscr{E}$-object a will be called *weakly extensional* if for any two distinct parallel arrows $f,g : a \to b$ with domain a there is a *partial* element $x : 1 \rightsquigarrow a$ that distinguishes them, i.e. has $f \circ x \neq g \circ x$. Thus the whole topos is weakly extensional, as defined in §12.1, just in case each $\mathscr{E}$-object is weakly extensional as just defined.

To see how this property obtains in $\mathbf{C\Omega\text{-}Set}$, suppose $f,g : \mathbf{A} \to \mathbf{B}$ are distinct strong arrows. Thus we have $f(a) \neq g(a)$ for some $a \in A$. But then assigning $a \restriction q$ to each $q \sqsubseteq Ea$ gives a strong arrow $x : \mathbf{1} \restriction Ea \to \mathbf{A}$ that distinguishes f and g (since $x(Ea) = a$). Here $\mathbf{1} \restriction Ea \hookrightarrow \mathbf{1}$ is the subobject of $\mathbf{1}$ based on the set $\Omega \restriction Ea = \{q \in \Omega : q \sqsubseteq Ea\}$ (*Ex.* 47), so that we have $x : \mathbf{1} \rightsquigarrow \mathbf{A}$.

EXERCISE 69. In any $\mathscr{E}$, given $u \rightarrowtail 1$ and $f : u \to \tilde{a}$ take the pullback g

$$\begin{tikzcd} b \arrow[r, tail] \arrow[d, "g"'] & u \arrow[d, "f"] \\ a \arrow[r, tail, "\eta_a"] & \tilde{a} \end{tikzcd}$$

of f along η_a and let $\tilde{g}$ be the unique arrow making the boundary of

$$\begin{tikzcd} b \arrow[r, tail] \arrow[d, "g"'] & u \arrow[r, tail] \arrow[dr, "f"'] & 1 \arrow[d, "\tilde{g}"] \\ a \arrow[rr, tail, "\eta_a"] & & \tilde{a} \end{tikzcd}$$

a pullback. Prove that the right-hand triangle of this last diagram commutes.

EXERCISE 70. Use the last exercise to show that if $\tilde{a}$ is weakly extensional then two distinct arrows with domain $\tilde{a}$ are distinguishable by a *global* element of $\tilde{a}$. □

We shall call an $\mathscr{E}$-object a *extensional* if any parallel pair $a \rightrightarrows b$ of distinct arrows with domain a are distinguished by a global element $1 \to a$. (Thus $\mathscr{E}$ is well-pointed precisely when all of its objects are extensional). The last exercise implies that an object $\tilde{a}$ of partial elements is always extensional whenever it is weakly so. Thus in a sheaf-category each object $\mathbf{A}$ is *sub-extensional*, i.e. is a subobject of an extensional object, (since we have a monic arrow $\mathbf{A} \rightarrowtail \tilde{\mathbf{A}}$).

The two conditions that $\mathscr{E}$ be weakly extensional and have coproducts of subobjects of 1 suffice to make $\mathscr{E}$ a sheaf-category, and indeed to make it equivalent to $\mathbf{C}\Omega_{\mathscr{E}}$-$\mathbf{Set}$. The original proof of this result used a great deal of heavy machinery in the form of "geometric morphisms" $\mathscr{E} \to \mathbf{Set}$ and such-like "abstract nonsense". However recent work by Michael Brockway has provided a proof that is much more accessible and has the conceptual advantage of making it possible to see just how an object becomes a sheaf and vice-versa.

The $\Omega_{\mathscr{E}}$-sheaf $\mathbf{A}_a$ corresponding to an $\mathscr{E}$-object a has A_a as the set of partial $\mathscr{E}$-elements of a, with the degree of equality of $x, y : 1 \rightsquigarrow a$ being obtained as the equaliser of the diagram

$$\begin{tikzcd} & \operatorname{dom} x \arrow[dr, "x"] & \\ \operatorname{dom} x \cap \operatorname{dom} y \arrow[ur, tail] \arrow[dr, tail] & & a \\ & \operatorname{dom} y \arrow[ur, "y"'] & \end{tikzcd}$$

Here the intersection of domains is given as usual as their pullback

$$\begin{tikzcd} \operatorname{dom} x \cap \operatorname{dom} y \arrow[r, tail] \arrow[d, tail] & \operatorname{dom} y \arrow[d] \\ \operatorname{dom} x \arrow[r] & 1 \end{tikzcd}$$

but since this is done over the terminal 1, the result is the product of $\operatorname{dom} x$ and $\operatorname{dom} y$. Now equalising a product is one way to obtain a pullback, so $[\![x \approx y]\!]$ is alternatively characterised as the pullback

$$\begin{tikzcd} {[\![x \approx y]\!]} \arrow[r, tail] \arrow[d, tail] & \operatorname{dom} y \arrow[d, "y"] \\ \operatorname{dom} x \arrow[r, "x"] & a \end{tikzcd}$$

of x and y.

In the case $\mathscr{E} = \mathbf{Top}(I)$, this construction produces the now familiar sheaf of local sections of a topological bundle (Example 2 of this section).

Even in the case $\mathscr{E} = \mathbf{Set}$ it has some interest in assigning to each set X its set $X \cup \{*\}$ of partial elements. Here of course Ω is the 2-element Boolean algebra consisting of $\top = 1$ and $\perp = \emptyset$.

When considered as a 2-sheaf, $X \cup \{*\}$ is actually the simple sheaf $\mathbf{X}^*$ obtained by completing the rigid 2-set $\bar{\mathbf{X}}$. It is not hard to see that the only singletons $s : \bar{\mathbf{X}} \to 2$ are those corresponding to elements of X, together with the unique singleton with empty extent ($Es = \perp$). The latter serves as the null entity $*$. In fact every object of **C2-Set** arises as a simple sheaf, for if $\mathbf{Y}$ is a 2-sheaf and $X = Y - \{\emptyset_{\mathbf{Y}}\}$ then the equality relation of $\mathbf{Y}$ is rigid on X (i.e. makes all elements of X global and distinct elements have $[\![x \approx y]\!] = \perp$.) Thus $\mathbf{X}^* = \mathbf{Y}$.

EXERCISE 71. Prove this last statement. □

In order to categorially recover the **Set**-object X from the **C2-Set** object $\mathbf{X}^* = X \cup \{*\}$ we form the coproduct (disjoint union) of the extents Ex for all $x \in X^*$. If $x \in X$, then $Ex = \top = \{0\}$, so we identify Ex with $\{x\}$. If $x = *$, then $Ex = \perp = \emptyset$, so that the coproduct becomes the union

$$\cup \{\{x\} : x \in X\} \cup \emptyset = X.$$

Thus we reconstruct X by representing each element of $\mathbf{X}^*$ by a disjoint copy of its own extent (which is a subobject of 1) and then putting these extents together. The reason why this procedure does faithfully reproduce X is that in $\mathbf{X}^*$ all elements are rigidly separated (disjoint). The same construction will not however work if elements of the sheaf overlap and so are to some extent equal. Consider for example the case $\mathscr{E} = \mathbf{Top}(I)$, where $\Omega_{\mathscr{E}} = \Theta$, the set of open subsets of I. Identifying local sections with their images in the stalk space we have the following sort of picture of the sheaf of partial elements of a bundle $A \to I$.

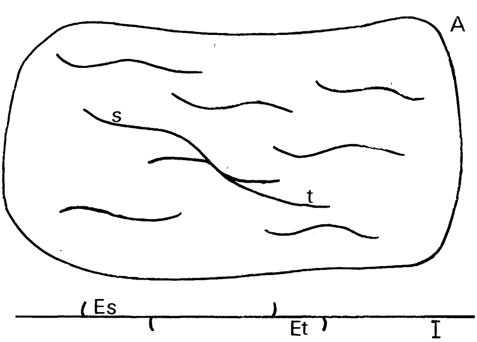


Fig. 14.3.

If we now identify each s with its extent Es and take the coproduct of these Es's we will construct a stalk space larger than A. The two displayed elements s and t will have disjoint copies in the new space, with

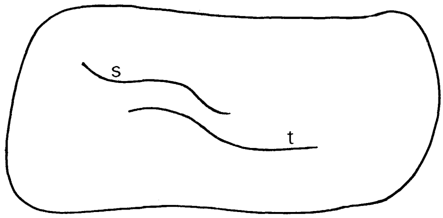


Fig. 14.4.

part of s being duplicated in the copy of t, and vice versa. To recover A we must "reduce" the coproduct by glueing together copies of s and t to the extent that they originally coincided.

