CHAPTER 11

ELEMENTARY TRUTH

> *"... a new theory, however special its range of application, is seldom or never just an increment to what is already known. Its assimilation requires the reconstruction of prior theory and the re-evaluation of prior fact, an intrinsically revolutionary process that is seldom completed by a single man and never overnight."*
>
> Thomas Kuhn.

This chapter marks a change in emphasis towards an approach that will be more descriptive than rigorous. Our major concern will as usual be to analyse classical notions and define their categorial counterparts, but the detailed attention to verification of previous chapters will often be foregone. The proof that these generalisations work "as they should" will thus at times be left to the reader.

11.1. The idea of a first-order language

The propositional language PL of §6.3 is quite inadequate to the task of expressing the most basic discourse about mathematical structures. Take for example a structure $\langle A, R\rangle$ consisting of a binary relation R on a set A (i.e. $R \subseteq A \times A$). Let c be a particular element of A and consider the sentence "if every x is related by R to c, then there is some x to which c is related by R". If the "range" of the variable x is A, then this sentence is certainly true. For, if everything is related to c, then in particular c is related to c, so c is related to something. To see the structure of the sentence a little more clearly let

α abbreviate "for all x, xRc"

and

β abbreviate "for some x, cRx".

Then the sentence is schematised as

$$\alpha \supset \beta.$$

Now the semantical theory developed for PL in Chapter 6 cannot analyse the above argument, i.e. it cannot tell us why $\alpha \supset \beta$ is true. To know the truth value of the whole sentence we must know the values of α and β. However these function as "atomic" sentences (like the letters π_i). Their structure cannot be expressed in the language PL, and the PL-semantics does not itself explain why β must have the value "true" if α does. In order then to formalise α and β we introduce the following symbols:

(i) a symbol $\forall$, known as the *universal quantifier*, and read "for all";

(ii) a symbol $\exists$, known as the *existential quantifier*, and read "for some" or "there exists";

(iii) a symbol $\mathbf{c}$, called an *individual constant*, which is a "name" for the element c;

(iv) a symbol $\mathbf{R}$, a (two placed) *relation symbol*, or *predicate letter*, which names the relation R;

(v) a symbol v, called an *individual variable* whose interpretation is, literally, variable. It may be taken to refer to any member of A. (We shall help ourselves to an infinite number of these variables shortly, but for now one will do).

We can now symbolise α as $(\forall v)v\mathbf{Rc}$, and β as $(\exists v)\mathbf{cR}v$.

A language of the type we are now developing is called a *first-order* or *elementary* language. The word "elementary" here means "of elements". The variables of a first-order language range over elements of a structure. In a *higher-order* language, quantifiers would be applied to variables ranging over, not just elements, but also sets of elements, sets of sets of elements, etc. However in saying that the sentence

$$(\forall v)v\mathbf{Rc} \supset (\exists v)\mathbf{cR}v$$

is true of the structure or "interpretation" $\langle A, R, c\rangle$ it is thereby understood that the variable v ranges over the elements of A. Thus we need not include in our first order language any symbolisations of locutions like "for all x *belonging* to A". That is, the use of an elementary language does not depend on a formalisation of set theory.

The language we have just sketched is but one among many first order languages. The one we use will depend on the nature of the mathematical

structure we wish to discuss. If we wanted to analyse **BA**'s we would need
— constants **0** and **1** to name zero and unit elements;
— *functions letters* for the Boolean operations. These would comprise a one-place letter **f** for complementation, with $\mathbf{f}(v)$ read "the complement of v", and a pair of two-placed function letters, **g** and **h**, for meets and joins, with $\mathbf{g}(v_1, v_2)$ read "the meet of v_1 and v_2", and $\mathbf{h}(v_1, v_2)$ read "the join of v_1 and v_2";
— the *identity symbol* $\approx$, with $v_1 \approx v_2$ read "v_1 is identical to v_2".
Then, for example, the sentences

$$(\forall v)(\mathbf{g}(v, \mathbf{f}(v)) \approx \mathbf{0})$$

and

$$(\forall v)(\mathbf{h}(v, \mathbf{f}(v)) \approx \mathbf{1})$$

would be true of any Boolean algebra – they simply express the defining property of the complement of an element.

In principle, functions can always be replaced by relations (their graphs). Correspondingly, instead of introducing a function letter, say **h** above, we could use a three place relation symbol **S**, with $\mathbf{S}(v_1, v_2, v_3)$ being read "v_1 is the join of v_2 and v_2". The last sentence would then be replaced by

$$(\forall v)\mathbf{S}(\mathbf{1}, v, \mathbf{f}(v))$$

The most important mathematical structure as far as this book is concerned is the notion of category. This too is a "first-order concept" and there is some choice in how we formalise it. We could introduce two different sorts of variables, one sort to range over objects and the other over arrows, and hence have what is called a "two-sorted language". Alternatively we could use one sort of variable and the following list of predicate letters:

$\mathbf{Ob}(v)$	"v is an object"
$\mathbf{Ar}(v)$	"v is an arrow"
$\mathbf{dom}(v_1, v_2)$	"$v_1 = \mathrm{dom}\, v_2$"
$\mathbf{cod}(v_1, v_2)$	"$v_1 = \mathrm{cod}\, v_2$"
$\mathbf{id}(v_1, v_2)$	"$v_1 = 1_{v_2}$"
$\mathbf{com}(v_1, v_2, v_3)$	"$v_1 = v_2 \circ v_3$"

Amongst the sentences we would need to formally axiomatise the concept of a category are

$$\forall v((\mathbf{Ob}(v) \vee \mathbf{Ar}(v)) \wedge \sim(\mathbf{Ob}(v) \wedge \mathbf{Ar}(v)))$$

$$(\forall v_2)(\mathbf{Ob}(v_2) \supset (\exists v_1)\mathbf{id}(v_1, v_2))$$

$$(\forall v_1)(\forall v_2)(\mathbf{dom}(v_1, v_2) \supset \mathbf{Ob}(v_1) \wedge \mathbf{Ar}(v_2))$$

$$(\forall v_1)\ldots(\forall v_6)(\mathbf{com}(v_4, v_1, v_2) \wedge \mathbf{com}(v_5, v_4, v_3) \wedge \mathbf{com}(v_6, v_2, v_3) \supset \mathbf{com}(v_5, v_1, v_6))$$

The last sentence expresses the associative law – $(v_1 \circ v_2) \circ v_3 = (v_1 \circ (v_2 \circ v_3)$. The interpretation of the others is left to the reader.

Notice that with the aid of the identity symbol we can express the statement $\psi(v_1)$ that an individual v_1 is the only one having a certain property φ (this of course is vital to the description of universal properties). We put $\psi(v_1) = (\varphi(v_1) \wedge (\forall v_2)(\varphi(v_2) \supset v_1 \approx v_2))$, i.e. "$v_1$ *has* the property, and anything having it is equal to v_1". The formula $\exists v_1 \psi(v_1)$ is sometimes written $(\exists! v_1)\varphi(v_1)$ which is read, "there is exactly one v_1 such that $\varphi(v_1)$".

The language just outlined is rather cumbersome in distinguishing arrows from objects. A simpler approach, mentioned earlier, is to eliminate objects in favour of their identity arrows, and so assume all individuals are arrows. We would then use the predicate **com** as before, as well as the function letters $\mathbf{D}(v)$ – "dom v", and $\mathbf{C}(v)$ – "cod v". Thus dom v is now an arrow, namely an identity arrow. But the dom and cod of an identity arrow ought to be itself, so we can *define* $\mathbf{Ob}(v)$ to be an abbreviation of the expression

$$(\mathbf{D}(v) \approx v) \wedge (\mathbf{C}(v) \approx v).$$

An extensive development of this type of first-order language for categories is presented by W. S. Hatcher [68], who uses it to discuss Lawvere's earlier work [64] on an elementary theory of the category of sets. Hatcher also gives a rigorous proof of the Duality Principle, which after all is a principle of logic (caveat – composites in Hatcher are written the other way around, i.e. what we have been calling "$g \circ f$" is written "fg").

EXERCISE 1. Express the Identity Law in the above languages.

EXERCISE 2. Write down a first order sentence expressing each of the axioms for the notion of an elementary topos.

11.2. Formal language and semantics

All of the examples just given have a common core, one shared by all such languages.

Basic alphabet for elementary languages

(i) an infinite list v_1, v_2, $v_3, \ldots$ of individual variables;
(ii) propositional connectives $\wedge$, $\vee$, $\sim$, $\supset$;
(iii) quantifier symbols $\forall$, $\exists$;
(iv) identity symbol $\approx$;
(v) brackets), (.

Given this stock of symbols we can specify a particular language, intended to describe a particular kind of structure, by listing its relation symbols, function letters, and individual constants. Hence a first-order language is, by definition, a set of symbols of these three kinds. For **BA**'s we employ the language $\{\mathbf{0}, \mathbf{1}, \mathbf{f}, \mathbf{g}, \mathbf{h}\}$, while for categories we could use $\{\mathbf{com}, \mathbf{C}, \mathbf{D}\}$. In order to discuss semantic theories for elementary logic we will work throughout with a particularly simple language, namely

$$\mathscr{L} = \{\mathbf{R}, \mathbf{c}\}$$

having just one (two-place) relation symbol, and one individual constant. This will suffice to illustrate the main points while avoiding complexities that are technical rather than conceptual.

TERMS: These are expressions denoting individuals. For $\mathscr{L}$ the terms are the variables v_1, $v_2, \ldots$ and the constant $\mathbf{c}$.

ATOMIC FORMULAE: These are the basic building blocks for sentences. For $\mathscr{L}$ they comprise all (and only) those expressions of the form $t \approx u$, and $t\mathbf{R}u$, where t and u are terms.

FORMULAE: These are built up inductively by the rules

(i) each atomic formula is a formula;

(ii) if φ and ψ are formulae, then so are $(\varphi \wedge \psi)$, $(\varphi \vee \psi)$, $(\varphi \supset \psi)$, $(\sim\varphi)$;

(iii) if φ is a formula and v an individual variable, then $(\forall v)\varphi$ and $(\exists v)\varphi$ are formulae.

SENTENCES: If a particular occurrence of a variable in a formula is within the scope of a quantifier, that is said to be a *bound* occurrence of the variable. Otherwise the occurrence is *free*. Thus the first occurrence of v_1 in $(v_1 \approx v_1) \vee \sim(\exists v_1)v_1\mathbf{R}v_1$ is free, while its third occurrence is bound. A *sentence* is a formula in which every occurrence of a variable is bound. A formula that is not a sentence, i.e. has at least one free occurrence of a variable, is called an *open formula*.

We will write $\varphi(v)$ to indicate that the variable v has a free occurrence in φ – thereby formalising a notation we have used all along. This may be extended to $\varphi(v_{i_1}, \ldots, v_{i_n})$ to indicate several (or perhaps all) of the free variables of φ.

INTERPRETATIONS OF $\mathscr{L}$: To ascribe meanings to $\mathscr{L}$-sentences we need to give an interpretation of the symbols $\mathbf{R}$ and $\mathbf{c}$, and then use these to define interpretations of formulae by induction over their rules of formation.

A *model* for $\mathscr{L}$, or a *realisation* of $\mathscr{L}$, is a structure $\mathfrak{A} = \langle A, R, c\rangle$ comprising

(i) a non-empty set A;
(ii) a relation $R \subseteq A \times A$;
(iii) a particular individual $c \in A$.

Now if φ is the sentence $(\forall v_1)v_1\mathbf{Rc}$, then we may ask whether φ is true or false with respect to $\mathfrak{A}$. The answer is – yes, if every element of A is R-related to c, and no otherwise. On the other hand if $\varphi(v_1)$ is the open formula $v_1\mathbf{Rc}$ it makes no sense to ask whether φ is true or false simpliciter. We would have to give some interpretation to the free variable v_1. We could thus ask whether φ is true when v_1 is interpreted as referring to the individual c. The answer then is – yes, if cRc, and no otherwise. The general point then is that to give an open formula a truth value relative to a model we have first to assign to its free variables specific "values" in that model.

We now introduce a method of interpreting the variables "all at once" in $\mathfrak{A}$. Let x be a function that assigns to each positive integer n an element $x(n)$, or simply x_n, of A. Such a function is called an $\mathfrak{A}$-valuation, and is represented as an infinite sequence $x = \langle x_1, x_2, \ldots, x_i, \ldots\rangle$. The i-th member x_i of this sequence is the interpretation of the variable v_i provided by the valuation x. In what follows we will have occasion to alter valuations like x in one place only. We denote by $x(i/a)$ the valuation obtained by replacing x_i by the element $a \in A$. Thus

$$x(i/a) = \langle x_1, x_2, \ldots, x_{i-1}, a, x_{i+1}, \ldots \rangle.$$

Once variables have been interpreted, we can discuss matters of truth. We are going to give a rigorous definition of the statement "*the formula* φ *is satisfied in* $\mathfrak{A}$ *by the valuation* x", which is symbolised

$$\mathfrak{A} \models \varphi[x].$$

The definition of satisfaction is intuitively almost obvious, but to set it out precisely is rather laborious. That such a rigorous definition really is needed was first realised by Alfred Tarski, who gave one in [36], thereby opening up what has become a substantial branch of mathematical logic, known as model theory.

Atomic Formulae: Given a valuation x, each term t determines an element x_t of A, defined by

$$x_t = \begin{cases} x_i & \text{if} \quad t \text{ is the variable } v_i \\ c & \text{if} \quad t \text{ is the constant } \mathbf{c}. \end{cases}$$

Then

(1) $\mathfrak{A} \models t \approx u[x]$ iff x_t is the same element as x_u

(2) $\mathfrak{A} \models t\mathbf{R}u[x]$ iff $x_t R x_u$.

Thus the symbol $\approx$ has a fixed interpretation on any model. It denotes the identity relation $\Delta = \{\langle x, y\rangle : x = y\}$.

Formulae:

(3) $\mathfrak{A} \models \varphi \wedge \psi[x]$ iff $\mathfrak{A} \models \varphi[x]$ and $\mathfrak{A} \models \psi[x]$

(4) $\mathfrak{A} \models \varphi \vee \psi[x]$ iff $\mathfrak{A} \models \varphi[x]$ or $\mathfrak{A} \models \psi[x]$

(5) $\mathfrak{A} \models \sim\varphi[x]$ iff not $\mathfrak{A} \models \varphi[x]$

(6) $\mathfrak{A} \models \varphi \supset \psi[x]$ iff either not $\mathfrak{A} \models \varphi[x]$ or $\mathfrak{A} \models \psi[x]$

(7) $\mathfrak{A} \models (\forall v_i)\varphi[x]$ iff for every $a \in A$, $\mathfrak{A} \models \varphi[x(i/a)]$

(8) $\mathfrak{A} \models (\exists v_i)\varphi[x]$ iff for some $a \in A$, $\mathfrak{A} \models \varphi[x(i/a)]$.