Notice that the extents of s and t are arranged thus

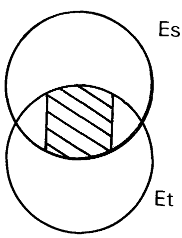
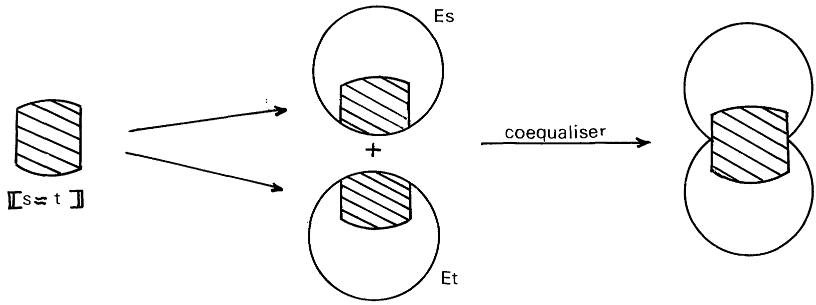


Fig. 14.5.

where the shaded area is the part $[\![s \approx t]\!]$ of $Es \cap Et$ on which s and t agree. This does not reflect the relationship between s and t faithfully either, since Es and Et overlap in places where s and t are distinct. The way to reduce the coproduct, and to build from Es and Et, an object that accurately represents the structure of s and t is to co-equalise the diagram

$$\begin{tikzcd}
& Es \arrow[dr] & \\
{[\![s \approx t]\!]} \arrow[ur, tail] \arrow[dr, tail] & & Es + Et \\
& Et \arrow[ur] &
\end{tikzcd}$$

giving



In the case of a general $\Omega_{\mathscr{E}}$-sheaf **A** we take the above diagram for each pair s, $t \in A$ and put them all together by the coproduct construction, yielding a pair of arrows

$$\varinjlim_{s,t \in A} [\![s \approx t]\!] \rightrightarrows \varinjlim_{s \in A} Es$$

Co-equalising this diagram gives an $\mathscr{E}$-object that has the original **A** as its sheaf of partial elements.

Brockway has developed these ideas to provide functors between $\mathscr{E}$ and **C**$\Omega_{\mathscr{E}}$**-Set** that establish that the latter is equivalent to the full subcategory of $\mathscr{E}$ consisting of the *sub-extensional* $\mathscr{E}$-objects (as defined above). This requires only that $\mathscr{E}$ have all coproducts of subobjects of 1, so that these constructions can be carried out at all. But if $\mathscr{E}$ is also weakly extensional then, by Exercise 70, each $\tilde{a}$ is extensional, so all objects a are sub-extensional, making **C**$\Omega_{\mathscr{E}}$**-Set** equivalent to $\mathscr{E}$ itself.

It can be shown that in order to have coproducts of all subsets of $\Omega_{\mathscr{E}}$ in a topos it suffices to have *arbitrary copowers of* 1, i.e. a coproduct for any set of terminal objects. Thus to put this characterisation in its strongest form (weakest hypothesis), in order to know that $\mathscr{E}$ is the category of sheaves over its subobjects of 1 (global truth-values) it suffices to know that $\mathscr{E}$ has arbitrary copowers of 1 and that each object of partial elements $\tilde{a}$ is weakly extensional.

14.8. Number systems as sheaves

In **Set**, the classical number systems have representations that are built up from the set ω of natural numbers to obtain the integers $\mathbb{Z}$, the rationals $\mathbb{Q}$, the reals $\mathbb{R}$, and finally the complex numbers $\mathbb{C}$. These constructions can be "internalised" to any topos $\mathscr{E}$ that has a natural-numbers object, and so any such category has within it analogues of all these number systems. The full development of this work is beyond our scope, but we will examine some aspects of it in relation to Ω-sets, where the constructions are accessible and the results rather striking.

In $\mathscr{E}$, the object N^+ of positive integers is obtained as the (image of the) subobject $\mathscr{s}: N \rightarrowtail N$ of N (in **Set**, the image of the monic successor function is $\omega^+ = \{1, 2, 3, \ldots\}$). The object of integers is the coproduct $Z = N + N^+$ ($\mathbb{Z}$ is the disjoint union of ω^+ and the isomorphic copy $\{\ldots -3, -2, -1, 0\}$ of ω). Classically the rationals arise as a quotient of $\mathbb{Z} \times \omega^+$, where, thinking of $\langle m, n\rangle$ as the rational m/n, we identify $\langle m, n\rangle$ and $\langle m', n'\rangle$ when $m \cdot n' = m' \cdot n$. Developing this within $\mathscr{E}$ produces the rational-numbers object Q.

In Ω-**Set**, these objects turn out to be the rigid structures $\bar{\omega}$, $\bar{\omega}^+$, $\bar{\mathbb{Z}}$ and $\bar{\mathbb{Q}}$, while in **CΩ-Set** they are the corresponding simple sheaves ω^*, ω^{+*}, $\mathbb{Z}^*$ and $\mathbb{Q}^*$. In particular for **CΘ-Set** we may take them to be the appropriate sheaves of locally constant functions on I.

EXERCISE 1. Define the (rigid) weak successor arrow $\mathscr{s}: \bar{\omega} \to \bar{\omega}$ by

$$\mathscr{s}(\langle m, n\rangle) = \begin{cases} \top & \text{if } \quad n = m+1 \\ \bot & \text{otherwise.} \end{cases}$$

Define the weak "zero arrow" $O: \mathbf{1} \to \bar{\omega}$ in Ω-**Set** analogously. Verify that Ω-**Set**$\models$NNO.

EXERCISE 2. Define $O: \mathbf{1} \to \omega^*$ and $\mathscr{s}: \omega^* \to \omega^*$ in **CΩ-Set** and verify NNO for that category. □

When we come to the reals, the situation is not so clear cut. Classically the two most familiar methods of defining real numbers are as equivalence classes of Cauchy-sequences of rationals, and on the other hand as Dedekind cuts of $\mathbb{Q}$. When carried out in $\mathscr{E}$, these approaches produce an object R_c of "Cauchy-reals" and an object R_d of "Dedekind-reals" *which in general are not isomorphic*! What we do have in general is that $R_c \rightarrowtail R_d$.

Now in Ω-**Set** the construction of Cauchy sequences $N \to Q$ of rationals uses basically the same entities as in **Set** and leads to the same conclusion: R_c is the rigid set $\bar{\mathbb{R}}$. The definition of R_d however, which also proceeds by analogy with the classical case, uses subsets of $\bar{\mathbb{Q}}$, i.e. functions $\mathbb{Q} \to \Omega$, and there may be many more of these than members of $\mathbb{R}$.

In **Set**, a real number $r \in \mathbb{R}$ is uniquely determined by the sets

$$U_r = \{c \in \mathbb{Q}: r < c\}$$
$$L_r = \{c \in \mathbb{Q}: r > c\},$$

called the *upper* and *lower cut* of *r*. In general an ordered pair $\langle U, L \rangle \in \mathcal{P}(\mathbb{Q}) \times \mathcal{P}(\mathbb{Q})$ of subsets of $\mathbb{Q}$ is called a *Dedekind real number* if it satisfies the sentences

$(\delta 1)$	$\exists v \exists w (v \boldsymbol{\varepsilon} \mathbf{U} \wedge w \boldsymbol{\varepsilon} \mathbf{L})$	"non-empty"
$(\delta 2)$	$\forall v \sim (v \boldsymbol{\varepsilon} \mathbf{U} \wedge v \boldsymbol{\varepsilon} \mathbf{L})$	"disjoint"
$(\delta 3)$	$\forall v (v \boldsymbol{\varepsilon} \mathbf{L} \equiv \exists w (w \boldsymbol{\varepsilon} \mathbf{L} \wedge w > v))$	"open lower cut"
$(\delta 4)$	$\forall v (v \boldsymbol{\varepsilon} \mathbf{U} \equiv \exists w (w \boldsymbol{\varepsilon} \mathbf{U} \wedge w < v))$	"open upper cut"
$(\delta 5)$	$\forall v \forall w (v > w \supset v \boldsymbol{\varepsilon} \mathbf{U} \vee w \boldsymbol{\varepsilon} \mathbf{L}))$	"close together"

where the symbols **U** and **L** denote the subsets U and L, $\boldsymbol{\varepsilon}$ denotes the standard membership relation, and the variables v and w range over the members of $\mathbb{Q}$. For such a pair $\langle U, L \rangle$ there is one and only one real number $r \in \mathbb{R}$ with $U = U_r$ and $L = L_r$.