In fact the satisfaction of a formula depends only on the interpretation of free variables in that formula, as shown by the

Exercise 1. If x and y are valuations with $x_i = y_i$ whenever v_i occurs free in φ, then

$$\mathfrak{A} \models \varphi[x] \quad \text{iff} \quad \mathfrak{A} \models \varphi[y].$$

□

In view of this fact, if φ is a sentence (no free variables) then one of two things can happen: either

(i) φ is satisfied by every valuation in $\mathfrak{A}$, or

(ii) φ is satisfied by no valuation in $\mathfrak{A}$.

In case (i), we simply write $\mathfrak{A} \models \varphi$, read "$\varphi$ *is true in* $\mathfrak{A}$", or "$\mathfrak{A}$ is *a model of* φ". In case (ii) we say that φ *is false in* $\mathfrak{A}$, or that φ *fails in* $\mathfrak{A}$.

Now there are some open formulae that we might want to say *are* simply true in $\mathfrak{A}$. One such example is $v_1 \approx v_1$ – it comes out true no matter how it is interpreted, i.e. it is satisfied by every valuation. To make this usage precise, and to reflect the fact that only interpretations of *free* variables are required we consider satisfaction of formulae by finite sequences. The *index* of a formula is defined to be the number of free variables that it has. If $\varphi(v_{i_1}, \ldots, v_{i_n})$ has index n, with $v_{i_1}, \ldots, v_{i_n}$ constituting all of its variables, we write $\mathfrak{A} \models \varphi[x_1, \ldots, x_n]$ if $\mathfrak{A} \models \varphi[y]$ for some (equivalently any) valuation y that has $y_{i_1} = x_1$, $y_{i_2} = x_2, \ldots, y_{i_n} = x_n$. This means that φ is satisfied when v_{i_1} is interpreted as x_1, v_{i_2} as x_2, etc. Then φ is said to be true in $\mathfrak{A}$, $\mathfrak{A} \models \varphi$, iff for any $x_1, \ldots, x_n \in A$, $\mathfrak{A} \models \varphi[x_1, \ldots, x_n]$.

EXERCISE 2. $\mathfrak{A} \models \varphi(v_{i_1}, \ldots, v_{i_n})$ iff $\mathfrak{A} \models (\forall v_{i_1})(\forall v_{i_2}) \ldots (\forall v_{i_n})\varphi$.

EXERCISE 3. $\mathfrak{A} \models (\forall v)\varphi[x]$ iff $\mathfrak{A} \models \sim(\exists v) \sim \varphi[x]$.

11.3. Axiomatics

An $\mathscr{L}$-formula φ is *valid* if it is true in all $\mathscr{L}$-models. To axiomatise the valid formulae we need to consider substitutions of a term t for a variable v in a formula φ. We write $\varphi(v/t)$ to denote the result of replacing every free occurrence of v in φ by t. This operation will "preserve truth" in general only if v is *free for* t *in* φ. This means either that t is the constant $\mathbf{c}$, or that t is a variable and no free occurrence of v is within the scope of a t-quantifier. This means then that t does not become bound when substituted for a free occurrence of v.

The classical axioms for $\mathscr{L}$ are of three kinds.

PROPOSITIONAL AXIOMS: *All formulae that are instances of the schemata* I–XII *of* §6.3 *are axioms.*

QUANTIFIER AXIOMS: *For each formula $\varphi(v)$, and term t for which v is free in φ,*

(UI) $\quad \forall v\varphi \supset \varphi(v/t),$

(EG) $\quad \varphi(v/t) \supset \exists v\varphi$

are axioms.

(The names stand for "universal instantiation" and "existential generalisation".)

IDENTITY AXIOMS: For any term t,

($I1$) $\quad t \approx t$ is an axiom.

For any $\varphi(v)$, and terms t and u, for which v is free in φ,

($I2$) $\quad (t \approx u) \wedge \varphi(v/t) \supset \varphi(v/u),$ is an axiom.

The rules of inference are,

DETACHMENT: *From φ and $\varphi \supset \psi$ infer ψ,*
and two quantifier rules:

($\forall$) $\quad$ *From $\varphi \supset \psi$ infer $\varphi \supset (\forall v)\psi$, provided v is not free in φ*

($\exists$) $\quad$ *From $\varphi \supset \psi$ infer $(\exists v)\varphi \supset \psi$, provided v is not free in ψ.*

Writing $\vdash_{\mathrm{CL}} \varphi$ to mean that φ is derivable from the above axioms by the above rules, we have

$$\vdash_{\mathrm{CL}} \varphi \quad \textit{iff} \quad \textit{for all } \mathscr{L}\textit{-models } \mathfrak{A},\ \mathfrak{A} \models \varphi.$$

This fact, that the class of valid $\mathscr{L}$-formulae is axiomatisable, is known as Gödel's Completeness Theorem, and was first proven for elementary logic by Gödel [30]. There are now several ways of proving it, and information about these may be found for example in Chang and Keisler [73] and Rasiowa and Sikorski [63].

EXERCISE. Show that the following are CL-theorems:

$$t \approx u \supset u \approx t, \qquad (t \approx u) \wedge (u \approx u') \supset (t \approx u'),$$

$$\sim(\exists v) \sim \varphi \supset (\forall v)\varphi, \qquad (\forall v)\varphi \supset \sim(\exists v) \sim \varphi.$$

11.4. Models in a topos

The interpretation of $\mathscr{L}$ in a topos is, like its classical counterpart, both natural in its conception, and arduous in its detail. It is based on a reformulation in arrow-language of the satisfaction relation

$$\mathfrak{A} \models \varphi[x_1, \ldots, x_n].$$

In fact it is convenient to deal first with a more general notion. An integer $m \geq 1$ will be called *appropriate* to φ if all of the variables of φ, free and bound, appear in the list $v_1, v_2, \ldots, v_m$. Notice that it is permitted that the list include other variables than those occurring in φ, so that if $m \leq l$, then l is also appropriate to φ. Now given an appropriate m, we can discuss satisfaction of φ by m-length sequences. We put $\mathfrak{A} \models \varphi[x_1, \ldots, x_m]$ iff $\mathfrak{A} \models \varphi[y]$ for some (equivalently any) valuation y that has $y_i = x_i$ whenever v_i is free in φ (such a v_i will then occur in the list $v_1, \ldots, v_m$).

Now given a model $\mathfrak{A} = \langle A, R, c\rangle$ and a particular m, each φ to which m is appropriate determines a subset, φ^m, of the m-fold product A^m. Namely,

$$\varphi^m = \{\langle x_1, \ldots, x_m\rangle : \mathfrak{A} \models \varphi[x_1, \ldots, x_m]\}$$

is the set of all m-length sequences satisfying φ in $\mathfrak{A}$.

To know all the φ^m's, for appropriate m's, is to know all about satisfaction of φ in $\mathfrak{A}$. Moreover the rules for satisfaction for the propositional connectives correspond to the Boolean set operations on subsets of A^m. Thus the complement of φ^m (i.e. the sequences not satisfying φ) is the set of sequences satisfying $\sim\varphi$, the intersecting of φ^m and ψ^m consists of the sequences satisfying $\varphi \wedge \psi$, and we get

$$(\sim\varphi)^m = -\varphi^m$$

$$(\varphi \wedge \psi)^m = \varphi^m \cap \psi^m$$

$$(\varphi \vee \psi)^m = \varphi^m \cup \psi^m \qquad \text{etc.}$$

(We see now the point of dealing with appropriate m's. If m is appropriate to φ and ψ it will be to $\varphi \wedge \psi$ also, although the three formulae might all have different indices.)

It would seem then that we could interpret φ in a topos as a subobject of a^m, for some object a, and then use the Heyting algebra structure of $\mathrm{Sub}(a^m)$ to interpret connectives, and hopefully quantifiers as well. This approach to categorial semantics has been set out in dissertations by students of Gonzalo Reyes and André Joyal at Montréal. The theory for elementary logic is presented by Monique Robitaille-Giguère [75].

The alternative approach is to switch from subobjects to their characteristic arrows. This accords with the propositional semantics of Chapter 6, and has the advantage for us that the interpretation of quantifiers is more accessible to a "first principles" treatment. This latter theory has been developed by Michael Brockway [76].

Returning to our $\mathscr{L}$-model $\mathfrak{A}$, we replace φ^m by its characteristic function $[\![\varphi]\!]^m : A^m \to 2$, where

$$[\![\varphi]\!]^m(\langle x_1, \ldots, x_m\rangle) = \begin{cases} 1 & \text{if } \mathfrak{A} \vDash \varphi[x_1, \ldots, x_m] \\ 0 & \text{otherwise} \end{cases}$$

Using the correspondence described in Theorem 1 of §7.1, we find that

$$\begin{aligned} [\![\sim\varphi]\!]^m &= \neg \circ [\![\varphi]\!]^m \\ [\![\varphi \wedge \psi]\!]^m &= [\![\varphi]\!]^m \cap [\![\psi]\!]^m \qquad (= \cap \circ \langle [\![\varphi]\!]^m, [\![\psi]\!]^m \rangle) \\ [\![\varphi \vee \psi]\!]^m &= [\![\varphi]\!]^m \cup [\![\psi]\!]^m \end{aligned}$$

where $\neg$, $\cap$, $\cup$ are the classical truth functions on 2.

To treat quantifiers in this manner we consider an example. Suppose that φ has just the variables v_1, v_2, and v_3 and (with $m = 3$), $[\![\varphi]\!]^3 : A^3 \to 2$ has been defined. We wish to define $[\![\forall v_2 \varphi]\!]^3 : A^3 \to 2$. So, take a triple $\langle x_1, x_2, x_3 \rangle \in A^3$ and let

$$\begin{aligned} B_2 &= \{x \in A : \mathfrak{A} \vDash \varphi[x_1, x, x_3]\} \\ &= \{x \in A : [\![\varphi]\!]^3(\langle x_1, x, x_3 \rangle) = 1\}. \end{aligned}$$

The satisfaction definition tells us that

$$\mathfrak{A} \vDash \forall v_2 \varphi[x_1, x_2, x_3] \quad \text{iff} \quad B_2 = A,$$

so we want

$$[\![\forall v_2 \varphi]\!]^3(\langle x_1, x_2, x_3 \rangle) = \begin{cases} 1 & \text{if } B_2 = A \\ 0 & \text{otherwise.} \end{cases}$$

Now the assignment of the subset B_2 of A to the triple $\langle x_1, x_2, x_3 \rangle$ establishes a function $|\varphi|_2^3$ from A^3 to $\mathscr{P}(A)$. We now define a new function $\forall_A : \mathscr{P}(A) \to 2$ by putting

$$\forall_A(B) = \begin{cases} 1 & \text{if } B = A \\ 0 & \text{if } B \neq A \quad (\text{i.e. } B \subset A) \end{cases}$$

Then the definition of $[\![\forall v_2 \varphi]\!]^3$ becomes

$$\begin{tikzcd} A^3 \arrow[r, "|\varphi|_2^3"] \arrow[dr, "[\![\forall v_2 \varphi]\!]^3"'] & \mathscr{P}(A) \arrow[d, "\forall_A"] \\ & 2 \end{tikzcd} \qquad [\![\forall v_2 \varphi]\!]^3 = \forall_A \circ |\varphi|_2^3$$

Under the isomorphism $\mathcal{P}(A) \cong 2^A$ we may construe $|\varphi|_2^3$ as a function $A^3 \to 2^A$, and hence it becomes the exponential adjoint (cf. §3.16) of a function $f: A^3 \times A \to 2$, i.e. $f: A^4 \to 2$. Then f assigns a 1 or a 0 to a 4-tuple $\langle x_1, x_2, x_3, x_4 \rangle \in A^4$ according as the function $|\varphi|_2^3(\langle x_1, x_2, x_3 \rangle) = \chi_{B_2}$ assigns a 1 or a 0 to x_4, i.e. according as $[\![\varphi]\!]^3(\langle x_1, x_4, x_3 \rangle)$ equals 1 or 0. Thus if we define $T_2^4: A^4 \to A^3$ by $T_2^4(\langle x_1, x_2, x_3, x_4 \rangle) = \langle x_1, x_4, x_3 \rangle$, we have that

$$\begin{tikzcd} A^4 \arrow[r, "T_2^4"] \arrow[dr, "f"'] & A^3 \arrow[d, "{[\![\varphi]\!]^3}"] \\ & 2 \end{tikzcd}$$

commutes. But T_2^4 can be given a categorial description. Recall from §3.8 that whenever $j \leqslant m$, we have a "j-th projection map" $pr_j^m: A^m \to A$ taking each m-sequence to its j-th member. In the present case, the effect of T_2^4 is to place the result of the 4-th projection of a 4-sequence in its 2nd position. But (§3.8) this process can be described as a product map – T_2^4 is the map

$$A^4 \xrightarrow{\langle pr_1^4, pr_4^4, pr_3^4 \rangle} A^3.$$

Consequently we get a categorial definition of f, and hence of $|\varphi|_2^3$. To complete the picture we need such a definition for $\forall_A$. This was given by Lawvere in [72], where he described $\forall_A$ as "the characteristic map of the name of $true_A$". In §4.2 we described $\ulcorner true_A \urcorner : 1 \to 2^A$, the name of $true_A$, as the arrow that picks $true_A$ out of 2^A. Since $true_A = \chi_A : A \to 2$, we identify $true_A$ with $\{A\} \subseteq \mathcal{P}(A)$. But the character of this last subobject is, by definition, $\forall_A$. $\ulcorner true_A \urcorner$ itself is the exponential adjoint of the composite

$$1 \times A \xrightarrow{pr_A} A \xrightarrow{true_A} 2, \quad \text{where} \quad pr_A(\langle 0, x \rangle) = x.$$

In summary then, $[\![\forall v_2 \varphi]\!]^3 = \forall_A \circ |\varphi|_2^3$, where $\forall_A$ is the character of the exponential adjoint of $true_A \circ pr_A$, while $|\varphi|_2^3$ is the exponential adjoint of $[\![\varphi]\!]^3 \circ \langle pr_1^4, pr_4^4, pr_3^4 \rangle$.

For existential quantifiers, by analogy we have

$$\mathfrak{A} \models \exists v_2 \varphi[x_1, x_2, x_3] \quad \text{iff} \quad B_2 \neq \emptyset$$

and so we put

$$[\![\exists v_2 \varphi]\!]^3(\langle x_1, x_2, x_3 \rangle) = \begin{cases} 1 & \text{if} \quad B_2 \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

and hence

$$\begin{tikzcd} A^3 \arrow[r, "|\varphi|_2^3"] \arrow[dr, "[\![\exists v_2\varphi]\!]^3"'] & \mathcal{P}(A) \arrow[d, "\exists_A"] \\ & 2 \end{tikzcd}$$

commutes where

$$\exists_A(B)=\begin{cases}1 & \text{if } B\neq\emptyset \\ 0 & \text{if } B=\emptyset.\end{cases}$$

It follows that $\exists_A$ is the character of the set

$$\begin{aligned} C &= \{B: B\neq\emptyset\} \\ &= \{B: \text{for some } x\in A, x\in B\}. \end{aligned}$$

But then if $\in_A \hookrightarrow \mathcal{P}(A)\times A$ is the membership relation on A (§4.7), i.e.

$$\in_A=\{\langle B, x\rangle : B\subseteq A, \text{ and } x\in B\},$$

we see that applying the first projection $p_A(\langle B, x\rangle)=B$ from $\mathcal{P}(A)\times A$ to $\mathcal{P}(A)$ yields $p_A(\in_A)=C$.

Thus $\exists_A$ is the character of the image of the composite

$$\in_A \hookrightarrow \mathcal{P}(A)\times A \xrightarrow{p_A} \mathcal{P}(A).$$

This places our account of quantifiers on an "arrows only" basis. The general definition of $[\![\forall v_i\varphi]\!]^m$, and $[\![\exists v_i\varphi]\!]^m$ comes from the above by putting m in place of 4, and i in place of 2.

The function $[\![t\approx u]\!]^m : A^m \to 2$ has

$$[\![t\approx u]\!]^m(\langle x_1, \ldots, x_m\rangle)=\begin{cases}1 & \text{if } x_t=x_u \\ 0 & \text{otherwise}\end{cases}$$

so

$$\begin{tikzcd} A^m \arrow[r, "\langle\rho_t^m, \rho_u^m\rangle"] \arrow[dr, "[\![t\approx u]\!]^m"'] & A^2 \arrow[d, "\delta_A"] \\ & 2 \end{tikzcd}$$

commutes where $\rho_t^m : A^m \to A$, $\rho_u^m : A^m \to A$, and δ_A have

$$\rho_t^m(\langle x_1, \ldots, x_m \rangle) = x_t$$
$$\rho_u^m(\langle x_1, \ldots, x_m \rangle) = x_u$$

and

$$\delta_A(\langle x, y \rangle) = \begin{cases} 1 & \text{if} \quad x = y \\ 0 & \text{if} \quad x \neq y \end{cases}, \qquad x, y \in A.$$

δ_A (the "Kronecker delta") is the character of the identity relation (diagonal) $\Delta = \{\langle x, y \rangle : x = y\} \subseteq A^2$. Notice that Δ can be identified with the monic $\langle 1_A, 1_A \rangle : A \to A^2$, that takes x to $\langle x, x \rangle$.

To define ρ_t^m, let $f_c : \{0\} \to A$ have $f_c(0) = c$.

Then

$$\rho_t^m = \begin{cases} pr_i^m : A^m \to A & \text{if} \quad t = v_i \\ f_c \circ ! : A^m \xrightarrow{\ !\ } 1 \xrightarrow{\ f_c\ } A & \text{if} \quad t = \mathbf{c}. \end{cases}$$

(Similarly for ρ_u^m).

To deal with the predicate letter $\mathbf{R}$, let $r : A^2 \to 2$ be the characteristic function of $R \subseteq A \times A$. Then

$$\begin{tikzcd} A^m \arrow[r, "\langle \rho_t^m, \rho_u^m \rangle"] \arrow[dr, "[\![t\mathbf{R}u]\!]^m"'] & A^2 \arrow[d, "r"] \\ & 2 \end{tikzcd}$$

commutes. The final notion to be re-examined is truth in a model. If $\varphi(v_{i_1}, \ldots, v_{i_n})$ has index n, then defining $[\![\varphi]\!]_{\mathfrak{A}} : A^n \to 2$ by

$$[\![\varphi]\!]_{\mathfrak{A}}(\langle x_1, \ldots, x_n \rangle) = \begin{cases} 1 & \text{if} \quad \mathfrak{A} \models \varphi[x_1, \ldots, x_n] \\ 0 & \text{otherwise} \end{cases}$$

we have

$$\begin{aligned} \mathfrak{A} \models \varphi \quad & \text{iff} \quad \text{for all } x_1, \ldots, x_n \in A, \qquad [\![\varphi]\!]_{\mathfrak{A}}(\langle x_1, \ldots, x_n \rangle) = 1 \\ & \text{iff} \quad [\![\varphi]\!]_{\mathfrak{A}} = \chi_{A^n} \\ & \text{iff} \quad [\![\varphi]\!]_{\mathfrak{A}} = true_{A^n}. \end{aligned}$$

To describe $[\![\varphi]\!]_{\mathfrak{A}}$ by arrows, we observe that if m is appropriate to φ, $\mathfrak{A} \models \varphi[x_1, \ldots, x_n]$ iff for any $y_1, \ldots, y_m$ having

$$y_{i_1} = x_1, \ldots, y_{i_n} = x_n,$$
$$\mathfrak{A} \models \varphi[y_1, \ldots, y_m].$$

Thus

$$\begin{tikzcd} A^n \arrow[r, "f"] \arrow[dr, "{[\![\varphi]\!]_{\mathfrak{A}}}"'] & A^m \arrow[d, "{[\![\varphi]\!]^m}"] \\ & 2 \end{tikzcd}$$

commutes for any f, provided only that

$$\begin{tikzcd} A^n \arrow[rr, "f"] \arrow[dr, "pr^n_k"'] & & A^m \arrow[dl, "pr^m_{i_k}"] \\ & A & \end{tikzcd} \qquad pr^m_{i_k} \circ f = pr^n_k \quad \text{for} \quad 1 \leqslant k \leqslant n.$$

This description fits in with the definition of truth of *sentences*. A member of A^n, i.e. an n-length sequence, can be thought of as a function from the ordinal $n = \{0, 1, \ldots, n-1\}$ to A. Thus, with $n = 0$, A^0 is the set of functions from the ordinal 0 (the initial object $\emptyset$) to A. Thus

$$A^0 = A^{\emptyset} = \{\emptyset\} = 1.$$

So if φ is a sentence, with index $n = 0$, $[\![\varphi]\!]_{\mathfrak{A}} : A^0 \to 2$ is a truth value $1 \to 2$. We have

$$[\![\varphi]\!]_{\mathfrak{A}} = \begin{cases} \mathit{true} & \text{if} \quad \mathfrak{A} \models \varphi \\ \mathit{false} & \text{if} \quad \text{not } \mathfrak{A} \models \varphi. \end{cases}$$

But then for any $m \geqslant 1$, *any* $f : 1 \to A^m$ makes

$$\begin{tikzcd} 1 \arrow[rr, "f"] \arrow[dr, "{[\![\varphi]\!]_{\mathfrak{A}}}"'] & & A^m \arrow[dl, "{[\![\varphi]\!]^m}"] \\ & 2 & \end{tikzcd}$$

commute, for if $\mathfrak{A} \models \varphi$ then $[\![\varphi]\!]^m$ is the "constant" function that outputs only 1's, while if not $\mathfrak{A} \models \alpha$, then $[\![\varphi]\!]^m$ outputs only 0's.

EXERCISE 1. Suppose that $\varphi(v_{i_1}, \ldots, v_{i_n})$ has index n, and m is appropriate to φ. Explain why

$$\begin{tikzcd} A^m \arrow[rr, "f"] \arrow[dr, "\llbracket\varphi\rrbracket^m"'] & & A^n \arrow[dl, "\llbracket\varphi\rrbracket_{\mathfrak{A}}"] \\ & 2 & \end{tikzcd}$$

commutes, where $f(\langle y_1, \ldots, y_m\rangle) = \langle y_{i_1}, \ldots, y_{i_n}\rangle$. □

The general definition

Let $\mathscr{E}$ be a topos, and a an $\mathscr{E}$-object. We define several arrows related to a.

DEFINITION 1. $\Delta_a : a \rightarrowtail a \times a$ is the product arrow $\langle 1_a, 1_a\rangle$
$\delta_a : a \times a \to \Omega$ is the character of Δ_a.

DEFINITION 2. $\forall_a : \Omega^a \to \Omega$ is the unique arrow making

$$\begin{tikzcd} 1 \arrow[r, "\ulcorner true_a \urcorner"] \arrow[d] & \Omega^a \arrow[d, "\forall_a"] \\ 1 \arrow[r, "true"] & \Omega \end{tikzcd}$$

a pullback, where $\ulcorner true_a \urcorner$ is the exponential adjoint of the composite $true_a \circ pr_a : 1 \times a \to a \to \Omega$.

DEFINITION 3. $\exists_a : \Omega^a \to \Omega$ is the character of the image arrow of the composite $p_a \circ \in_a : \in \rightarrowtail \Omega^a \times a \to \Omega^a$, where p_a is the first projection arrow, and $\in_a$ (§4.7) is the subobject of $\Omega^a \times a$ whose character is the evaluation arrow $ev_a : \Omega^a \times a \to \Omega$. Thus we have a diagram

$$\begin{tikzcd} \in \arrow[r, tail, "\in_a"] \arrow[d, two heads] & \Omega^a \times a \arrow[d, "p_a"] \\ p_a \circ \in_a(\in) \arrow[r, tail, "\mathrm{im}(p_a \circ \in_a)"] \arrow[d] & \Omega^a \arrow[d, "\exists_a"] \\ 1 \arrow[r, "true"] & \Omega \end{tikzcd}$$

where the bottom square is a pullback, and the top an epi-monic factorisation.

DEFINITION 4. For each m and i, with $1 \leqslant i \leqslant m$, $T_i^{m+1} : a^{m+1} \to a^m$ is the product arrow

$$\langle pr_1^{m+1}, \ldots, pr_{i-1}^{m+1}, pr_{m+1}^{m+1}, pr_{i+1}^{m+1}, \ldots, pr_m^{m+1} \rangle.$$

An $\mathscr{E}$-model for $\mathscr{L}$ is a structure

$$\mathfrak{A} = \langle a, r, f_c \rangle, \quad \text{where}$$

(i) a is an $\mathscr{E}$-object that is non-empty, i.e. $\mathscr{E}(1, \mathrm{a}) \neq \emptyset$;
(ii) $r : a \times a \to \Omega$ is an $\mathscr{E}$-arrow;
(iii) $f_c : 1 \to a$ is an "$\mathscr{E}$-element" of a.

Then given a term t we associate with each appropriate m an arrow ρ_t^m, where,

$$\rho_t^m = \begin{cases} pr_i^m : a^m \to a & \text{if} \quad t = v_i \\ f_c \circ ! : a^m \to a & \text{if} \quad t = \mathbf{c}. \end{cases}$$

Then for each $\mathscr{L}$-formula φ and appropriate m we define an $\mathscr{E}$-arrow $[\![\varphi]\!]^m : a^m \to \Omega$ inductively as follows:

(1) $[\![t \approx u]\!]^m = \delta_a \circ \langle \rho_t^m, \rho_u^m \rangle$

$$\begin{tikzcd} a^m \arrow[r, "\langle p_t^m, p_u^m \rangle"] \arrow[dr, "{[\![t \approx u]\!]^m}"'] & a^2 \arrow[d, "\delta_a"] \\ & \Omega \end{tikzcd}$$

(2) $[\![t\mathbf{R}u]\!]^m = r \circ \langle \rho_t^m, \rho_u^m \rangle$

(3) $[\![\varphi \wedge \psi]\!]^m = [\![\varphi]\!]^m \cap [\![\psi]\!]^m = \cap \circ \langle [\![\varphi]\!]^m, [\![\psi]\!]^m \rangle$

$$\begin{tikzcd} & a^m \arrow[dl, "{[\![\varphi]\!]^m}"'] \arrow[d, dashed] \arrow[dr, "{[\![\psi]\!]^m}"] & \\ \Omega & \Omega \times \Omega \arrow[l] \arrow[r] \arrow[d, "\cap"] & \Omega \\ & \Omega & \end{tikzcd}$$

(4) $[\![\varphi \vee \psi]\!]^m = [\![\varphi]\!]^m \cup [\![\psi]\!]^m$
(5) $[\![\sim\varphi]\!]^m = \neg \circ [\![\varphi]\!]^m$

(6) $[\![\varphi \supset \psi]\!]^m = [\![\varphi]\!]^m \Rightarrow [\![\psi]\!]^m$

(7) $[\![\forall v_i \varphi]\!]^m = \forall_a \circ |\varphi|_i^m$

$$\begin{tikzcd}
a^m \arrow[r, "|\varphi|_i^m"] \arrow[dr, "{[\![\forall v_i \varphi]\!]^3}"'] & \Omega^a \arrow[d, "\forall_a"] \\
& \Omega
\end{tikzcd}$$

where $|\varphi|_i^m$ is the exponential adjoint of the composite of

$$a^{m+1} \xrightarrow{T_i^{m+1}} a^m \xrightarrow{[\![\varphi]\!]^m} \Omega$$

(8) $[\![\exists v_i \varphi]\!]^m = \exists_a \circ |\varphi|_i^m$.

Now let $\varphi(v_{i_1}, \ldots, v_{i_n})$ have index n. Then let g be any arrow from a^n to a. Choose a φ-appropriate m, and let $f : a^n \to a^m$ be the product arrow $\langle p_1, \ldots, p_m \rangle$, where

$$p_j = \begin{cases} pr_k^n : a^n \to a, & \text{if } \ j = i_k, \ \text{ some } \ 1 \leqslant k \leqslant n \\ g & \text{otherwise.} \end{cases}$$

Then define $[\![\varphi]\!]_{\mathfrak{A}} : a^n \to \Omega$ by

$$\begin{tikzcd}
a^n \arrow[r, "f"] \arrow[dr, "{[\![\varphi]\!]_{\mathfrak{A}}}"'] & a^m \arrow[d, "{[\![\varphi]\!]^m}"] \\
& \Omega
\end{tikzcd}$$

i.e. $[\![\varphi]\!]_{\mathfrak{A}} = [\![\varphi]\!]^m \circ f$. Then we define "$\mathfrak{A}$ is an $\mathscr{E}$-model of φ" by

$$\mathfrak{A} \overset{\mathscr{E}}{\models} \varphi \quad \textit{iff} \quad [\![\varphi]\!]_{\mathfrak{A}} = true_{a^n}.$$

Notice that if $n \geqslant 1$, we could take g as any of the projection arrows $a^n \to a$, while if $n = 0$, we need the assumption that a is non-empty for there to be a $g : 1 \to a$ at all.

The demonstration that the definition of $[\![\varphi]\!]_{\mathfrak{A}}$ does not depend on which g is chosen, or which appropriate m, depends on some lengthy but straightforward exercises:

EXERCISE 2. If $f, h : a^n \rightrightarrows a^m$ have

$$pr^m_{i_k} \circ f = pr^m_{i_k} \circ h = pr^n_k, \quad \text{for all} \quad 1 \leqslant k \leqslant n,$$

then $[\![\varphi]\!]^m \circ f = [\![\varphi]\!]^m \circ h$, for $\varphi(v_{i_1}, \ldots, v_{i_n})$ of index n.