Now the conjunction of the sentences $(\delta 1)$–$(\delta 5)$ may be thought of as a sentence $\delta(\mathbf{r})$, where $\mathbf{r} = \langle \mathbf{U}, \mathbf{L} \rangle$ is an "ordered-pairs symbol" denoting members $r = \langle U, L \rangle$ of $(\mathcal{P}(\mathbb{Q}))^2$. Thus in an axiomatic development of classical set theory the Dedekind real-number system is defined by the Comprehension principle as the set

$$\mathbb{R}_d = \{r: \delta(\mathbf{r}) \text{ is true}\} \subseteq \mathcal{P}(\mathbb{Q})^2.$$

By analogy then, in ***Ω*-Set** we obtain R_d as the subobject of $\mathcal{P}(\bar{\mathbb{Q}}) \times \mathcal{P}(\bar{\mathbb{Q}})$ defined by $\delta(\mathbf{r})$. According to our earlier discussion of Comprehension, this is the set

$$R_d = \{r: Er \sqsubseteq [\![\delta(\mathbf{r})]\!]\} \subseteq \mathcal{P}(\bar{\mathbb{Q}}) \times \mathcal{P}(\bar{\mathbb{Q}}).$$

Now in ***Ω*-Set**, power objects, and hence their products, have only global elements (§11.9, Ex. 8), so this simplifies to

$$R_d = \{r: [\![\delta(\mathbf{r})]\!] = \top\}.$$

In order to compute the truth-value $[\![\delta(\mathbf{r})]\!]$ for a given *r*, we observe that the quanitfied variables v, w in δ range over the rigid set $\bar{\mathbb{Q}}$, i.e. over standard rationals, so that we need to know the "atomic" truth-values $[\![\mathbf{c} < \mathbf{d}]\!]$, $[\![\mathbf{c} > \mathbf{d}]\!]$, $[\![\mathbf{c} \boldsymbol{\varepsilon} \mathbf{U}]\!]$, $[\![\mathbf{c} \boldsymbol{\varepsilon} \mathbf{L}]\!]$, for c, $d \in \mathbb{Q}$. The numerical orderings are interpreted as the standard (rigid) ones

$$[\![\mathbf{c} < \mathbf{d}]\!] = \begin{cases} \top & \text{if } \ c < d \\ \bot & \text{otherwise,} \end{cases}$$

and similarly for $>$. Moreover $r = \langle U, L\rangle$ is a pair of subsets of $\bar{\mathbb{Q}}$, i.e. strict extensional functions $\mathbb{Q} \to \Omega$, so we put

$$[\![\mathbf{c}\ \boldsymbol{\varepsilon}\ \mathbf{U}]\!] = U(c)$$
$$[\![\mathbf{c}\ \boldsymbol{\varepsilon}\ \mathbf{L}]\!] = L(c)$$

in accordance with our interpretation of subsets developed in §11.9.

Next we notice that since $\bar{\mathbb{Q}}$ is rigid, *every* function $\mathbb{Q} \to \Omega$ is strict and extensional. Thus, putting all of these pieces together with the semantical rules of §11.9 and general lattice-theoretic properties of Ω, it follows that a Dedekind-real number in Ω-**Set** is a pair $r = \langle U, L\rangle$ of functions $\mathbb{Q} \to \Omega$ such that

(δi) $\quad \bigsqcup\{U(c) \sqcap L(d) \colon c, d \in \mathbb{Q}\} = \top$

(δii) $\quad U(c) \sqcap L(c) = \bot, \quad$ all $\quad c \in \mathbb{Q}$

(δiii) $\quad L(c) = \bigsqcup\{L(d) \colon d > c\}, \quad$ all $\quad c \in \mathbb{Q}$

(δiv) $\quad U(c) = \bigsqcup\{U(d) \colon d < c\}, \quad$ all $\quad c \in \mathbb{Q}$

(δv) $\quad U(c) \sqcup L(d) = \top, \quad$ all $\quad c > d \in \mathbb{Q}$

(remember $Ec = \top$, all $c \in \bar{\mathbb{Q}}$).

Now in the case $\Omega = \Theta$, we can obtain such a pair by starting with a real-valued function $f \colon I \to \mathbb{R}$ on I and defining

$$\begin{aligned} U_f(c) = [\![\mathbf{c}\boldsymbol{\varepsilon}\mathbf{U_f}]\!] &= \{i \colon c \in U_{f(i)}\} \\ &= \{i \colon\ f(i) < c\} = f^{-1}(-\infty, c) \end{aligned}$$

and

$$\begin{aligned} L_f(c) = [\![\mathbf{c}\boldsymbol{\varepsilon}\mathbf{L_f}]\!] &= \{i \colon c \in L_{f(i)}\} \\ &= \{i \colon f(i) > c\} = f^{-1}(c, \infty) \end{aligned}$$

where

$$(-\infty, c) = \{x \in \mathbb{R} \colon c > x\}$$

and

$$(c, \infty) = \{x \in \mathbb{R} \colon c < x\}.$$

Now if f is continuous (which means precisely that the inverse images of open sets are open) with respect to the usual topology on $\mathbb{R}$, then $r_f = \langle U_f, L_f\rangle$ will be a pair of functions from $\mathbb{Q}$ to Θ satisfying (δi)–(δv).

Conversely, given a Dedekind-real $r = \langle U_r, L_r\rangle$ in Θ-**Set**, and an element $i \in I$, we put

$$U_i = \{c \in \mathbb{Q} \colon i \in [\![\mathbf{c}\boldsymbol{\varepsilon}\mathbf{U_r}]\!]\}$$

and

$$L_i = \{c \in \mathbb{Q} : i \in [\![\mathbf{c} \varepsilon \mathbf{L_r}]\!]\}.$$

Then $\langle U_i, L_i \rangle$ proves to be a classical Dedekind cut in $\mathbb{Q}$, determining a unique real number $r_i \in \mathbb{R}$. Putting $f_r(i) = r_i$ defines a function $f_r : I \to \mathbb{R}$.

EXERCISE 3. (Compulsory). Verify that

(i) r_f satisfies (δi)–(δv).

(ii) $\langle U_i, L_i \rangle$ is a classical Dedekind cut.

(iii) The operations $f \mapsto r_f$ and $r \mapsto f_r$ are mutually inverse.

(iv) f_r is continuous (remember that the sets (c, ∞), $(-\infty, c)$ generate the usual topology on $\mathbb{R}$). □

Thus we have established that in Θ-**Set**, R_d can be represented as the set of all *globally defined continuous real-valued functions on I.* A "Dedekind-real" is a continuous function of the form $I \to \mathbb{R}$, which we envisage as a standard real number "varying continuously" (through the stalks of a bundle) over I. In particular, these "global reals" include, for each $a \in \mathbb{R}$, the totally constant function with output a, and in this way we determine that $R_c \rightarrowtail R_d$.

The analysis just given adapts immediately (in fact reverses) to give a representation of continuous *partial* functions on I. If $V \in \Theta$ then, as defined in §14.2, the set

$$\Theta_V = \{W \in \Theta : W \subseteq V\}$$

is the subspace topology on V, making $\langle V, \Theta_V \rangle$ a topological space in its own right. Notice that, in the terminology of Exercise 47, Θ_V is the **CHA** $\Theta \restriction V$ of all elements "below V" in the **CHA** Θ. We shall also introduce the symbol $\Theta_{\mathbb{R}}$ to denote the open subsets of $\mathbb{R}$ for the usual topology.

Now in saying that $f : V \to \mathbb{R}$ is a *continuous* partial function on I, where I has topology Θ_I, we have meant that for each $W \in \Theta_{\mathbb{R}}$ we have

$$f^{-1}(W) = \{i \in V : f(i) \in W\} \in \Theta_I.$$

But in fact this last condition is equivalent to

$$f^{-1}(W) \in \Theta_V,$$

and so the partial continous $\mathbb{R}$-valued functions on (I, Θ_I) that have domain V are precisely the global continuous $\mathbb{R}$-valued functions on (V, Θ_V). But the latter, by the above construction, correspond precisely to the Dedekind-reals in the topos Θ_V-**Set**!

In other words, if we take a partial continuous $f: I \rightsquigarrow \mathbb{R}$ and relativise to $\Theta \restriction Ef$, within which context f is global, we find that f becomes a Dedekind-real in $\Theta \restriction Ef$-**Set**. And all members of R_d in the latter category arise in this way.

Now let us move to the topos **CΩ-Set** of complete Ω-sets. Here the object R_c is the simple sheaf $\mathbb{R}^*$, so that Cauchy-reals are locally-constant $\mathbb{R}$-valued functions. R_d is again defined by our axioms for Dedekind cuts as the subobject

$$\{r: Er \sqsubseteq [\![\delta(\mathbf{r})]\!]\} \subseteq \mathcal{P}(Q) \times \mathcal{P}(Q).$$

This time Q is the simple sheaf $\mathbb{Q}^*$, and we saw, in analysing models on simple sheaves that we have $\mathbb{Q} \subseteq \mathbb{Q}^*$ as a generating set for $\mathbb{Q}^*$. This means that in determining the truth-value of $\delta(\mathbf{r})$ we can confine the quantifiers to range over the (global) elements of $\mathbb{Q}$ (cf. the equations (†) given earlier).