EXERCISE 3. If m and l are both appropriate to φ, then

$$\begin{tikzcd} a^m \arrow[rr, "f"] \arrow[dr, "{[\![\varphi]\!]^m}"'] & & a^l \arrow[dl, "{[\![\varphi]\!]^l}"] \\ & \Omega & \end{tikzcd}$$

commutes provided that $pr^l_i \circ f = pr^m_i$, whenever v_i is free in φ. Show that such an f exists.

EXERCISE 4. If $\varphi(v_{i_1}, \ldots, v_{i_n})$ has index n, and m is appropriate to φ, then

$$\begin{tikzcd} a^m \arrow[rr, "{\langle pr^m_{i_1}, \ldots, pr^m_{i_n} \rangle}"] \arrow[dr, "{[\![\varphi]\!]^m}"'] & & a^n \arrow[dl, "{[\![\varphi]\!]_{\mathfrak{A}}}"] \\ & \Omega & \end{tikzcd}$$

commutes (cf. Exercise 1). □

From these results we obtain:

THEOREM. *If φ has index n, and m is appropriate to φ, then*

$$\mathfrak{A} \models^{\mathscr{E}} \varphi \quad \textit{iff} \quad [\![\varphi]\!]^m = true_{a^m}.$$

PROOF. By Exercise 3 of §4.2, any arrow that "factors through *true* is *true*", i.e. if

$$\begin{tikzcd} b \arrow[rr, "g"] \arrow[dr, "h"'] & & c \arrow[dl, "true_c"] \\ & \Omega & \end{tikzcd}$$

commutes, then $h = true_b$. But by the definition of $[\![\varphi]\!]_{\mathfrak{A}}$, and Exercise 4, each of $[\![\varphi]\!]_{\mathfrak{A}}$ and $[\![\varphi]\!]^m$ factor through each other, hence

$$[\![\varphi]\!]_{\mathfrak{A}} = true_{a^n} \quad \text{iff} \quad [\![\varphi]\!]^m = true_{a^m}.$$ □

11.5. Substitution and soundness

An $\mathscr{L}$-formula φ is called *$\mathscr{E}$-valid*, $\mathscr{E} \models \varphi$, if $\mathfrak{A} \models^{\mathscr{E}} \varphi$ holds for every $\mathscr{E}$-model $\mathfrak{A}$.

THEOREM 1. *If $\mathscr{E} \models \varphi$ and $\mathscr{E} \models \varphi \supset \psi$, then $\mathscr{E} \models \psi$.*

PROOF. Let $\mathfrak{A}$ be any $\mathscr{E}$-model. Then $\mathfrak{A} \models \varphi$ and $\mathfrak{A} \models \varphi \supset \psi$, and so taking an m appropriate to $(\varphi \supset \psi)$, we have $[\![\varphi]\!]^m \Rightarrow [\![\psi]\!]^m = [\![\varphi \supset \psi]\!]^m = true_{a^m}$ (by the Theorem at the end of the last section). But $true_{a^m}$ is the unit of the **HA** $\mathscr{E}(a^m, \Omega)$, so (Exercise 8.3.8) in that **HA**, $[\![\varphi]\!]^m \sqsubseteq [\![\psi]\!]^m$. But since m is also appropriate to φ, and $\mathfrak{A} \models \varphi$, we also have $[\![\varphi]\!]^m = true_{a^m}$. Thus in $\mathscr{E}(a^m, \Omega)$, $[\![\psi]\!]^m = true_{a^m}$ and so as m is appropriate to ψ, $\mathfrak{A} \models \psi$. □

So the rule of *Detachment* preserves $\mathscr{E}$-validity. Since the propositional connectives are interpreted as the truth arrows in a topos it should come as no surprise that any instance of the schemata I–XI is valid in any $\mathscr{E}$, while there are topos models in which XII fails (an example will be given later). We shall write $\vdash_{\mathrm{IL}} \varphi$ to mean that φ is derivable in the system that has all the rules and axioms of §11.3 except for XII. Without *I*1 and *I*2, this is the system of intuitionistic predicate logic of Heyting [66]. Axioms for identity equivalent to the ones given here are discussed by Rasiowa and Sikorski [63].

SOUNDNESS THEOREM. *If $\vdash_{\mathrm{IL}} \varphi$, then for any $\mathscr{E}$, $\mathscr{E} \models \varphi$.*

We will not prove all the Soundness Theorem, but will concentrate on setting up the machinery that lies behind it. The method as always is to show that the axioms are $\mathscr{E}$-valid and the rules of inference preserve this property. The strategy for the first part is to show that if φ is an axiom then relative to $\mathfrak{A}$, $[\![\varphi]\!]^m = true_{a^m}$, for some (or any) appropriate m. The Theorem of the last section then gives $\mathfrak{A} \models \varphi$.

To establish validity of the quantifier and identity axioms we must look at the categorial content of the substitution process. If $\psi = \varphi(v_i/t)$, then in **Set**, interpreting t in ψ as x_t is the same as interpreting v_i in φ as x_t, i.e. $\mathfrak{A} \models \psi[x_1, \ldots, x_m]$ iff $\mathfrak{A} \models \varphi[x_1, \ldots, x_{i-1}, x_t, x_{i+1}, \ldots, x_m]$, and so

$$\begin{tikzcd} A^m \arrow[rr, "f"] \arrow[dr, "{[\![\psi]\!]^m}"'] & & A^m \arrow[dl, "{[\![\varphi]\!]^m}"] \\ & 2 & \end{tikzcd}$$

commutes, where $f(\langle x_1, \ldots, x_m\rangle) = \langle x_1, \ldots, x_{i-1}, x_t, x_{i+1}, \ldots, x_m\rangle$.

Correspondingly, in a general topos $\mathscr{E}$, if $i \leqslant m$ and t is a term to which m is appropriate (i.e. if $t = v_j$ then $j \leqslant m$), the arrow $\delta^m[i/t]: a^m \to a^m$ is defined to be the product arrow

$$\langle pr_1^m, \ldots, pr_{i-1}^m, \rho_t^m, pr_{i+1}^m, \ldots, pr_m^m\rangle.$$

SUBSTITUTION LEMMA. *In any topos, the diagram*

$$\begin{tikzcd} a^m \arrow[rr, "\delta^m[i/t]"] \arrow[dr, "\llbracket\varphi(v_i/t)\rrbracket^m"'] & & a^m \arrow[dl, "\llbracket\varphi\rrbracket^m"] \\ & \Omega & \end{tikzcd}$$

commutes whenever v_i is free for t in φ. □

EXERCISE 1. $pr_i^m \circ \delta[i/v_j] = pr_j^m \circ \delta^m[i/v_j] = pr_j^m$.

EXERCISE 2. If $f: b \to a^m$ has $pr_i^m \circ f = pr_j^m \circ f$, then

$$\begin{tikzcd} & b \arrow[dl, "f"'] \arrow[dr, "f"] & \\ a^m \arrow[rr, "\delta^m[i/v_j]"] & & a^m \end{tikzcd}$$

commutes. (Interpret this in **Set**.)

EXERCISE 3. For i, $j \leqslant m$,

$$\begin{tikzcd} a^{m+1} \arrow[r, "T_j^{m+1}"] \arrow[d, "\langle T_j^{m+1}, pr_{m+1}^{m+1}\rangle"'] & a^m \arrow[d, "\delta[i/v_j]"] \\ a^{m+1} \arrow[r, "T_i^{m+1}"] & a^m \end{tikzcd}$$

commutes.

EXERCISE 4. If v_j does not occur in φ, then

$$\llbracket\varphi(v_i/v_j)\rrbracket^m \circ T_j^{m+1} = \llbracket\varphi\rrbracket^m \circ T_i^{m+1},$$

and hence

$$[\![\exists v_j \varphi(v_i/v_j)]\!]^m = [\![\exists v_i \varphi]\!]^m$$
$$[\![\forall v_j \varphi(v_i/v_j)]\!]^m = [\![\forall v_i \varphi]\!]^m.$$

Consequently $[\![\varphi]\!]^m = [\![\psi]\!]^m$ if φ and ψ are "bound alphabetical variants" of each other. □

To use the Substitution Lemma to show validity of the identity axioms we examine the properties of the Kronecker delta.

THEOREM 2. *For any pair* $f, g : b \to a$, $\delta_a \circ \langle f, g\rangle$ *is the character of the equaliser of* f *and* g.

PROOF. Consider

$$\begin{tikzcd} c \arrow[r, tail, "h"] \arrow[d] & b \arrow[d, "\langle f, g\rangle"] \\ a \arrow[r, "\langle 1_a, 1_a\rangle"] \arrow[d] & a \times a \arrow[d, "\delta_a"] \\ 1 \arrow[r, "true"] & \Omega \end{tikzcd}$$

The top square is obtained by pulling $\langle 1_a, 1_a\rangle = \Delta_a$ back along $\langle f, g\rangle$. By the universal property of that square qua pullback, it is an easy exercise to show that h equalises f and g. But the bottom square is the pullback defining δ_a, so by the PBL and the Ω-axiom, $\delta_a \circ \langle f, g\rangle = \chi_h$. □

COROLLARY. $\delta_a \circ \langle f, f\rangle = true_b$, *for* $f : b \to a$.

PROOF. $true_b = \chi_{1_b}$ and 1_b equalises the pair $\langle f, f\rangle$. □

From this Corollary we obtain immediately the validity of $I1$, i.e. $\mathscr{E} \models t \approx t$. For, $[\![t \approx t]\!]^m = \delta_a \circ \langle \rho_t^m, \rho_t^m\rangle$, where $\rho_t^m : a^m \to a$.

Now in **Set**, the formula $(t \approx u)$ determines the set

$$D_{tu} = \{\langle x_1, \ldots, x_m\rangle : \mathfrak{A} \models (t \approx u)[x_1, \ldots, x_m]\}$$
$$= \{\langle x_1, \ldots, x_m\rangle : x_t = x_u\}.$$

Correspondingly in $\mathscr{E}$ we define $d_{tu} : d \rightarrowtail a^m$ to be the subobject whose character is $[\![t \approx u]\!]^m$.

THEOREM 3. *For appropriate m,*

$$\delta^m[i/t] \circ d_{tu} = \delta^m[i/u] \circ d_{tu}.$$

PROOF. Since $[\![t \approx u]\!]^m = \delta_a \circ \langle \rho_t, \rho_u \rangle : a^m \to \Omega$, Theorem 2 tells us that d_{tu} equalises ρ_t and ρ_u, hence $\rho_t \circ d_{tu} = \rho_u \circ d_{tu}$. Then

$$\begin{aligned}
&\langle pr_1, \ldots, \rho_t, \ldots, pr_m \rangle \circ d_{tu} \\
&\quad = \langle pr_1 \circ d_{tu}, \ldots, \rho_t \circ d_{tu}, \ldots, pr_m \circ d_{tu} \rangle \\
&\quad = \langle pr_1 \circ d_{tu}, \ldots, \rho_u \circ d_{tu}, \ldots, pr_m \circ d_{tu} \rangle \\
&\quad = \langle pr_1, \ldots, \rho_u, \ldots, pr_m \rangle \circ d_{tu}.
\end{aligned}$$

□

COROLLARY. *If m is appropriate to t, u, and $\varphi(v_i)$, with v_i free for t and u in φ, then*

$$[\![\varphi(v_i/t)]\!]^m \sqcap [\![t \approx u]\!]^m = [\![\varphi(v_i/u)]\!]^m \sqcap [\![t \approx u]\!]^m$$

PROOF. Using the Substitution Lemma, we have

$$\begin{aligned}
[\![\varphi(v_i/t)]\!]^m \circ d_{tu} &= [\![\varphi]\!]^m \circ \delta^m[i/t] \circ d_{tu} \\
&= [\![\varphi]\!]^m \circ \delta^m[i/u] \circ d_{tu} \\
&= [\![\varphi(v_i/u)]\!]^m \circ d_{tu}.
\end{aligned}$$

Since $\chi_{d_{tu}} = [\![t \approx u]\!]^m$, Lemma 1(2) of §7.5 yields the desired result. □

Now in order to have $\mathfrak{A} \models [(t \approx u) \wedge \varphi(v_i/t)] \supset \varphi(v_i/u)$ we require that for some appropriate m,

$$[\![t \approx u]\!]^m \sqcap [\![\varphi(v_i/t)]\!]^m \sqsubseteq [\![\varphi(v_i/u)]\!]^m$$

in the **HA** $\mathscr{E}(a^m, \Omega)$. But this follows from the Corollary, by lattice properties, and so the schema $I2$ is $\mathscr{E}$-valid.

We turn now to the validity of the quantifier axioms. For this we elicit the basic properties of the quantifier arrows.

THEOREM 4. (1) $(\forall_a \circ p_a) \Rightarrow ev_a = true_{\Omega^a \times a}$

(2) $ev_a \Rightarrow (\exists_a \circ p_a) = true_{\Omega^a \times a}$

PROOF. (1) Consider

$$\begin{tikzcd}
d \arrow[r, tail, "f"] \arrow[d] & \Omega^a \times a \arrow[d, "p_a"] \\
1 \arrow[r, "\ulcorner true_a \urcorner"] \arrow[d] & \Omega^a \arrow[d, "\forall_a"] \\
1 \arrow[r, "true"] & \Omega
\end{tikzcd}$$

The top square is obtained by pulling $\ulcorner true_a \urcorner$ back along p_a. A now familiar argument tells then that $\chi_f = \forall_a \circ p_a$. But by definition of $\ulcorner true_a \urcorner$ as the exponential adjoint of $true_a \circ pr_a$, the diagram

$$\begin{tikzcd}
1 \times a \arrow[r, "\ulcorner true_a \urcorner \times 1_a"] \arrow[dr, "true_a \circ pr_a"'] & \Omega^a \times a \arrow[d, "ev_a"] \\
& \Omega
\end{tikzcd}$$

commutes, which says precisely that the perimeter of

$$\begin{tikzcd}
1 \times a \arrow[drr, bend left, "\ulcorner true_a \urcorner \times 1_a"] \arrow[dr, dashed] \arrow[d, bend right, "pr_a"'] & & \\
a \arrow[dr, bend right] & \in \arrow[r, tail, "\in_a"] \arrow[d] & \Omega^a \times a \arrow[d, "ev_a"] \\
& 1 \arrow[r, "true"] & \Omega
\end{tikzcd}$$

commutes, yielding an arrow $1 \times a \rightarrow \in$ that makes $\ulcorner true_a \urcorner \times 1_a$ factor through $\in_a$.

But consider the diagram

$$\begin{tikzcd}
d \arrow[dr, tail, "f"] \arrow[d, "\langle 1_d, p_2 \circ f \rangle"'] & \\
1 \times a \arrow[r, "\ulcorner true_a \urcorner \times 1_a"'] & \Omega^a \times a
\end{tikzcd}$$

where $p_2 : \Omega^a \times a \rightarrow a$ is the 2nd projection.

Using Exercise 8 of §3.8 we find that

$$
\begin{aligned}
&(\ulcorner true_a \urcorner \times 1_a) \circ \langle 1_d, p_2 \circ f\rangle \\
&\quad = \langle \ulcorner true_a \urcorner \circ 1_d, 1_a \circ p_2 \circ f\rangle \\
&\quad = \langle p_a \circ f, p_2 \circ f\rangle \\
&\quad = \langle p_a, p_2\rangle \circ f && \text{(Exercise 3.8.2)} \\
&\quad = 1_{\Omega^a \times a} \circ f && \text{(Exercise 3.8.3)} \\
&\quad = f
\end{aligned}
$$

Thus f factors through $\ulcorner true_a \urcorner \times 1_a$. Since the latter factors through $\in_a$, in $\mathrm{Sub}(\Omega^a \times a)$ we have $f \subseteq \in_a$. Hence (Theorem 7.5.1),

$$\chi_f \Rightarrow \chi_{\in_a} = true_{\Omega^a \times a},$$

which is the desired result.

(2) Exercise – use the diagram given with the definition of $\exists_a$ to show $\in_a \subseteq g$, where $\chi_g = \exists_a \circ p_a$. □

Now in **Set**, if we take the sequence $\langle x_1, \ldots, x_m\rangle$, form $\langle x_1, \ldots, x_m, x_t\rangle$, and then apply T_i^{m+1} we end up with $\langle x_1, \ldots, x_{i-1}, x_t, x_{i+1}, \ldots, x_m\rangle$ – the overall effect is to perform $\delta[i/t]$. Abstracting, we have

THEOREM 5. *Let $U_t^m : a^m \to a^{m+1}$ be the product arrow $\langle 1_{a^m}, \rho_t^m\rangle$, Then*

(1)

$$
\begin{tikzcd}
a^m \arrow[rr, "U_t^m"] \arrow[dr, "{\delta^m[i/t]}"'] & & a^{m+1} \arrow[dl, "T_i^{m+1}"] \\
& a^m &
\end{tikzcd}
$$

commutes, and

(2)

$$
\begin{tikzcd}
a^m \times a \arrow[r, "f \times 1_a"] & \Omega^a \times a \arrow[d, "p_a"] \\
a^m \arrow[u, "U_t^m"] \arrow[r, "f"] & \Omega^a
\end{tikzcd}
$$

commutes for any f as shown.

PROOF. (1) Exercise – you will need to know $1_{a^m} = \langle pr_1^m, \ldots, pr_m^m\rangle$.

(2) By definition of the product arrow $f \times 1_a$,

$$p_a \circ (f \times 1_a) \circ \langle 1_{a^m}, \rho_t\rangle = f \circ pr \circ \langle 1_{a^m}, \rho_t\rangle$$

(where $pr : a^m \times a \to a^m$ is projection)

$$= f \circ 1_{a^m}$$
$$= f \qquad \square$$

Part (1) of this theorem, with the Substitution Lemma, gives

$$[\![\varphi(v_i/t)]\!]^m = [\![\varphi]\!]^m \circ T_i^{m+1} \circ U_i^m,$$

and since

$$\begin{tikzcd} \Omega^a \times a \arrow[dr, "ev_a"] & \\ & \Omega \\ a^{m+1} \arrow[uu, "|\varphi|_i^m \times 1_a"] \arrow[ur, "[\![\varphi]\!]^m \circ T_i^{m+1}"'] & \end{tikzcd}$$

commutes, by definition of $|\varphi|_i^m$ as exponential adjoint to $[\![\varphi]\!]^m \circ T_i^{m+1}$, we get

$$[\![\varphi(v_i/t)]\!]^m = ev_a \circ (|\varphi|_i^m \times 1_a) \circ U_i^m.$$

Moreover by taking $f = |\varphi|_i^m$ in Theorem 5(2), we have

$$|\varphi|_i^m = p_a \circ (|\varphi|_i^m \times 1_a) \circ U_i^m.$$

Using these last two equations, and putting $(|\varphi|_i^m \times 1_a) \circ U_i^m = g$, we calculate

$$\begin{aligned} [\![\forall v_i \varphi \supset \varphi(v_i/t)]\!]^m &= \Rightarrow \circ \langle [\![\forall v_i \varphi]\!]^m, [\![\varphi(v_i/t)]\!]^m \rangle \\ &= \Rightarrow \circ \langle \forall_a \circ |\varphi|_i^m, ev_a \circ g \rangle \\ &= \Rightarrow \circ \langle \forall_a \circ p_a \circ g, ev_a \circ g \rangle \\ &= \Rightarrow \circ \langle \forall_a \circ p_a, ev_a \rangle \circ g \\ &= (\forall_a \circ p_a \Rightarrow ev_a) \circ g \\ &= true_{\Omega^a \times a} \circ g && \text{(Theorem 4)} \\ &= true_{a^m} && (a^m \xrightarrow{g} \Omega^a \times a) \end{aligned}$$

Hence the axiom UI is valid. $\square$

EXERCISE 5. Show that EG is valid by an anologous argument using the second part of Theorem 4. $\square$

The soundness of the rules ($\forall$) and ($\exists$) are left for the enthusiastic reader. The details have been worked out in Brockway [76].

11.6. Kripke Models

The algebraic and topological interpretations of intuitionistic propositional logic extend readily to first-order logic. The truth-value of a formula becomes a function $[\![\varphi]\!]^m : A^m \to \mathbf{H}$, where $\mathbf{H}$ is a suitable Heyting algebra, e.g. the lattice of open sets of some topological space. A comprehensive study of this type of model is undertaken by Rasiowa and Sikorski [63] (cf. also its application to intuitionistic analysis by Dana Scott [68].)

In his 1965 paper, Kripke gave a semantics for first-order IL that generalises the classical notion of $\mathscr{L}$-model described earlier in this chapter. The basic idea is (or can be seen to be) that for a given poset $\mathbf{P}$, a model assigns to each $p \in P$ a classical model $\mathfrak{A}_p$. Atomic formulae have their truth value at p determined by their classical truth value in $\mathfrak{A}_p$, and then the connectives can be dealt with as in the propositional case (§8.4). In fact Kripke's theory did not discuss individual constants, or the identity predicate, so in order to do so ourselves we introduce a slightly more general notion of model than that considered previously.

Let $\mathbf{P}$ be a poset. An *$\mathscr{L}$-model based on* $\mathbf{P}$ is defined to be a structure $\mathfrak{A}$ consisting of

(a) for each $p \in P$ a classical $\mathscr{L}$-model $\mathfrak{A}_p = \langle A_p, R_p, c_p \rangle$;

(b) for each arrow $p \sqsubseteq q$ in $\mathbf{P}$, a function $A_{pq} : A_p \to A_q$, such that

(i) if $p \sqsubseteq q$ then $A_{pq}(c_p) = c_q$

(ii) if $p \sqsubseteq q$ then $xR_p y$ only if $A_{pq}(x) R_q A_{pq}(y)$

(iii) A_{pp} is the identity $\mathbf{1} : A_p \to A_p$

(iv) if $p \sqsubseteq q \sqsubseteq r$, then

$$\begin{tikzcd} A_p \arrow[rr, "A_{pq}"] \arrow[dr, "A_{pr}"'] & & A_q \arrow[dl, "A_{qr}"] \\ & A_r & \end{tikzcd}$$

commutes. Thus (i) requires that A_{pq} take the interpretation of $\mathbf{c}$ at p to its interpretation at q, while by (ii) A_{pq} "preserves" the truth of atomic formulae of the form $t\mathbf{R}u$. Notice that the collection $\{A_p : p \in P\}$ of sets together with the transition maps A_{pq} constitute a functor $A : \mathbf{P} \to \mathbf{Set}$, i.e. an object in the topos $\mathbf{Set}^{\mathbf{P}}$. This is a consequence of the definition, rather than the motivation for it. The reason why $\mathscr{L}$-models are defined as above is that this seems to be the natural way to treat $\approx$ as the relation of identity of individuals. Kripke's definition has in place of (b) the requirement that

$$p \sqsubseteq q \quad \text{implies} \quad A_p \subseteq A_q \quad \text{and} \quad R_p \subseteq R_q.$$

This amounts to putting A_{pq} as the inclusion $A_p \hookrightarrow A_q$. As pointed out by Richmond Thomason in [68], if $\approx$ is interpreted as identity, such a model would validate $(t \approx u) \vee \sim(t \approx u)$, for distinct individuals are left distinct by inclusions, and so remain "distinct forever". Thomason's solution is to interpret $\approx$ as an equivalence relation E_p on A_p, with perhaps $E_p \neq \Delta$. However by introducing the transitions A_{pq} we are able to give $\approx$ its natural interpretation and still not have the above instance of XII come out valid. For it is quite possible to have $x_t \neq x_u$ in A_p, but $A_{pq}(x_t) = A_{pq}(x_u)$. We thus give an account of the notion that things not known to be identical could come to be so known later, and also formalise some of the discussion of §10.1.

Now if φ is an $\mathscr{L}$-formula to which m is appropriate, we may define the relation

$$\mathfrak{A} \models_p \varphi[x_1, \ldots, x_m]$$

for $x_1, \ldots, x_m \in A_p$, of satisfaction of φ in $\mathfrak{A}$ at p.

In the interest of legibility we will abbreviate $A_{pq}(x)$ to x^q.

(1) If φ is atomic, $\mathfrak{A} \models_p \varphi[x_1, \ldots, x_m]$ iff $\mathfrak{A}_p \models \varphi[x_1, \ldots, x_m]$ in the classical sense.

(2) $\mathfrak{A} \models_p \varphi \wedge \psi[x_1, \ldots, x_m]$ iff $\mathfrak{A} \models_p \varphi[x_1, \ldots, x_m]$ and $\mathfrak{A} \models_p \psi[x_1, \ldots, x_m]$.

(3) $\mathfrak{A} \models_p \varphi \vee \psi[x_1, \ldots, x_m]$ iff $\mathfrak{A} \models_p \varphi[x_1, \ldots, x_m]$ or $\mathfrak{A} \models_p \psi[x_1, \ldots, x_m]$.

(4) $\mathfrak{A} \models_p \sim\varphi[x_1, \ldots, x_m]$ iff for all q with $p \sqsubseteq q$, not $\mathfrak{A} \models_q \varphi[x_1^q, \ldots, x_m^q]$.

(5) $\mathfrak{A} \models_p \varphi \supset \psi[x_1, \ldots, x_m]$ iff for all q with $p \sqsubseteq q$, if $\mathfrak{A} \models_q \varphi[x_1^q, \ldots, x_m^q]$ then $\mathfrak{A} \models_q \psi[x_1^q, \ldots, x_m^q]$.

(6) $\mathfrak{A} \models_p \exists v_i \varphi[x_1, \ldots, x_m]$ iff for some $a \in A_p$, $\mathfrak{A} \models_p \varphi[x_1, \ldots, x_{i-1}, a, x_{i+1}, \ldots, x_m]$.

(7) $\mathfrak{A} \models_p \forall v_i \varphi[x_1, \ldots, x_m]$ iff for every q with $p \sqsubseteq q$, and every $a \in A_q$, $\mathfrak{A} \models_q \varphi[x_1^q, \ldots, x_{i-1}^q, a, x_{i+1}^q, \ldots, x_m^q]$.

Thus $\exists v \varphi$ is to be true at stage p iff φ is true of some individual present at stage p, while the truth of $\forall v \varphi$ at p requires φ to be true not only of all individuals present at p but also all that occur at later stages.

If $\varphi(v_{i_1}, \ldots, v_{i_n})$ has index n, we put $\mathfrak{A} \models_p \varphi[x_1, \ldots, x_n]$ iff $\mathfrak{A} \models_p \varphi[y_1, \ldots, y_m]$ for some (hence any) appropriate m and $y_1, \ldots, y_m$ having $y_{i_1} = x_1, \ldots, y_{i_n} = x_n$.

Then we put $\mathfrak{A} \models_p \varphi$ (*φ is true at p*) iff $\mathfrak{A} \models_p \varphi[x_1, \ldots, x_n]$ for all $x_1, \ldots, x_n \in A_p$, and finally $\mathfrak{A} \models \varphi$ (*$\mathfrak{A}$ is a model of φ*) iff for all $p \in P$, $\mathfrak{A} \models_p \varphi$.

EXERCISE. 1. Show that this definition reduces to the classical notion of $\mathscr{L}$-model when P has only one member.

EXERCISE 2. Show that if $\mathfrak{A} \models_p \varphi[x_1, \ldots, x_m]$ and $p \sqsubseteq q$, then $\mathfrak{A} \models_q \varphi[x_1^q, \ldots, x_m^q]$, any φ. □

Now the **P**-model $\mathfrak{A}$ is turned into a $\mathbf{Set}^{\mathbf{P}}$ model $\mathfrak{A}^* = \langle A, r, f_c \rangle$, by taking

(i) $A : \mathbf{P} \to \mathbf{Set}$ as the functor associated with $\mathfrak{A}$ described earlier.

(ii) $r : A \times A \dot{\to} \Omega$ as the natural transformation with components $r_p : A_p \times A_p \to \Omega_p$ given by

$$r_p(\langle x, y \rangle) = \{q : p \sqsubseteq q \text{ and } A_{pq}(x) R_q A_{pq}(y)\}.$$

(iii) $f_c : 1 \dot{\to} A$ as the arrow with components $(f_c)_p : \{0\} \to A_p$ having $(f_c)_p(0) = c_p$.

EXERCISE 3. Show that $r_p(\langle x, y \rangle)$ is an hereditary subset of $[p)$.

EXERCISE 4. Show that xR_py iff $r_p(\langle x, y \rangle) = [p)$ and hence (cf. §10.3)

$$\begin{tikzcd} R_p \arrow[r, hook] \arrow[d] & A_p^2 \arrow[d, "r_p"] \\ 1 \arrow[r, "true_p"] & \Omega_p \end{tikzcd}$$

is a pullback.

EXERCISE 5. Verify that r and f_c *are* natural transformations. □

The exercises tell us how to reverse the construction. Given a $\mathbf{Set}^{\mathbf{P}}$ model $\langle A, r, f_c \rangle$ we specify $\mathfrak{A}_p$ by *defining* c_p by the equation (iii) in Exercise 2, and *defining* R_p by the equation in Exercise 4. This establishes a bijective correspondence between $\mathscr{L}$-models $\mathfrak{A}$ based on **P** and $\mathbf{Set}^{\mathbf{P}}$-models $\mathfrak{A}^*$ for $\mathscr{L}$.