A typical element r of $\mathcal{P}(\mathbb{Q}^*)^2$ is now a pair $\langle U, L\rangle \in \mathcal{P}(\mathbb{Q}^*) \cdot \mathcal{P}(\mathbb{Q}^*)$ of elements of $\mathcal{P}(Q)$ *with the same extent*, i.e. $Er = EU = EL = e$, say. U itself will be a pair $\langle U_r, e\rangle$, where $U_r : \mathbb{Q}^* \to \Omega$ satisfies

(i) $U_r(a) \sqsubseteq e$,
(ii) $U_r(a \restriction p) = U_r(a) \sqcap p$

all $a \in \mathbb{Q}^*$, $p \in \Omega$.

We put

$$[\![\mathbf{c} \varepsilon \mathbf{U}]\!] = U_r(c), \quad \text{all} \quad c \in \mathbb{Q}.$$

Similarly, we have $L = \langle L_r, e\rangle$, and put

$$[\![\mathbf{c} \varepsilon \mathbf{L}]\!] = L_r(c).$$

In fact the condition (ii) is immaterial to our purposes, since we observed in Exercise 14.7.56 that it means precisely that U_r is a strict extensional function on $\mathbb{Q}^*$, and in Exercise 14.7.41 that such functions correspond uniquely to strict extensional functions on $\bar{\mathbb{Q}}$. But we know that the latter are simply *all* Ω-valued functions on $\mathbb{Q}$, and anyway we are only interested in the U_r-values of members of the generating set $\mathbb{Q}$. Thus for the present exercise we may simply regard $\mathcal{P}(\mathbb{Q}^*)$ as the set of all pairs $\langle f, e\rangle$, where $f: \mathbb{Q} \to \Omega$ has $f(c) \sqsubseteq e$, all $c \in \mathbb{Q}$.

Having determined the truth-values of atomic sentences, we can establish that the defining condition

$$Er \sqsubseteq [\![\delta(\mathbf{r})]\!]$$

for R_d is equivalent to the satisfaction of the following (remembering that the values of U_r and L_r are "bounded above" by $e = Er$).

(δi$_e$) $\bigsqcup\{U_r(c)\sqcap L_r(d): c, d \in \mathbb{Q}\} = e$

(δii$_e$) $U_r(c)\sqcap L_r(d) = \perp$

(δiii$_e$) $L_r(c) = \bigsqcup\{L_r(d): d > c\}$, all $c \in \mathbb{Q}$

(δiv$_e$) $U_r(c) = \bigsqcup\{U_r(d): d < c\}$, all $c \in \mathbb{Q}$

(δv$_e$) $U_r(c)\sqcup L_r(d) = e$, all $c > d \in \mathbb{Q}$

EXERCISE 4. Verify (δi$_e$)–(δv$_e$). (The Heyting-algebra involved is a little more complex than for (δi)–(δv).) □

The correspondence between (δi)–(δv) and (δi$_e$)–(δv$_e$) is apparent. The only difference is that in (δi) and (δv) we have the unit $\top$ of Ω where in (δi$_e$) and (δv$_e$) we have e. But by passing to the **CHA**

$$\Omega \restriction e = \{q : q \sqsubseteq e\} \qquad \text{(Ex. 14.7.47)}$$

we relativise to an algebra in which e *is* the unit. So we see that what (δi$_e$)–(δv$_e$) means is precisely that *the pair* $\langle U_r, L_r\rangle$ *is a Dedekind-real in the topos* $\Omega \restriction e$-**Set**.

Conversely, if a pair $\langle U_r, L_r\rangle$ of functions $\mathbb{Q} \to \Omega \restriction e$ is a Dedekind-real in $\Omega \restriction e$-**Set**, then they satisfy (δi$_e$)–(δv$_e$), and of course (i), so that $r = \langle\langle U_r, e\rangle, \langle L_r, e\rangle\rangle$ is a Dedekind-real in **C**Ω-**Set**, with $Er = e$.

In summary then we have established that for any given **CHA** Ω, and given $e \in \Omega$,

> *the set of Dedekind-reals in* **C**Ω-**Set** *that have extent e can be identified with the set of* all *Dedekind reals in* $\Omega \restriction e$-**Set**.

Returning to the topological case $\Omega = \Theta_I$ again, an element $r \in R_d$ in **C**Θ_I-**Set** that has $Er = V$, say, is essentially a Dedekind real in $\Theta \restriction V$-**Set**, i.e. a continuous $\mathbb{R}$-valued function defined on all of (V, Θ_V). Thus

> *the Dedekind-reals in* **C**Θ_I-**Set** *with extent V are precisely the continuous* $\mathbb{R}$*-valued partial functions on I that have domain V.*

Putting these "local-reals" together for all $V \in \Theta$ allows us to conclude that

> *in* **C**Θ_I-**Set**, R_d *is the sheaf* $\mathbf{C}_{\mathbb{R}}$ *of all* $\mathbb{R}$*-valued continuous partial functions on I.*

Amongst the continuous $\mathbb{R}$-valued functions on I are the locally constant ones of course, and that observation confirms that we have $R_c \rightarrowtail R_d$ in **CΘ-Set**.

For an arbitrary **CHA**, there is also a representation of R_d available relating to the classical reals $\mathbb{R}$, for which we need the notion of an $\sqcap$-$\bigsqcup$ *map*. This as a function between two **CHA**'s that preserves the operations $\sqcap$ and $\bigsqcup$, i.e. has

$$f(x \sqcap y) = f(x) \sqcap f(y)$$

and

$$f(\bigsqcup B) = \bigsqcup\{f(b) \colon b \in B\}.$$

Such functions are natural objects of study in this generalised topological context, since a topology on a set I is precisely a subset of the lattice $\langle \mathscr{P}(I), \subseteq \rangle$ that is closed under $\sqcap$ and $\bigsqcup$.

EXERCISE 5. Prove that the restriction operator $g_e : \Omega \to \Omega \restriction e$, where $g_e(p) = p \sqcap e$ is a (surjective) $\sqcap$-$\bigsqcup$ map.

EXERCISE 6. Let $f : I \to \mathbb{R}$ be continuous. Define $g_f : \Theta_{\mathbb{R}} \to \Theta_I$ by

$$g_f(W) = f^{-1}(W), \quad \text{all} \quad W \in \Theta_{\mathbb{R}}$$

Show that g_f is an $\sqcap$-$\bigsqcup$ map, and that for any $V \in \Theta_I$,

$$\begin{tikzcd} \Theta_{\mathbb{R}} \arrow[r, "g_f"] \arrow[dr, "g_{f \restriction V}"'] & \Theta_I \arrow[d, "g_V"] \\ & \Theta_I \restriction V \end{tikzcd}$$

commutes (which may be written $g_{f \restriction V} = g_f \restriction V$). □

In the light of the last exercise, and the earlier representation of R_d in **Θ-Set** we make the following definition:

if $g : \Theta_{\mathbb{R}} \to \Omega$ (Ω an arbitrary **CHA**) is an $\sqcap$-$\bigsqcup$ map ("and-Or" map), put

$$[\![\mathbf{c} \varepsilon \mathbf{U_g}]\!] = g(-\infty, c)$$

and

$$[\![\mathbf{c}\varepsilon\mathbf{L_g}]\!] = g(c, \infty).$$

Then the pair $r_g = \langle U_g, L_g\rangle$ satisfies (δi)–(δv) and is a Dedekind-real in Ω-**Set**.

Conversely, given $r = \langle U_r, L_r\rangle$ satisfying (δi)–(δv) we define an $\sqcap$-$\bigsqcup$ map $g_r : \Theta_{\mathbb{R}} \to \Omega$. Intuitively, g_r assigns to each open subset $W \subseteq \mathbb{R}$ the truth-value of "$r \in W$", and the definition uses the fact that each such W is a union of intervals (c, d) with rational end points. For $c,d \in \mathbb{Q}$ we put

$$[\![\mathbf{r}\varepsilon(\mathbf{c},\mathbf{d})]\!] = [\![\mathbf{c}\varepsilon\mathbf{L_r}]\!] \sqcap [\![\mathbf{d}\varepsilon\mathbf{U_r}]\!]$$

(since, classically,

$$\begin{aligned} r \in (c, d) \quad &\text{iff} \quad r > c \quad \text{and} \quad r < d \\ &\text{iff} \quad c \in L_r \quad \text{and} \quad d \in U_r). \end{aligned}$$

Then the general definition of g_r is

$$g_r(W)(= [\![\mathbf{r}\varepsilon\mathbf{W}]\!]) = \bigsqcup\{[\![\mathbf{r}\varepsilon(\mathbf{c},\mathbf{d})]\!] : c,d \in \mathbb{Q} \text{ and } (c, d) \subseteq W\}.$$

g_r can be shown to be an $\sqcap$-$\bigsqcup$ map by an argument that uses the compactness of closed intervals $[c, d]$ in $\mathbb{R}$. The constructions $g \mapsto r_g$ and $r \mapsto g_r$ are, as always, mutually inverse, and so we have the presentation of $\boldsymbol{R}_d$ in Ω-**Set** as the set of all $\sqcap$-$\bigsqcup$ maps of the form $\Theta_{\mathbb{R}} \to \Omega$.

EXERCISE 7. You should by now be able to guess what this exercise says.

EXERCISE 8. Prove that in **CΩ-Set**, the members of $\boldsymbol{R}_d$ with extent e are precisely the $\sqcap$-$\bigsqcup$ maps $\Theta_{\mathbb{R}} \to \Omega \restriction e$.

It should be emphasised that it is by no means determinate what object the term "the real-number continuum" denotes in a topos $\mathscr{E}$. One classical property that may fail for $\boldsymbol{R}_d$ is *order-completeness*, i.e. the property

> *every non-empty set of reals with a $\leqslant$-upper-bound has a* least *$\leqslant$-upper-bound.*

A counter-example to this (from Stout [76]) is available in Θ_I-**Set**, where I is the unit interval $[0, 1] \subseteq \mathbb{R}$. The basic idea is conveyed by the following picture.

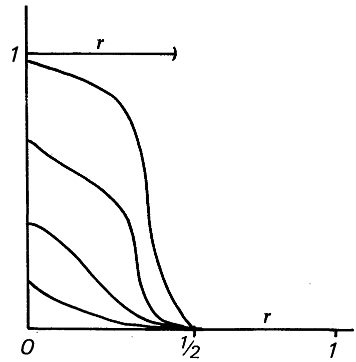


Fig. 14.7.

The ordering $\leqslant$ in R_d (= global continuous functions $I \to \mathbb{R}$) is the Θ-valued relation

$$[\![f \leqslant g]\!] = \{i : f(i) \leqslant g(i)\}^{\mathrm{o}}.$$

In this model, since everything is global, we will have that "$f \leqslant g$" is true, i.e. $[\![f \leqslant g]\!] = \top$, just in case

(iii) $\quad f(i) \leqslant g(i) \quad$ for all $\quad i \in I.$

In the picture, $r : I \to \mathbb{R}$ is the characteristic function of $[0, \frac{1}{2})$, i.e.

$$r(i) = \begin{cases} 1 & \text{if} \quad 0 \leqslant i < \frac{1}{2} \\ 0 & \text{if} \quad \frac{1}{2} \leqslant i \leqslant 1. \end{cases}$$

Now consider the set $B \subseteq R_d$ of all continuous functions that are $\leqslant$-below r in the sense of (iii). B has $\leqslant$-upper-bounds (e.g. the function with constant output 1). But it is evident that r can be approximated "arbitrarily closely from below" by members of B, and so the only possible l.u.b. for B is r itself. But r has a "jump discontinuity" at $i = \frac{1}{2}$, and so does not exist at all in R_d.

EXERCISE 9. Write out a formal sentence $\varphi(\mathbf{B})$ that expresses "B has a $\leqslant$-l.u.b." and show that $[\![\varphi(\mathbf{B})]\!] = I - \{\frac{1}{2}\} \neq \top$. □

It is patent that the counter-example applies to the sheaf R_d in $\mathbf{C}\Theta_I$-**Set**, since what we have been dealing with is just the set of global elements of the latter.

It is possible in fact to "plug the holes" in R_d and expand it to its *order completion* $^*R$ which satisfies the least-upper-bounds principle. A Dedekind-cuts-style definition of an order-complete extension of the rationals within constructive analysis seems to have first been given by John Staples [71]. Christopher Mulvey has modified this approach to show that $^*R$ can be obtained by replacing axiom ($\delta 5$) by the sentences

$$(\delta 6) \qquad \forall v(v \varepsilon \mathbf{L} \equiv \exists w(v < w \wedge \forall u(u \varepsilon \mathbf{U} \supset w < u)))$$

$$(\delta 7) \qquad \forall v(v \varepsilon \mathbf{U} \equiv \exists w(v > w \wedge \forall u(u \varepsilon \mathbf{L} \supset w > u))).$$

The object $^*R$ has been used by Charles Burden to derive a version of the Hahn–Banach Theorem in categories of sheaves and more general topoi like Ω-**Set**. A characterisation of those topoi in which the Dedekind-reals, as we have defined them, are order-complete has been given by Peter Johnstone, in a way that graphically illustrates how the underlying logic determines the structure of number systems. This result is that $R_d = {}^*R$ iff the internal logic of the topos validates De Morgan's law

$$\sim(\alpha \wedge \beta) \equiv (\sim\alpha \vee \sim\beta).$$

We will return to the subject of Dedekind cuts and order-completeness below, where the δ-axioms will be put into a rather more perspicacious form.

As for *complex* numbers, they are represented classically as ordered pairs $\langle x, y\rangle = x + iy$ of real numbers, and so given a real-numbers object R, we define an associated complex-numbers object $C = R \times R$. Since in particular a pair of continuous $\mathbb{R}$-valued functions with the same domain can be construed as a single $\mathbb{C}$-valued function on that domain, it transpires that in **CΘ-Set**, $C_d = R_d \times R_d$ is the sheaf of continuous complex-valued functions on I.

Complex analysis in a topos has been developed by Christiane Rousseau, who derives in $\mathscr{E}$ a version of the Weierstrass Division Theorem for functions of a single complex variable, and establishes that when interpreted in the topos of sheaves over $\mathbb{C}^{n-1}$ the result is equivalent to the classical theorem for functions of n variables. She has also observed that the concept of "holomorphic function" gives rise to an object H that has

$$R_c^2 \rightarrowtail H \rightarrowtail R_d^2,$$

and that H is a suitable "object of complex numbers" upon which to develop complex analysis, although it cannot itself be written as R^2 for any $R \rightarrowtail R_d$.

To close this particular segment about the use of formal logic to construct number systems, we return to the natural numbers once more and record the following version of the Peano Postulates, taken from Fourman [74]:

$$\mathbf{E}(0)$$

$$\forall v \mathbf{E}(\mathscr{s}(v))$$

$$\forall v \sim (\mathscr{s}(v) \approx 0)$$

$$\forall v \forall w (\mathscr{s}(v) \approx \mathscr{s}(w) \supset v \approx w)$$

$$\forall S((0 \varepsilon S \wedge \forall v(v \varepsilon S \supset \mathscr{s}(v) \varepsilon S)) \supset \forall v(v \varepsilon S)),$$

Here the symbol S is a second-order variable whose range is the set of all subsets of the set of natural numbers, i.e. the range of S is $\mathscr{P}(N)$.

EXERCISE 10. Let μ be the conjunction of the above sentences. Show that in $\mathbf{C}\Omega$-**Set**, $\langle \mathbf{A}, \mathscr{s}_{\mathbf{A}}, O_{\mathbf{A}} \rangle$ is a model of μ iff it is isomorphic in a unique way to $\langle \omega^*, \mathscr{s}_{\omega^*}, O_{\omega^*} \rangle$. □

Ordering the continuum

The standard orderings $\neq$, $<$, $\leqslant$, $>$, $\geqslant$, can be lifted to Θ-valued relations on a Θ-set whose elements are functions $r : I \to \mathbb{R}$ by putting

$$[\![r \neq s]\!] = \{i : r(i) \neq s(i)\}^{\mathrm{o}}$$

$$[\![r < s]\!] = \{i : r(i) < s(i)\}^{\mathrm{o}}$$

$$[\![r \leqslant s]\!] = \{i : r(i) \leqslant s(i)\}^{\mathrm{o}}$$

and similarly for $>$, $\geqslant$ (if both r and s continuous then the interior operator in the definitions of $\neq$, $<$, $>$, are redundant, as the set within the brackets is already open).

We will identify each rational $c \in \mathbb{Q}$ with the constant continuous function $I \to \mathbb{R}$ having c as its sole output. It will simplify matters if we commit a series of abuses of language by using letters c, r, . . . , indiscriminately as informal symbols to refer to elements, as individual constants in formal sentences, and even as variables in such sentences. c, d, b, e, refer always to rationals, and r, s, t, to general reals.