Undoubtedly the reader has anticipated that corresponding models have the same formulae true in them. Indeed the connection is much finer than that. Let us calculate $[\![\varphi]\!]^m$, relative to $\mathfrak{A}^*$, for φ an atomic formula.

We have

$$\begin{tikzcd} A^m \arrow[rr, "\langle\rho_t, \rho_u\rangle"] \arrow[dr, "[\![t\mathbf{R}u]\!]^m"'] & & A^2 \arrow[dl, "r"] \\ & \Omega & \end{tikzcd}$$

where A^m is the product functor having $A_p^m = (A_p)^m$ etc., and $\rho_t : A^m \dot{\to} A$ has components

$$(\rho_t)_p : A_p^m \to A_p,$$

where

$$(\rho_t)_p(\langle x_1, \ldots, x_m\rangle) = x_t.$$

From this we see that the component $[\![t\mathbf{R}u]\!]_p^m : A_p^m \to \Omega_p$ assigns to $\langle x_1, \ldots, x_m\rangle$ the set

$$\begin{aligned} r_p(\langle x_t, x_u\rangle) &= \{q : p \sqsubseteq q \text{ and } x_t^q R_q x_u^q\} \\ &= \{q : p \sqsubseteq q \text{ and } \mathfrak{A} \models_q t\mathbf{R}u[x_1^q, \ldots, x_m^q]\}. \end{aligned}$$

This situation is quite typical, as expressed in the:

Truth Lemma. *For any φ, and appropriate m, then relative to $\mathfrak{A}^*$ the* **Set**$^{\mathbf{P}}$*-arrow $[\![\varphi]\!]^m : A^m \to \Omega$ has p-th component*

$$[\![\varphi]\!]_p^m : A_p^m \to \Omega_p,$$

where

$$[\![\varphi]\!]_p^m(\langle x_1, \ldots, x_m\rangle) = \{q : p \sqsubseteq q \textit{ and } \mathfrak{A} \models_q \varphi[x_1^q, \ldots, x_m^q]\}.$$

Given the analysis of **Set**$^{\mathbf{P}}$ in Chapter 10, the proof of the Truth Lemma for the inductive cases of the connectives should be evident. For identities and quantification we need to examine the arrows δ_A, $\forall_A$, and $\exists_A$, for a **Set**$^{\mathbf{P}}$-object $A : \mathbf{P} \to \mathbf{Set}$.

Theorem 1. *$\delta_A : A \times A \dot{\to} \Omega$ has*

$$(\delta_A)_p : A_p \times A_p \to \Omega_p$$

given by

$$(\delta_A)_p(\langle x, y\rangle) = \{q : p \sqsubseteq q \text{ and } x^q = y^q\}.$$

Proof. $\Delta_A : A \dot{\to} A \times A$ has $(\Delta_A)_p$ as the map $\langle 1_{A_p}, 1_{A_p}\rangle : A_p \to A_p^2$. $(\Delta_A)_p$

then can be identified with the identity relation $\Delta_p = \{\langle x, y\rangle : x = y\} \subseteq A_p \times A_p$. The characteristic function of this set is $(\delta_A)_p$, and so (cf. §10.3)

$$(\delta_A)_p(\langle x, y\rangle) = \{q : p \sqsubseteq q \text{ and } \langle A_{pq}(x), A_{pq}(y)\rangle \in \Delta_q\}$$

as required. □

EXERCISE 6. Use Theorem 1 to prove the Truth Lemma for the case that φ has the form $(t \approx u)$. □

The definition of $\forall_A$ uses the operation of exponentiation in $\mathbf{Set}^{\mathbf{P}}$. Given functors F and G from $\mathbf{P}$ to $\mathbf{Set}$, this operation produces a functor $G^F : \mathbf{P} \to \mathbf{Set}$ consisting of a collection $\{(G^F)_p : p \in P\}$ of sets indexed by P, together with transitions $(G^F)_{pq} : (G^F)_p \to (G^F)_q$ whenever $p \sqsubseteq q$. Now for each p we define the *restriction* of F to the category $[p)$ to be the functor $F \restriction p : [p) \to \mathbf{Set}$ that assigns to each object $q \in [p)$ the set F_q, and to each arrow $q \to r$ in $[p)$ (i.e. $p \sqsubseteq q \sqsubseteq r$) the function F_{qr}. Similarly we define the functor $G \restriction p$, and then put

$$(G^F)_p = \{\sigma : F \restriction p \xrightarrow{\sigma} G \restriction p\}$$

to be the set of all natural transformations from $F \restriction p$ to $G \restriction p$. Thus an element σ of $(G^F)_p$ may be directly described as a collection $\{\sigma_q : p \sqsubseteq q\}$ of functions, indexed by the members of $[p)$, with $\sigma_q : F_q \to G_q$, such that

$$\begin{tikzcd} F_q \arrow[r, "\sigma_q"] \arrow[d, "F_{qr}"'] & G_q \arrow[d, "G_{qr}"] \\ F_r \arrow[r, "\sigma_r"] & G_r \end{tikzcd}$$

commutes, whenever $p \sqsubseteq q \sqsubseteq r$.

Now one way of obtaining such a σ would be to take an arrow $\tau : F \dot{\to} G$ in $\mathbf{Set}^{\mathbf{P}}$ and restrict it to the subcategory $[p)$, i.e. let $\sigma = \{\tau_q : p \sqsubseteq q\}$. This process also yields the transition map $(G^F)_{pq}$ when $p \sqsubseteq q$. For $\sigma \in (G^F)_p$ we put

$$(G^F)_{pq}(\sigma) = \{\sigma_r : q \sqsubseteq r\}.$$

The arrow $ev : G^F \times F \dot{\to} G$ has p-th component

$$ev_p : (G^F)_p \times F_p \to G_p$$

given by

$$ev_p(\langle \sigma, x\rangle) = \sigma_p(x),$$

for each

$$\sigma \in (G^F)_p \quad \text{and} \quad x \in F_p.$$

EXERCISE 7. Verify that $(G^F)_{pq}(\sigma)$ is a natural transformation $F \upharpoonright q \rightarrow G \upharpoonright q$.

EXERCISE 8. Relate this construction to its analogue for $\mathbf{Set}^{\mathscr{C}}$ in Chapter 9. □

Now for an arrow $\tau : H \times F \rightarrow G$ the exponential adjoint

$$\hat{\tau} : H \rightarrow G^F$$

has as p-th component a function

$$\hat{\tau}_p : H_p \rightarrow (G^F)_p.$$

For each y in H_p,

$$\hat{\tau}_p(y) = \{\tau_q^y : p \sqsubseteq q\}$$

is a natural transformation

$$F \upharpoonright p \rightarrow G \upharpoonright p.$$

Its q-th component

$$\tau_q^y : F_q \rightarrow G_q$$

has, for each $x \in F_q$,

$$\tau_q^y(x) = \tau_q(H_{pq}(x), x).$$

The reader should now take a deep breath and go through that again. Having done so he may test his understanding of the definition in some further exercises:

EXERCISE 9. $true_A \circ pr_A : 1 \times A \rightarrow \Omega$ has as p-th component $\{0\} \times A_p \rightarrow \Omega_p$ the function assigning $[p)$ to each input $\langle 0, x \rangle$.

EXERCISE 10. The p-th component $\ulcorner true_A \urcorner_p : \{0\} \rightarrow (\Omega^A)_p$ of $\ulcorner true_A \urcorner : 1 \rightarrow \Omega^A$ may be identified with the natural transformation $\sigma : A \upharpoonright p \rightarrow \Omega \upharpoonright p$ that has $\sigma_q : A_q \rightarrow \Omega_q$, where $p \sqsubseteq q$, given by $\sigma_q(x) = [q)$, all $x \in A_q$. Thus $\sigma_q = true_q \circ !_{A_q}$, i.e. $\ulcorner true_A \urcorner_p(0) = \{true_q \circ !_{A_q} : p \sqsubseteq q\}$. □

THEOREM 2. *$\forall_A : \Omega^A \rightarrow \Omega$ has*

$$(\forall_A)_p : (\Omega^A)_p \rightarrow \Omega_p$$

given by

$$(\forall_A)_p(\sigma)=\{q:p\sqsubseteq q, \text{ and for every } r \text{ with } q\sqsubseteq r, \text{ and every } x\in A_r,\ \sigma_r(x)=[r)\}.$$

PROOF. For $\sigma\in(\Omega^A)_p$, since $\forall_A$ is the character of $\ulcorner true_A \urcorner$ we have

$$\begin{aligned}(\forall_A)_p(\sigma)&=\{q: p\sqsubseteq q \text{ and } (\Omega^A)_{pq}(\sigma)=\ulcorner true_A \urcorner_q(0)\}\\&=\{q: p\sqsubseteq q \text{ and } \{\sigma_r: q\sqsubseteq r\}=\{true_r\circ !_{A_r}: q\sqsubseteq r\}\}\\&=\{q: p\sqsubseteq q, \text{ and if } q\sqsubseteq r \text{ then } \sigma_r=true_r\circ !_{A_r}\}\end{aligned}$$

from which the theorem follows. □

If, for each p, we define $\in_p\subseteq(\Omega^A)_p\times A_p$ to be the set $\in_p=\{\langle\sigma,x\rangle: \sigma_p(x)=[p)\}$ then

$$\begin{array}{ccc}\in_p & \xrightarrow{(\in_A)_p} & (\Omega^A)_p\times A_p\\ \downarrow & & \downarrow{\scriptstyle (ev_A)_p}\\ 1 & \xrightarrow{true_p} & \Omega_p\end{array}$$

is a pullback, by §10.3, and the description of ev_A given above. Thus the inclusions $(\in_A)_p$ are the components of the "membership relation" on A, i.e. the arrow $\in_A:\in\rightarrowtail\Omega^A\times A$ whose character is ev_A.

EXERCISE 11. The collection $\{\in_p: p\in P\}$ gives rise to a functor (**Set**$^{\mathbf{P}}$-object) $\in$ as just mentioned. What are its transitions $\in_{pq}$?

EXERCISE 12. Show that the component $(p_A\circ\in_A)_p$ of the composite of $\in_A$ and the first projection $p_A:\Omega^A\times A\to\Omega^A$ has $(p_A\circ\in_A)_p(\langle\sigma,x\rangle)=\sigma$.

EXERCISE 13. Let ι be the image arrow of $p_A\circ\in_A$. Show that the p-th component of ι is the inclusion

$$\iota_p\hookrightarrow(\Omega^A)_p,$$

where

$$\iota_p=\{\sigma: \text{for some } x\in A_p, \langle\sigma,x\rangle\in\in_p\}.$$ □

THEOREM 3. $\exists_A:\Omega^A\to\Omega$ *has*

$$(\exists_A)_p:(\Omega^A)_p\to\Omega_p$$

given by

$$(\exists_A)_p(\sigma) = \{q\colon p \sqsubseteq q \textit{ and for some } x \in A_q,\ \sigma_q(x) = [q)\}$$

PROOF. $\exists_A$ is the character of the image arrow of $p_A \circ \in_A$. Using Exercise 13 then,

$$\begin{aligned}(\exists_A)_p(\sigma) &= \{q\colon p \sqsubseteq q \text{ and } (\Omega^A)_{pq}(\sigma) \in \iota_q\}\\ &= \{q\colon p \sqsubseteq q \text{ and for some } x \in A_q,\ \langle \sigma', x\rangle \in \in_q\}\\ &\qquad\qquad (\text{where } \sigma' = \Omega^A_{pq}(\sigma) = \{\sigma_r\colon q \sqsubseteq r\})\\ &= \{q\colon p \sqsubseteq q \text{ and for some } x \in A_q,\ \sigma'_q(x) = [q)\},\end{aligned}$$

and since $\sigma'_q = \sigma_q$, the result follows. □

The descriptions of $\forall_A$ and $\exists_A$ in Theorems 2 and 3 reflect the structure of the satisfaction clauses for $\forall$ and $\exists$ in Kripke models. The explicit link is given by

THEOREM 4. *For each $\mathscr{L}$-formula φ and appropriate m, the* $\mathbf{Set}^{\mathbf{P}}$*-arrow*

$$|\varphi|^m_i\colon A^m \dot{\to} \Omega^A$$

has as p-th component the function

$$f_p\colon A^m_p \to (\Omega^A)_p,$$

which assigns to $\langle x_1, \ldots, x_m\rangle \in A^m_p$ the natural transformation

$$f_p(\langle x_1, \ldots, x_m\rangle) = \{\sigma_q : p \sqsubseteq q\} \textit{ from } A \upharpoonright p \textit{ to } \Omega \upharpoonright p,$$

with $\sigma_q\colon A_q \to \Omega_q$ having

$$\sigma_q(x) = [\![\varphi]\!]^m_q(\langle x^q_1, \ldots, x^q_{i-1}, x, x^q_{i+1}, \ldots, x^q_m\rangle)$$ □

EXERCISE 14. Prove Theorem 4.

EXERCISE 15. Show that $[\![\exists v_i \varphi]\!]^m_p\colon A^m_p \to \Omega_p$ assigns to $\langle x_1, \ldots, x_m\rangle \in A^m_p$ the collection

$$\{q\colon\ p \sqsubseteq q \text{ and for some } x \in A_q,\ [\![\varphi]\!]^m_q(\langle x^q_1, \ldots, x^q_{i-1}, x, x^q_{i+1}, \ldots, x^q_m\rangle) = [q)\}.$$

EXERCISE 16. Derive the corresponding description of $[\![\forall v_i \varphi]\!]^m_p$ in terms of the $[\![\varphi]\!]^m_q$'s.

EXERCISE 17. Hence complete the inductive proof of the Truth Lemma. □

11.7. Completeness

Our first application of the Truth Lemma is a description of $[\![\varphi]\!]_{\mathfrak{A}}: A^n \to \Omega$, in $\mathbf{Set}^{\mathbf{P}}$, where φ has index n.

THEOREM 1. $[\![\varphi]\!]_p : A_p^n \to \Omega_p$ has

$$[\![\varphi]\!]_p(\langle x_1, \ldots, x_n \rangle) = \{q \colon p \sqsubseteq q \textit{ and } \mathfrak{A} \models_q \varphi[x_1^q, \ldots, x_n^q]\}.$$

PROOF. Exercise – use the fact that there is a commuting triangle

$$\begin{tikzcd} A_p^n \arrow[rr] \arrow[dr, "{[\![\Phi]\!]_p}"'] & & A_p^m \arrow[dl, "{[\![\Phi]\!]_p^m}"] \\ & \Omega_p & \end{tikzcd}$$

whenever m is appropriate to φ. □

THEOREM 2. *For any $\mathscr{L}$-model $\mathfrak{A}$ based on* $\mathbf{P}$, *and associated* $\mathbf{Set}^{\mathbf{P}}$ *model* $\mathfrak{A}^*$, *we have for all $\mathscr{L}$-formulae* φ,

$$\mathfrak{A} \models \varphi \quad \textit{iff} \quad \mathfrak{A}^* \overset{\mathbf{Set}^{\mathbf{P}}}{\models\!\!=} \varphi.$$

PROOF. Take any p, and $x_1, \ldots, x_n \in A_p^n$, where n is the index of φ. Then

$$p \in [\![\varphi]\!]_p(\langle x_1, \ldots, x_n \rangle) \quad \text{iff} \quad [\![\varphi]\!]_p(\langle x_1, \ldots, x_n \rangle) = [p)$$

by properties of hereditary sets (§10.2, Exercise 3(ii)). Thus by Theorem 1

$$\mathfrak{A} \models_p \varphi[x_1, \ldots, x_n] \quad \text{iff} \quad [\![\varphi]\!]_p(\langle x_1, \ldots, x_n \rangle) = (\mathit{true}_{A^n})_p \\ (\langle x_1, \ldots, x_n \rangle).$$

Since this is the case for all n-length sequences from A_p, we have

$$\mathfrak{A} \models_p \varphi \quad \text{iff} \quad [\![\varphi]\!]_p = (\mathit{true}_{A^n})_p.$$

Since that is the case for all $p \in P$,

$$\mathfrak{A} \models \varphi \quad \text{iff} \quad [\![\varphi]\!]_{\mathfrak{A}} = \mathit{true}_{A^n}.$$

□

Now by the methods used by Thomason [68] (and also by Fitting [69]), we can construct a canonical poset $\mathbf{P}_{\mathscr{L}}$, and a canonical model $\mathfrak{A}_{\mathscr{L}}$ based on $\mathbf{P}_{\mathscr{L}}$ such that for any φ,

$$\mathfrak{A}_{\mathscr{L}} \models \varphi \quad \textit{iff} \quad \vdash_{\overline{\mathrm{IL}}} \varphi.$$

(Thomason's models interpret $\approx$ as an equivalence relation E_p on A_p. However by taking A_p instead to be the set of E_p-equivalence classes, and A_{pq} the transition that maps the E_p-equivalence class of x to the E_q-equivalence class of x, we realise $\mathfrak{A}_{\mathscr{L}}$ as a canonical IL-model on which $\approx$ is interpreted as the diagonal relation Δ.)

Now if $\mathfrak{A}^*_{\mathscr{L}}$ is the associated model in the topos $\mathscr{E}_{\mathscr{L}} = \mathbf{Set}^{\mathbf{P}_{\mathscr{L}}}$, by Theorem 2 we have

$$\mathfrak{A}^*_{\mathscr{L}} \models\!\!\!\!\overset{\mathscr{E}_{\mathscr{L}}}{=\!=} \varphi \quad \textit{iff} \quad \vdash_{\overline{\mathrm{IL}}} \varphi.$$

Hence, with the Soundness Theorem we get

$$\mathscr{E}_{\mathscr{L}} \models \varphi \quad \textit{iff} \quad \vdash_{\overline{\mathrm{IL}}} \varphi.$$

From this follows a general

COMPLETENESS THEOREM. *If φ is valid in every topos, then $\vdash_{\mathrm{IL}} \varphi$.*

An example of a topos model in which the Law of Excluded Middle fails is now readily given. We take $\mathbf{P}$ as the ordinal poset $\mathbf{2} = \langle\{0, 1\}, \leqslant\rangle$. $\mathfrak{A}$ has

$$\mathfrak{A}_0 = \langle\{b, c\}, R_0, c\rangle$$

$$\mathfrak{A}_1 = \langle\{c\}, R_1, c\rangle,$$

where b and c are two distinct entities, R_0 and R_1 are any relations on $A_0 = \{b, c\}$ and $A_1 = \{c\}$ of the reader's fancy, and $A_{01} : \{b, c\} \to \{c\}$ is the only map it can be. Then if φ is the sentence $(\forall v_1)(v_1 \approx \mathbf{c})$, φ is true at $\mathfrak{A}_1$ but false at $\mathfrak{A}_0$.

Thus we have not $\mathfrak{A} \models_0 \varphi$, but we do have $\mathfrak{A} \models_1 \varphi$, so not $\mathfrak{A} \models_0 {\sim}\varphi$, hence not $\mathfrak{A} \models_0 \varphi \vee {\sim}\varphi$.

Now we saw in §7.4 that, for propositional logic, a topos can validate all instances of $\alpha \vee {\sim}\alpha$ (since Sub(1) is a **BA**) but still not be Boolean (since Sub(Ω) is not a **BA**). This occurs for example in the topos $\mathbf{M_2}$. Similarly we have $\mathbf{M_2} \models \varphi \vee {\sim}\varphi$ whenever φ is an $\mathscr{L}$-sentence, since then $[\![\varphi]\!]_{\mathfrak{A}}$ is a truth-value $1 \to \Omega$. However the situation is not the same for open formulae.

THEOREM 3. *If* $\mathscr{E} \models \varphi \vee {\sim}\varphi$ *for every* $\mathscr{L}$*-formula* φ*, then* $\mathscr{E}$ *is a Boolean topos.*

PROOF. Let $\mathfrak{A}$ be a $\mathscr{L}$-model of the form $\langle \Omega, r, true \rangle$, i.e. a model in which $\mathbf{c}$ is interpreted as the element $true: 1 \to \Omega$ of Ω. Let $\varphi(v_1)$ be the formula $(v_1 \approx \mathbf{c})$. Then $[\![\varphi]\!]_{\mathfrak{A}}: \Omega \to \Omega$ is $\delta_\Omega \circ \langle \mathbf{1}_\Omega, true_\Omega \rangle$. By Exercise 2 of §5.1, the equaliser of $\mathbf{1}_\Omega$ and $true_\Omega$ is $true$ $\mathbf{1} \to \Omega$ so (*Theorem* 2, §11.5) $[\![\varphi]\!]_{\mathfrak{A}} = \chi_{true} = \mathbf{1}_\Omega$. But $\mathfrak{A} \models \varphi \vee {\sim}\varphi$, so $[\![\varphi \vee {\sim}\varphi]\!]_{\mathfrak{A}} = true_\Omega$, i.e.

$$[\![\varphi]\!]_{\mathfrak{A}} \cup (\neg \circ [\![\varphi]\!]_{\mathfrak{A}}) = true_\Omega$$

i.e.

$$\mathbf{1}_\Omega \cup (\neg \circ \mathbf{1}_\Omega) = true_\Omega$$

which by Theorem 3 of §7.4 implies that Sub(Ω) is a **BA**. □

EXERCISE. The proof of Theorem 3 used the fact that $\mathscr{L}$ had an individual constant. Show that this assumption is not needed, by considering the process of "adjoining" a constant to a language. □

11.8. Existence and free logic

The assumption of non-emptiness, $(\mathscr{E}(\mathbf{1}, a) \neq \emptyset)$, for $\mathscr{L}$-models in a topos has been needed, not just for interpreting constants, but also for our definition of $[\![\varphi]\!]_{\mathfrak{A}}$ and hence of truth in a model. In **Set** of course the only empty object is the null set $\emptyset$, and if *that* is admitted as a model, then as Andrzej Mostowski [51] observed, the rule of DETACHMENT no longer preserves validity. Informally we regard any universal sentence $\forall v\varphi$, or any open formula $\varphi(v)$, as being true of $\emptyset$, since there is nothing in $\emptyset$ of which φ is false. On the other hand an existential statement $\exists v\varphi$ is false in $\emptyset$ since the latter has no element of which φ is true. More formally, since $2^0 = \{0\}$, $\forall_\emptyset: \{\emptyset\} \to 2$ is simply the map *true*, while $\exists_\emptyset: \{\emptyset\} \to 2$ is the map *false*. Moreover if φ has index $n \geqslant 1$, then $\emptyset^n = \emptyset$, so $[\![\varphi]\!]_{\mathfrak{A}}: \emptyset \to 2$ is the empty map, i.e. the map $true_\emptyset$. Thus, e.g., the open formulae

$$(v_1 \approx v_1) \supset \exists v_1(v_1 \approx v_1)$$

and

$$(v_1 \approx v_1)$$

and true in $\emptyset$, while the sentence

$$\exists v_1(v_1 \approx v_1)$$

is false.

There are two basic methods that have been developed of doing logic when empty models are allowed (so called "free" logic). Mostowski modified the rule of DETACHMENT to read:

> *From φ and $\varphi \supset \psi$ infer ψ, provided that all variables free in φ are free in ψ.*

(Alternatively we allow ψ to be detached only if $\exists v(v \approx v)$ has also been derived for each variable v that is free in φ.)

This approach is used in the topos setting by the Montréal school (cf. Robitaille-Giguère [75], Boileau [75]). The other method is to introduce a special *existence* predicate $\mathbf{E}$, with $\mathbf{E}(t)$ read "t exists", and to modify the definition of satisfaction to accommodate the possibility that "t may not denote anything". This notion has been studied by Dana Scott and Michael Fourman [74], and has a very interesting interpretation for sheaves and bundles, as well as Kripke models.

Let us consider an object $a = (A, f)$ in the topos $\mathbf{Bn}(I)$ of bundles over I. An element $s : 1 \to a$ of a is a *global* section $s : I \to A$ of the bundle, picking one "germ" $s(i)$ out of each stalk A_i. But if the stalk is empty, $A_i = \emptyset$, then no such $s(i)$ exists. So we see that if a has at least one empty stalk (because f is not epic), that is enough to prevent there being any elements $1 \to a$. (We also see that $\mathbf{Bn}(I)$ has many significant and non-isomorphic objects that are empty in the categorial sense). At best we can consider *local* sections $s : D \to A$, with $f \circ s = D \hookrightarrow I$, defined on some subset D of I. This possible if $A_i \neq \emptyset$ for all $i \in D$. Recall (§4.4 Example 6) that the set $D \subseteq I$ can be regarded as a subobject of the terminal object 1 under the isomorphism

$$\mathcal{P}(I) \cong \mathbf{Bn}(I)(1, \Omega) \cong \mathrm{Sub}(1)$$

that obtains for $\mathbf{Bn}(I)$.

A similar situation arises in the context of a $\mathbf{Set}^{\mathbf{P}}$ model $\langle A, r, f_c \rangle$. If the object (functor) A has element $f_c : 1 \dashrightarrow A$, then for each p, $(f_c)_p(0) \in A_p$, so $A_p \neq \emptyset$. So if just one A_p were empty, A would have no elements. However even if A does have elements, it may be undesirable to interpret a constant as an arrow of the form $1 \dashrightarrow A$. We may for instance wish to expand our language $\mathcal{L}$ to include a "name" $\mathbf{c}_0$ for a particular element c_0 of some A_p. $\mathbf{c}_0$ would then be interpreted (as c_0^q) only in those $\mathfrak{A}_q$ for $q \in [p)$. Notice that $[p)$ being hereditary can be identified (Exercise 2, §10.6) with a subobject $D \rightarrowtail 1$ of the terminal object in $\mathbf{Set}^{\mathbf{P}}$. The interpretation of $\mathbf{c}_0$ then yields an arrow $f_{c_0} : D \dashrightarrow A$ with $(f_{c_0})_q : D_q \to A_q$ picking out c_0^q whenever $p \sqsubseteq q$, i.e. $D_q = \{0\}$, and $(f_{c_0})_q = ! : \emptyset \to A_q$ otherwise.

We are thus lead to replace elements $1 \to a$ of an object a by arrows $d \to a$ whose domains are subobjects $d \rightarrowtail 1$ of the terminal 1. Such things are called *partial elements* of a. This comes from the more general notion of *partial arrow*. In **Set** we say that f is a *partial function* from A to B, written $f: A \rightsquigarrow B$, if f is a function from a subset of A to B, i.e. $\operatorname{dom} f \subseteq A$ and $\operatorname{cod} f = B$. In a general category $\mathscr{C}$ we put $f: a \rightsquigarrow b$ if f is a $\mathscr{C}$-arrow with $\operatorname{cod} f = b$, and there is a $\mathscr{C}$-monic $\operatorname{dom} f \rightarrowtail a$. Thus a partial element of a is an arrow $s: 1 \rightsquigarrow a$.

Now in the **Set** case, if $f: A \rightsquigarrow B$ there may be some elements $x \in A$ with $x \notin \operatorname{dom} f$. This is often expressed as "$f(x)$ is undefined". But if we introduce some new entity $*$, with $* \notin B$, and write "$f(x) = *$" whenever $x \notin \operatorname{dom} f$ then we can regard f as being defined on all of A (we need $* \notin B$, or else "$f(x) = *$" could be compatible with $x \in \operatorname{dom} f$). A convenient choice for $*$ would be the null set $\emptyset$ ($f(x) = \emptyset$ means "x has null denotation"). However it may be that $\emptyset \in B$. We can get around this by replacing each element y of B by the singleton subset $\{y\}$ and replacing B by the collection of these singletons, i.e. we replace B by its isomorphic copy $B' = \{\{y\} : y \in B\}$. Then $\emptyset \notin B'$ so we add $\emptyset$ to B' to form

$$\tilde{B} = \{\{y\} : y \in B\} \cup \{\emptyset\}.$$

Then given $f: D \to B$, with $D \subseteq A$, define $\tilde{f}: A \to \tilde{B}$ by

$$\tilde{f}(x) = \begin{cases} \{f(x)\} & \text{if} \quad x \in \operatorname{dom} f = D \\ \emptyset & \text{otherwise} \end{cases}$$

It is clear then that

$$\begin{tikzcd} D \arrow[r, hook] \arrow[d, "f"'] & A \arrow[d, "\tilde{f}"] \\ B \arrow[r, tail, "\eta_B"] & \tilde{B} \end{tikzcd}$$

commutes, where $\eta_B(y) = \{y\}$, all $y \in B$.

Moreover the pullback of η_B and $\tilde{f}$ has domain

$$\begin{aligned} \{\langle y, x \rangle : \{y\} = \tilde{f}(x)\} &= \{\langle y, x \rangle : x \in D \text{ and } y = f(x)\} \\ &= \{\langle f(x), x \rangle : x \in D\} \cong D. \end{aligned}$$

Thus, knowing $\tilde{f}$, we pull it back along η_B to recover f. In fact (exercise) it can be shown that $\tilde{f}$ as defined is the only map $A \to \tilde{B}$ making this diagram a pullback. Thus the arrow $\eta_B : B \to \tilde{B}$ is a generalisation of *true*: $1 \to 2$. It acts as a "partial function classifier", providing a bijective correspondence between (equivalence classes of) partial maps $f: A \rightsquigarrow B$ with codomain B, and "total" maps $\tilde{f}: A \to \tilde{B}$ with codomain $\tilde{B}$.