EXERCISE 11. Show that the above definitions yield

(i) $[\![c < d]\!] = [\![\sim \sim (c < d)]\!]$

(ii) $[\![(c < d]\!] = \top$ or $[\![\sim (c < d)]\!] = \top$

(iii) $[\![(c < d)]\!] = \top$ or $[\![c \approx d]\!] = \top$ or $[\![c > d]\!] = \top$

(iv) $[\![c \leqslant d]\!] = [\![(c < d) \vee (c \approx d)]\!]$. □

The import of these facts is that the structure of the rationals is rigidly determined (constant). This is often expressed by saying that the theory of the ordering of rationals is *decidable*. In other words we may reason with them as if we were working in $\mathbb{Q}$ and applying classical logic (e.g. (ii) above gives the law of excluded middle).

The relation $\not\approx$ is called *apartness*, a word that comes from intuitionistic mathematics, where it denotes a relation that conveys a positive, constructive sense of difference between elements. (i.e. to be apart is to have been constructively demonstrated to be different). Indeed structures of this kind were first devised by Dana Scott [68, 70] to provide models of intuitionistic theories of the real-number continuum, and in particular to obtain a model that validates Brouwer's theorem on continuity that states: *all* functions $\boldsymbol{R} \to \boldsymbol{R}$ are uniformly continuous on closed intervals.

EXERCISE 12. Show that the following formulae are *true* (i.e. are assigned truth-value $\top = I$).

(i) $\sim(r < s \wedge s < r)$
(ii) $(r < s) \wedge (s < t) \supset (r < t)$
(iii) $((r < s) \vee (s < r)) \supset (r \not\approx s)$
(iv) $(r \leqslant s) \equiv \sim(s < r)$
(v) $(r \approx s) \equiv \sim(r \not\approx s)$
(vi) $\sim(r \not\approx r)$
(vii) $(r < s) \equiv ((r \leqslant s) \wedge (r \not\approx s))$
(viii) $(r \approx s) \equiv ((r \leqslant s) \wedge (s \leqslant r))$
(ix) $(r \leqslant s \leqslant t) \supset (r \leqslant t)$
(x) $(r \leqslant s < t) \supset (r < t)$
(xi) $(r < s \leqslant t) \supset (r < t)$.

EXERCISE 13. If r and s are continuous, then

$$[\![(r \not\approx s) \supset ((r < s) \vee (s < r))]\!] = \top$$

(converse to (iii) above).

EXERCISE 14. If r, s, t are continuous, then

(i) $[\![(r < s) \supset (r < t) \vee (t < s)]\!] = \top$
(ii) $[\![(r < s) \supset \exists c (r < c < s)]\!] = \top$.

EXERCISE 15. If r is continuous then the following are true

(i) $\exists c,d(c<r<d)$

(ii) $(c<r)\equiv\exists d(c<d<r)$

(iii) $(c\leqslant r)\equiv\forall d(r<d\supset c<d)$

(iv) $(c\leqslant r)\equiv\forall d(d<c\supset d<r)$

(where $\exists c,d$ abbreviates $\exists c\ \exists d$ etc.) □

A word of caution: we are dealing throughout this exposition with *global* elements. For local reals some of these statements must be modified. 12.(vi) is true whether or nor r is global, since invariably $[\![r\not\approx r]\!]=\emptyset$. But (v) and (vi) together yield $[\![r\approx r]\!]=\top$, which is, by definition, false for non-global elements. What we do have in place of (v) is

$$Er\sqcap Es\sqsubseteq[\![(r\approx s)\equiv\sim(r\not\approx s)]\!],$$

the point being that for local elements we need to take account of their extents, and so what is true is the *universal closure*

$$\forall r\forall s((r\approx s)\equiv\sim(r\not\approx s)).$$

EXERCISE 16. Check out the rest of Exercises 12–15 in regard to local elements. □

The principles

$$(r<s)\vee(r\approx s)\vee(r>s)$$

and

$$(r\leqslant s)\equiv((r<s)\vee(r\approx s))$$

both fail in general. A counter example to both is provided by taking the two displayed continuous functions on $I=[0, 1]$

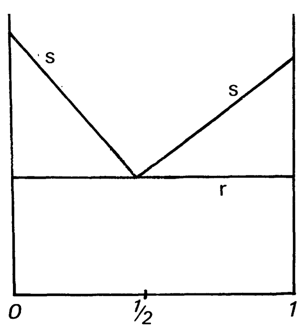


Fig. 14.8.

We have $[\![r \leqslant s]\!] = I$, $[\![r < s]\!] = I - \{\frac{1}{2}\}$, $[\![r \approx s]\!] = \{\frac{1}{2}\}^0 = \emptyset$ and $[\![r > s]\!] = \emptyset$.

Let us now return to the axioms ($\delta 1$)–($\delta 5$) that characterise those pairs $\langle U, L\rangle$ of sets of rationals that are the pair $\langle U_r, L_r\rangle$ of cuts for a unique real number r. By invoking the definitions of U_r and L_r we can rewrite these after appropriate conversions as

01: $\exists c,d(c < r < d)$

02: $\forall c \sim (c < r < c)$

03: $\forall c((c < r) \equiv \exists d(c < d < r))$

04: $\forall c((r < c) \equiv \exists d(r < d < c))$

05: $\forall c,d((c < d) \supset (c < r \vee r < d))$.

We have in fact observed in the above Exercises that 01–05 hold for any continuous $r : I \to \mathbb{R}$. As one would expect from our previous work, these axioms characterise the continuous functions.

EXERCISE 17. Suppose $r : I \to \mathbb{R}$ satisfies 01–05. Prove that for all i,

$$\begin{aligned} r(i) &= \text{g.l.b.}\{c : i \in [\![r < c]\!]\} \\ &= \text{l.u.b.}\{d : i \in [\![d < r]\!]\}. \end{aligned}$$

Hence show that

$$r^{-1}(-\infty, c) = [\![r < c]\!]$$

and

$$r^{-1}(d, \infty) = [\![d < r]\!]$$

and so r is continuous. □

In sum then, if $0(r)$ is the conjunction of 01–05, we find that the subset of the Θ-set $\mathbf{A}_{\mathbb{R}}$ of *all* $\mathbb{R}$-valued functions on I that is defined by $0(r)$ is precisely the object R_d of Dedekind-reals for Θ-**Set**.

The necessity of continuity for 05 is illustrated by our earlier example that showed R_d was not order-complete for $I = [0, 1]$. With r the characteristic function of $[0, \frac{1}{2})$ we have $[\![\frac{1}{2} < 1]\!] = I$, while $[\![\frac{1}{2} < r]\!] \cup [\![r < 1]\!] = [0, \frac{1}{2}) \cup (\frac{1}{2}, 1] = I - \{\frac{1}{2}\}$. Indeed for any rational d that has $0 \leqslant d \leqslant 1$ we find that

$$[\![(d < r) \vee (r < d)]\!] \neq \top,$$

so that d is not rigidly determined to belong to either $L_r = \{c : c < r\}$ or $U_r = \{c : r < c\}$. There is a big gap between these two cuts.

Notice also in this example that if d is *strictly* between 0 and 1 we have $[\![d \neq r]\!] = \top$, and this shows that the continuity assumption in Exercise 13 is essential. This means that in considering non-continuous reals we move away from the intuitionistic theory of the continuum. The latter has the "close together" property 05, and introduces apartness *by definition* as meaning $(r < s) \vee (s < r)$.

Let us now move to a more abstract axiomatic level and explore the order properties that are implicit in the 0-axioms. We shall assume only that we are dealing with an extension R of the rationals that has a binary relation $<$ on it that satisfies 01–04, and when restricted to Q is identical with the classical decidable theory of order for the rationals. The point will be to see what properties of $<$ can be derived using only principles of intuitionistic logic.

Axiom 01 implies that the sets L_r and U_r are not empty. The word *inhabited* is often used here, an intuitionistic term conveying a positive sense of membership. To know that A is inhabited is to have constructively proven $\exists a(a \in A)$, whereas to know that A is non-empty is to have proven only $\sim(A = \emptyset)$ i.e. $\sim\forall a(a \notin A)$, which is equivalent to

$$\sim\sim\exists a(a \in A).$$

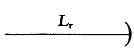
03 implies two things about L_r. First it gives

$$(c < d < r) \supset (c < r),$$

which means that L_r is unbounded on the left (anything to the left of a member of L_r is also in L_r). Secondly, from

$$(c < r) \supset \exists d(c < d < r)$$

it implies that L_r has no end-point to the right, and so must be all of Q, or else look like



04 gives a dual description of U_r, and 02 implies that the two sets are disjoint, hence neither can be Q and we must have

The linear picture is perhaps misleading, in that we do not have the trichotomy law

$$(c < r) \vee (c \approx r) \vee (r < c).$$

Indeed, we shall take the gap in the line between L_r and U_r to consist only of the points that we know positively to be between the two cuts, i.e.

those that we know to be less than every member of U_r and greater than every member of L_r. The gap then is defined by the conjunction of the sentences

$$\forall d(r<d \supset c<d)$$

and

$$\forall d(d<r \supset d<c).$$

To consider negative membership $(\sim(c \in U_r))$ we introduce the symbol $\leqslant$ by stipulating that $s \leqslant t$ is an abbreviation for $\sim(t<s)$ (cf. Exercise 12 (iv)), and then define

$$L_{\leqslant r}=\{c\colon c \leqslant r\}, \qquad U_{\geqslant r}=\{c\colon r \leqslant c\}.$$

By 02 we get

$$(c<r) \supset (\sim(r<c)),$$

which implies $L_r \subseteq L_{\leqslant r}$. Similarly $U_r \subseteq U_{\geqslant r}$. Since $L_{\leqslant r}$ is defined negatively $(\sim(r<c))$ in terms of the members of U_r, its order properties depend on the axiom 04. From the latter we obtain

$$(d<c) \supset ((r<d) \supset (r<c)),$$

which by contraposition gives

$$(d<c) \supset (\sim(r<c) \supset \sim(r<d)),$$

leading to the transitivity law

$$(d<c \leqslant r) \supset d \leqslant r.$$

This states that $L_{\leqslant r}$ is unbounded on the left. The dual property for $U_{\geqslant r}$ is given by the derivation of

$$(r \leqslant c<d) \supset (r \leqslant d)$$

from 03. Thus far, the picture is

It is easy to see that all members of $L_{\leqslant r}$ are positively to the left of U_r. For if we have $c \leqslant r$, and $r<d$ (hence $d \neq c$) but not $c<d$, we get $d<c$ (Q is decidable), and so $r<c$ by 04. But this contradicts $\sim(r<c)$. Thus we have proven

$$(c \leqslant r) \supset \forall d((r<d) \supset (c<d)).$$

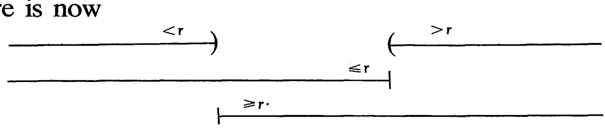
But conversely, if c is less than every member of U_r, assuming $r<c$ would lead to the contradiction $c<c$. Therefore we have $\sim(r<c)$. This establishes

$$(c \leqslant r) \equiv \forall d((r<d) \supset (c<d)),$$

and dually

$$(r \leqslant c) \equiv \forall d((d<r) \supset (d<c)).$$

The picture is now



Recall the axioms $\delta 6$ and $\delta 7$ given earlier for the order-complete reals $^*R$. In the present notation $\delta 6$ becomes

06: $\quad \forall c((c<r) \equiv \exists d(c<d \wedge \forall b((r<b) \supset (d<b))))$

and dually for 07. But by the above, $\forall b((r<b) \supset (d<b))$ is equivalent to $d \leqslant r$ so we have

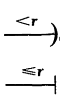
06: $\quad \forall c((c<r) \equiv \exists d(c<d \leqslant r))$

07: $\quad \forall c((r<c) \equiv \exists d(r \leqslant d<c))$.

Then from 06 we obtain

$$(d \leqslant r) \supset ((c<d) \supset (c<r)),$$

which means that any member of $L_{\leqslant r}$ has the property that everything to the left of it is in L_r. This has the effect of reducing the gap between L_r and $L_{\leqslant r}$ to (at most) a single point



and so closes the gap in the line. Alternatively by contraposition on the last formula we get

$$(d \leqslant r \leqslant c) \supset (d \leqslant c)$$

which we can interpret as reducing the overlap of $L_{\leqslant r}$ and $U_{\geqslant r}$ to a point.

EXERCISE 18. Assuming only decidability of Q, prove that 02 is implied by

02′ $\quad \forall c,d(c<r<d \supset c<d)$.

Show that 02 together with either 03 or 04 implies 02′.

EXERCISE 19. Forget the negative definition of $\leqslant$, and assume only that $c \leqslant r$ and $r \leqslant c$ have their positive meanings

$$\forall d(r<d \supset c<d)$$

and

$$\forall d(d<r \supset d<c)$$

respectively. Prove

(i) 02′ is equivalent to each of

$$\forall c(c<r \supset c \leqslant r)$$

and

$$\forall c(r<c \supset r \leqslant c)$$

(ii) 06 and 02′ together imply 03

(iii) 07 and 02′ together imply 04

(iv) Each of 06 and 07 implies 02′.

EXERCISE 20. Show that

(i) 02′, 03 and 05 together give 06.

(ii) 02′, 04 and 05 together give 07. □

The discussion preceding these exercises could be summarised by saying that the axioms for $^*R$ ensure that there is no *positive* gap at the cut determined by a real number. To see how these axioms lead also to order-completeness we continue the derivation of order properties using only principles of intuitionistic logic.

Let us *define* $^*R$ to be the set of all pairs $r=\langle U_r, L_r\rangle$ of subsets of Q that satisfy 01, 06 and 07, where $r<c$ and $c<r$ mean that $c \in U_r$ and $c \in L_r$ respectively, and $c \leqslant r$ and $r \leqslant c$ have their positive meanings as in Exercise 19. It then follows from Exercises 18 and 19 that r satisfies 02, 02′, 03 and 04 and hence we could recover the negative characterisation of $\leqslant$. The advantage of the present approach is of course that we have fewer axioms to deal with. Notice also by Exercise 20 that $R_d \subseteq {}^*R$.

EXERCISE 21. If r, $s \in {}^*R$, show that

(a) $$\forall c(s<c \supset r<c)$$

together with 06 implies

(b) $\forall c(c<r\supset c<s)$,

and dually (b) and 07 together give (a). □

We now define $r\leqslant s$, for $r,s\in{}^*R$, to mean that either of the equivalent conditions of the last Exercise obtains.

EXERCISE 22. Prove that $r\leqslant s$ iff every rational upper bound of L_s is an upper bound of L_r, i.e.

$$\forall d(d<s\supset d\leqslant c)\supset\forall d(d<r\supset d\leqslant c).$$

Show that this is equivalent to the statement that every rational lower bound of U_r is a lower bound of U_s. □

If B is a subset of ${}^*R$, we put $B\leqslant s$ to mean that s is an upper bound of B, i.e. that

$$\forall t(t\in B\supset t\leqslant s).$$

Suppose that B is inhabited ($\exists s(s\in B)$) and has an upper bound. To define a least upper bound r_0 for B we have to give its upper and lower cuts. Writing $B<d$, for *rational* d, to mean that

$$\forall t(t\in B\supset t<d)$$

we put

$$r_0<c \quad \text{iff} \quad \exists d(B<d<c)$$
$$c<r_0 \quad \text{iff} \quad \exists d(c<d\wedge\sim(B<d)).$$

The first thing we have to prove about r_0 is that it is in ${}^*R$:–

Verification of 01: The upper cut of r_0 is inhabited: there exists an s with $B\leqslant s$, and by 01 applied to s there is some $d>s$. Then if $t\in B$ we get $t\leqslant s<d$, so $t<d$ by definition of $t\leqslant s$. This establishes $B<d$, so taking any $c>d$ puts $r_0<c$.

Dually, we use the fact that there is a $t\in B$. By 01 again there is a $d<t$. Then $B<d$ would imply $t<d$, in contradiction with 02. Hence $\sim(B<d)$, so any $c<d$ gives $c<r_0$. □

Verification of 06: Suppose $c<r_0$. Then for some $d,c<d$ and $\sim(B<d)$. We prove that $d\leqslant r_0$. For , if $r_0<e$, there is an e_0 with $B<e_0<e$. Now if

$e \leqslant d$, then any $t \in B$ would have $t < e_0 < d$, hence $t < d$ by 04. But that would imply $B < d$, contrary to $\sim(B < d)$. Thus if $r_0 < e$, we must have $d < e$, which means that $d \leqslant r_0$ as required.

Conversely, suppose $c < d \leqslant r_0$ for some d. Take any rational e with $c < e < d$. Then if $B < e$, we have $B < e < d$, implying $r_0 < d$, which in turn by $d \leqslant r_0$ gives the contradiction $d < d$. Thus it must be that $c < e$ and $\sim(B < e)$, giving $c < r_0$. □

Verification of 07: If $r_0 < c$ then $B < d < c$ for some d. To show that $r_0 \leqslant d$, take any $e < r_0$. Then for some e_0, $e < e_0$ and $\sim(B < e_0)$. But then $d \leqslant e$ would imply $B < d < e_0$, leading by 04 to the contradiction $B < e_0$. Hence we must have $e < d$ as required.

Conversely, if $r_0 \leqslant d < c$, take an e with $d < e < c$. If we can show $B < e$, this will yield our desideratum $r_0 < c$. So let $t \in B$. If $e_0 < t$ then by 03 $e_0 < e_1 < t$ for some e_1. But then $B < e_1$ would give the contradiction $t < e_1$. So we have $\sim(B < e_1)$, implying $e_0 < r_0$, which by $r_0 \leqslant d$ gives $e_0 < d$. This establishes $t \leqslant d$. But since $d < e$, 07 for t gives $t < e$ as required. □

The role of r_0 as least upper bound of B is given by the fact that for any $s \in {}^*R$,

$$B \leqslant s \quad \text{iff} \quad r_0 \leqslant s$$

PROOF. Suppose $B \leqslant s$. Then $s < c$ implies $s < d < c$ for some d (04). But then if $t \in B$ we get $t \leqslant s < d$, hence $t < d$. This shows that $B < d < c$, putting $r_0 < c$.

Conversely, assume $r_0 \leqslant s$, and let $t \in B$. Then if $s < c$ we have $r_0 < c$ and so for some $d, B < d < c$. Hence $t < d < c$, giving $t < c$ by 04. This proves $t \leqslant s$. □

EXERCISE 23. Show that 01 and the "close together" axiom 05 yield the property

$$05' \qquad \forall n\, \exists c,d \left(c < r < d \wedge d - c < \frac{1}{n} \right)$$

where n is a symbol for positive integers (assume the classical theory of arithmetic for rationals).

Show that 05′, 03, and 04 together imply 05.

EXERCISE 24. Show that each of 06 and 07 implies that for each integer $n>0$, the set

$$\left\{\langle c, d\rangle : c<r<d \wedge d-c<\frac{1}{n}\right\}$$

is non-empty in the weak sense. That is,

$$05'': \qquad \forall n \sim\sim \exists c,d\left(c<r<d \wedge d-c<\frac{1}{n}\right).$$

EXERCISE 25. Construct examples of $r,s \in {}^*R$ satisfying
(i) 01, 04, 06, 05″, but not 07.
(ii) 01, 03, 07, 05″, but not 06. □

Let us return now to the result stated earlier that ${}^*R=R_d$ if De Morgan's law

$$\sim(\alpha \wedge \beta) \equiv (\sim\alpha \vee \sim\beta)$$

is valid. Since $R_d \subseteq {}^*R$, the various results given earlier imply that it suffices to show that any $r \in {}^*R$ satisfies 05. Given the present set up, the proof is quite brief. For any rational e we have (02)

$$\sim(e<r \wedge r<e)$$

and so De Morgan's law gives

$$\sim(e<r) \vee \sim(r<e)$$

which by the earlier analysis of the consequences of 03 and 04 is equivalent to

$$(r \leqslant e) \vee (e \leqslant r).$$

Now to derive 05, suppose $c<d$. Taking any e with $c<e<d$, we then have either $e \leqslant r$, hence $c<e \leqslant r$ and so $c<r$ by 06, or $r \leqslant e$ and so $r<d$ by 07. □

To date we have studiously avoided reference to the ordering $<$ for general members of ${}^*R$. In the classical case, the density of $\mathbb{Q}$ in $\mathbb{R}$ guarantees that $r<s$ just in case

$$\exists c(r<c<s),$$

and this last condition is used to define $<$ on R_d in general (cf. Exercise 28 below). It will not do however for ${}^*R$, and the procedure adopted there is to invoke the arithmetical structure of Q to put

$$r<s \quad \text{iff} \quad \exists d(d>0 \wedge r+d \leqslant s)$$

where $r+d$ is defined by specifying its upper and lower cuts by the (obvious?) clauses

$$c<r+d \quad \text{iff} \quad c-d<r$$

$$r+d<c \quad \text{iff} \quad r<c-d.$$

EXERCISE 26. Show that $r+d \in {}^*R$ if $r \in {}^*R$.

EXERCISE 27. Give examples of $r,s \in {}^*R$ with $r+d \leqslant s$ for some $d>0$ but for which $r<c<d$ fails for all rationals c.

EXERCISE 28. Use the above density condition to define $<$ on R_d in **CΩ-Set**, giving $\leqslant$ by either its positive or negative description, and $r \not\approx s$ by $(r<s) \vee (s<r)$. Show that in the topological case $\Omega = \Theta$ these lead to the Θ-valued relations with which we began this subsection.

EXERCISE 29. Let X be any of ω $\mathbb{Z}, \mathbb{Q}$, $\mathbb{R}$. Show that the standard rigid relations $\neq$, $<$ etc. on $\bar{\mathbf{X}}$ lift to the simple sheaf $\mathbf{X}^*$ to satisfy

$$[\![s \not\approx t]\!] = \bigsqcup\{[\![s \approx a]\!] \sqcap [\![t \approx b]\!] \colon a \neq b \in X\}$$

$$[\![s < t]\!] = \bigsqcup\{[\![s \approx a]\!] \sqcap [\![t \approx b]\!] \colon a < b \in X\}$$

etc. Investigate the properties of these Ω-valued order relations. □

Points

An important feature of the study of number systems in **CΩ-Set** is a generalisation of the notion of a *point* in a topological space. A given $i \in I$ determines the function $f_i : \Theta_I \to 2$ that has

$$f_i(V) = \begin{cases} 1 & \text{if} \quad i \in V \\ 0 & \text{if} \quad i \notin V, \end{cases}$$

f_i is an $\sqcap$-$\bigsqcup$ map, and so in general a *point of a* **CHA** Ω is defined to be an $\sqcap$-$\bigsqcup$ map of the form $\Omega \to 2$. The abstraction from Θ to Ω is a movement to view a generalised "space" as being made up of its *parts* (open sets) rather than its points (Lawvere [76]). In some classical topological spaces every point $\Theta_I \to 2$ is of the form f_i for some $i \in I$. Such spaces are called *sober* (all points are in focus). These include all Hausdorff spaces, so in particular $\mathbb{R}$ is sober.

There is a categorial duality between the category of sober spaces with continuous functions and the category of **CHA**'s with $\sqcap$-$\sqcup$ maps that gives a natural isomorphism between the former and **CHA**'s of the type Θ. For an arbitrary **CHA** Ω, let $\beta(\Omega)$ be the set of all points $\Omega \to 2$ of Ω. (A sober space is one having $I \cong \beta(\Theta_I)$). For $p \in \Omega$ let

$$\begin{aligned} V_p &= \{f \in \beta(\Omega) : f(p) = 1\} \\ &= \{f : \text{“}f \in p\text{”}\} \end{aligned}$$

be the set of points f that “belong to p”.

EXERCISE 30. Prove that

$$V_p \cap V_q = V_{p \sqcap q} \quad \text{all} \quad p,q \in \Omega$$

$$\bigcup_{p \in C} V_p = V_{\sqcup C} \quad \text{all} \quad C \subseteq \Omega \qquad \square$$

This result implies that the collection

$$\Theta_\Omega = \{V_p : p \in \Omega\}$$

is closed under finite intersections and arbitrary unions, i.e. is a topology on $\beta(\Omega)$.

EXERCISE 31. Given a point $g : \Theta_\Omega \to 2$ define $f_g : \Omega \to 2$ by

$$f_g(p) = g(V_p).$$

Show that $f_g \in \beta(\Omega)$ and

$$g(V_p) = 1 \quad \text{iff} \quad f_g \in V_p. \qquad \square$$

Thus we see that $\langle \beta(\Omega), \Theta_\Omega \rangle$ is a sober topological space. Moreover the previous exercise implies that the function $p \mapsto V_p$ is a surjective $\sqcap$-$\sqcup$ map (**CHA**-homomorphism). Ω will be said to *have enough points* if it satisfies

$$V_p = V_q \quad \text{only if} \quad p = q, \quad \text{all} \quad p,q \in \Omega.$$

This is an extensionality principle, asserting that if two parts have the same points in them (”$f \in p$ iff $f \in q$”) then they are equal. Obviously a topology Θ has enough points, and conversely the condition implies that the function $p \mapsto V_p$ is an isomorphism between Ω and Θ_Ω.

Thus the *spatial* **CHA**'s (the topologies) are precisely those that have enough points. At the other extreme there exist **CHA**'s that are quite pointless, and the associated sheaf categories of such structures can exhibit extremely pathological behaviour. For instance, Michael Fourman and Martin Hyland have constructed topoi along these lines that fail to satisfy such standard mathematical "facts" as "every complex number has a square root", "the equation $x^3+ax+b=0$ has a real solution for $a,b \in \mathbb{R}$", and "the unit interval [0, 1] is compact".

An account of the construction of number systems in topoi is given in Chapter 6 of Johnstone [77] and further details of the order and topological properties of R_d may be found in Stout [76] (cf. also Mulvey [74] for spatial sheaves). The major source of information in this area is the Proceedings of the Durham Conference on sheaf theory (Fourman, Mulvey, Scott [79]) which contains details of all the results that have been mentioned in this section without references.