PARTIAL ARROW CLASSIFIER THEOREM. *If $\mathscr{E}$ is any topos, then for each $\mathscr{E}$-object b there is an $\mathscr{E}$-object $\tilde{b}$ and an arrow $\eta_b : b \rightarrowtail \tilde{b}$ such that given any pair (f, g) of arrows as in the following diagram, there is one and only one arrow $\tilde{f}$ as shown that makes the diagram a pullback.*

$$\begin{tikzcd} d \arrow[r, tail, "g"] \arrow[d, "f"'] & a \arrow[d, "\tilde{f}"] \\ b \arrow[r, tail, "\eta_b"] & \tilde{b} \end{tikzcd}$$

□

The proof of this theorem is given in detail by Kock and Wraith [71]. To define η_b, the arrow $\{\cdot\}_b : b \to \Omega^b$ is introduced as the exponential adjoint to $\delta_b : b \times b \to \Omega$ (in **Set** $\{\cdot\}_b$ maps y to $\{y\}$). $\{\cdot\}_b$ proves to be monic, and so is $\langle\{\cdot\}_b, \mathbf{1}_b\rangle : b \to \Omega^b \times b$. The latter has a character $h : \Omega^b \times b \to \Omega$ and this in turn has an exponential adjoint $\hat{h} : \Omega^b \to \Omega^b$ (in **Set** $\hat{h}$ is the identity on singletons and maps all other subsets of b to $\emptyset$). It is then shown that $\hat{h} \circ \{\cdot\}_b = \{\cdot\}_b$, so

$$\begin{tikzcd} \tilde{b} \arrow[r, tail] & \Omega^b \arrow[r, shift left, "1"] \arrow[r, shift right, "\hat{h}"'] & \Omega^b \\ b \arrow[u, dashed, "\eta_b"] \arrow[ur, "\{\cdot\}_b"'] & & \end{tikzcd}$$

defining $\tilde{b}$ as the (domain of the) equaliser of $\mathbf{1}_{\Omega^b}$ and $\hat{h}$, η_b is the unique arrow factoring $\{\cdot\}_b$ through $\tilde{b}$.

EXERCISE 1. Examine the details of this construction in **Set**.

EXERCISE 2. Show that

$$\eta_1 : 1 \to \tilde{1}$$

is a subobject classifier in any topos. □

Returning now to free logic, a semantical theory in the classical case may be developed by allowing variables and constants to be interpreted in a model $\mathfrak{A}=\langle A,\ldots\rangle$ as elements of $A\cup\{*\}$. The existence predicate $\mathbf{E}$ is interpreted as the set (one-place relation) A, i.e. for $a\in A\cup\{*\}$

$$\mathfrak{A}\models \mathbf{E}(v)[a] \quad \text{iff} \quad a\in A,$$

while the range of quantification remains A itself, i.e.

$$\mathfrak{A}\models \forall v\varphi \quad \text{iff for all} \quad a\in A, \quad \mathfrak{A}\models\varphi[a].$$

Under this semantics, DETACHMENT preserves validity, while the axioms UI and EG are modified to

$$(\forall v)\varphi\wedge\mathbf{E}(t)\supset\varphi(v/t)$$

and

$$\varphi(v/t)\wedge\mathbf{E}(t)\supset(\exists v)\varphi$$

More details of this type of theory may be found in Scott [67] – where, as is often done, $\mathbf{E}(t)$ is taken to stand for a formula of the form $\exists v(v\approx t)$.

Moving to models $\mathfrak{A}=\langle a,\ldots\rangle$ in a general topos, we see that instead of dealing with partial elements $1 \rightsquigarrow a$ as suggested by the examples discussed earlier, we may deal with elements $1\to\tilde{a}$ of the "object of partial elements of a" ($\tilde{a}$ always has elements, since a has at least the partial element $0\rightarrowtail a$). The interpretation of the predicate $\mathbf{E}$ becomes the character $e:\tilde{a}\to\Omega$ of the monic $\eta_a:a\rightarrowtail\tilde{a}$, and each formula φ determines an arrow $[\![\varphi]\!]_{\mathfrak{A}}:(\tilde{a})^n\to\Omega$. Then given a partial element $f_c:1\rightsquigarrow a$,

$$\begin{tikzcd}
\operatorname{dom} f_c \arrow[r, tail] \arrow[d, "f_c"'] & 1 \arrow[d, "\tilde{f}_c"] \\
a \arrow[r, tail, "\eta_a"] \arrow[d] & \tilde{a} \arrow[d, "e"] \\
1 \arrow[r, "true"] & \Omega
\end{tikzcd}$$

we have $[\![\mathbf{E}(\mathbf{c})]\!]=e\circ\tilde{f}_c$, and so as the diagram indicates,

$$[\![\mathbf{E}(\mathbf{c})]\!] \text{ is the character of } \operatorname{dom} f_c\rightarrowtail 1.$$

Hence

$$\begin{aligned}
\mathfrak{A}\models\mathbf{E}(\mathbf{c}) \quad &\text{iff} \quad [\![\mathbf{E}(\mathbf{c})]\!]_{\mathfrak{A}}=true\\
&\text{iff} \quad \operatorname{dom} f_c\rightarrowtail 1\simeq 1_1 \text{ in } \mathrm{Sub}(1)\\
&\text{iff} \quad f_c \text{ is a "total" element of } a.
\end{aligned}$$

In the case of a bundle $a=\langle A, f\rangle$, $\tilde{a}$ is a bundle of (disjoint) copies of the sets $\tilde{A}_i$, with η_a acting on the stalk A_i being the map $\eta_{A_i}: A_i \rightarrow \tilde{A}_i$. An element $\tilde{f}_c: 1 \rightarrow \tilde{a}$ is essentially a partial element $f_c: 1 \rightsquigarrow a$, i.e. a local section $f_c: I \rightsquigarrow A$, with

$$\operatorname{dom} f_c=\{i: \tilde{f}_c(i) \neq \emptyset \text{ in } \tilde{A}_i\}$$

Identifying truth values with subsets of I we may then simply say that

$$[\![\mathbf{E}(\mathbf{c})]\!]_{\mathfrak{A}}=\operatorname{dom} f_c,$$

and

$$\mathfrak{A} \models \mathbf{E}(\mathbf{c}) \quad \text{iff} \quad f_c \textit{ is a global section.}$$

Now the set $\tilde{A}$ is isomorphic in **Set** to $A+1$, the latter being the disjoint union of A and $\{0\}$. The iso arrow in question is the co-product arrow $[\eta_A, \emptyset_A]$, where $\emptyset_A:\{0\} \rightarrow \tilde{A}$ has $\emptyset_A(0)=\emptyset$. Thus $\emptyset_A$ "is" the element of $\tilde{A}$ corresponding to the partial element $!: \emptyset \rightarrow A$ of A. The obvious question then arises as to whether $\tilde{a}$ is isomorphic to $a+1$ in general. If this were so, we would have in particular $\tilde{1} \cong 1+1$. But (Exercise 2 above) $\tilde{1}$ is an object of truth values, and we know that $\Omega \cong 1+1$ only in *Boolean* topoi.

To formulate the situation precisely, let $\emptyset_a: 1 \rightarrow \tilde{a}$, where a is an object of topos $\mathscr{E}$, be the unique $\mathscr{E}$-arrow making

$$\begin{tikzcd}
0 \arrow[r, tail] \arrow[d] & 1 \arrow[d, "\emptyset_a"] \\
a \arrow[r, tail, "\eta_a"'] & \tilde{a}
\end{tikzcd}$$

a pullback, and form the co-product arrow

$$\begin{tikzcd}
a \arrow[r] \arrow[dr, "\eta_a"'] & a+1 \arrow[d, dashed, "{[\eta_a, \emptyset_a]}"] & 1 \arrow[l] \arrow[dl, "\emptyset_a"] \\
& \tilde{a} &
\end{tikzcd}$$

LEMMA. *In* $\operatorname{Sub}(\tilde{a})$, $\emptyset_a$ *is the pseudo-complement of* η_a.

PROOF. If $-\eta_a : -a \rightarrowtail \tilde{a}$ is the pseudo-complement of η_a, then $\eta_a \cap -\eta_a \simeq 0_{\tilde{a}}$ (§7.2) and so

$$\begin{tikzcd} 0 \arrow[r, tail, "!"] \arrow[d, tail, "!"'] & -a \arrow[d, tail, "-\eta_a"] \\ a \arrow[r, tail, "\eta_a"] & \tilde{a} \end{tikzcd}$$

is a pullback. But the Partial Arrow Classifier Theorem then implies that $-\eta_a$ is the only arrow that makes the diagram thus a pullback.

Now consider

$$\begin{tikzcd} 0 \arrow[r, "!"] \arrow[d] & -a \arrow[d, "!"] \\ 0 \arrow[r, "!"] \arrow[d] & 1 \arrow[d, "\emptyset_a"] \\ a \arrow[r, "\eta_a"] & \tilde{a} \end{tikzcd}$$

The top square is a pullback (exercise), and the bottom square is the pullback defining $\emptyset_a$. Hence by the PBL the outer rectangle is a pullback. In view of the unique role of $-\eta_a$ just mentioned, it follows that

$$\begin{tikzcd} -a \arrow[d, "!"'] \arrow[dr, "-\eta_a"] & \\ 1 \arrow[r, "\emptyset_a"'] & \tilde{a} \end{tikzcd}$$

commutes, showing that $-\eta_a \subseteq \emptyset_a$. But the pullback square defining $\emptyset_a$ shows that $\eta_a \cap \emptyset_a \simeq 0_{\tilde{a}}$. In view of the description of $-\eta_a$ as the largest element of Sub($\tilde{a}$) disjoint from η_a, we get then $\emptyset_a \subseteq -\eta_a$, and altogether $\emptyset_a \simeq -\eta_a$. □

THEOREM. *In any topos $\mathscr{E}$, the following are equivalent*

(1) *For all $\mathscr{E}$-objects a, $[\eta_a, \emptyset_a] : a + 1 \to \tilde{a}$ is iso*
(2) $[\eta_1, \emptyset_1] : 1 + 1 \to \tilde{1}$ *is iso*
(3) $\mathscr{E}$ *is Boolean.*

PROOF. Clearly (1) implies (2). But $\emptyset_1$ is defined by the pullback

$$\begin{tikzcd} 0 \arrow[r, "!"] \arrow[d] & 1 \arrow[d, "\phi_1"] \\ 1 \arrow[r, "\eta_1"] & \tilde{1} \end{tikzcd}$$

which shows that when η_1 is used as subobject classifier, i.e. $\eta_1 = true$, then $\emptyset_1$ is the arrow *false*. Hence (2) asserts that the co-product $[true, false]$ is iso, which yields Booleanness as we saw in §7.3.

Finally, if (3) holds, then applying the Lemma to any $\mathscr{E}$-object a, we have

$$\eta_a \cup \emptyset_a \simeq \eta_a \cup -\eta_a \simeq \mathbf{1}_{\tilde{a}}.$$

But η_a and $\emptyset_a$ are disjoint monics, so the Lemma following Theorem 3 of §5.4 implies that $[\eta_a, \emptyset_a]$ is monic, and hence is its own epi-monic factorisation, i.e. $\eta_a \cup \emptyset_a \simeq [\eta_a, \emptyset_a]$ in $\mathrm{Sub}(\tilde{a})$. Thus $\mathbf{1}_{\tilde{a}} \simeq [\eta_a, \emptyset_a]$, and so the latter is iso (Exercise, 7.2.1). □

EXERCISE 3. Let $a = f : A \to B$ be an object in the topos $\mathbf{Set}^{\to}$ of set functions. Form the co-product function

$$A + B \xrightarrow{[f, \mathrm{id}_B]} B,$$

and let $[f, \mathrm{id}_B]^{\sim} : (A+B)^{\sim} \to \tilde{B}$ be defined by the $\sim$-construction in **Set**. Then

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] & (A+B)^{\sim} \arrow[d, "{[f, \mathrm{id}_B]^{\sim}}"] \\ B \arrow[r, "\eta_B"] & \tilde{B} \end{tikzcd}$$

commutes, where g is the composite of $i_A : A \to A+B$ and η_{A+B}.

Show that $\eta_a : a \to \tilde{a}$ is a partial arrow classifier with respect to a in $\mathbf{Set}^{\to}$, where $\tilde{a}$ is the function $[f, \mathrm{id}_B]^{\sim}$ and η_a is the pair (g, η_B).

Apply the construction just given to the terminal 1 in $\mathbf{Set}^{\to}$ to recover the description of the subobject classifier for $\mathbf{Set}^{\to}$ given in Chapter 4.

11.9. Heyting-valued sets

Building on the ideas of the previous section, we might regard an object in a topos as a "set-like" entity consisting of *potentially* existing (partially defined) elements, only some of which possess *actual* existence (are totally defined). The variables in a formula that are bound by quantifiers are then taken to range over actually existing elements. In the context of this "logic of partial elements" we distinguish two concepts of sameness. The sentence $\exists v(v \approx \mathbf{c})$ is tantamount to the assertion that the individual c exists, in that it asserts that there actually exists an individual that is equal to c. So the sentence

(i) $\quad \mathbf{E(c)} \equiv \exists v(v \approx \mathbf{c})$

is valid on this account. Here the symbol $\equiv$ is the *biconditional* connective read "if and only if". The expression $\varphi \equiv \psi$ is formally introduced as an abbreviation for the formula

$$(\varphi \supset \psi) \wedge (\psi \supset \varphi).$$

In arriving at (i) we have implicitly invoked the principle that anything equal to an existing entity must itself exist. But more strongly than this we are going to require that elements can only be equal if they exist. Equality implies existence, and we thus have

(ii) $\quad (v \approx w) \supset \mathbf{E}(v) \wedge \mathbf{E}(w)$

The other notion of sameness, for which we use the symbol $\approxeq$, is a weaker concept of *equivalence* which does not differentiate elements in regard to their lack of existence. Thus v and w will be equivalent if neither of them exists, or if they both exist and are equal ($\approx$). We can express this in a positive form as "if either of them exists then they are equal" (and hence the other exists by (ii)). Thus equivalence is characterised by

(iii) $\quad (v \approxeq w) \equiv (\mathbf{E}(v) \vee \mathbf{E}(w) \supset v \approx w).$

But then we see, conversely, that we may describe equality in terms of equivalence, since equal elements are those that exist and are equivalent, i.e.

(iiia) $\quad (v \approx w) \equiv ((v \approxeq w) \wedge \mathbf{E}(v) \wedge \mathbf{E}(w)).$

These notions are simply illustrated in the topos $\mathbf{Bn}(I)$. Let f and g be two partial elements $I \rightsquigarrow A$ of a bundle $A \to I$ over I, and put

$$[\![f \approx g]\!] = \{i \in I \colon f(i) = g(i)\}$$

Then $[\![f \approx g]\!]$, being a subset of I, is a truth-value in $\mathbf{Bn}(I)$. We regard it as the truth-value of the statement "$f = g$", or alternatively as a measure of the extent to which f and g are equal. The expression "$f = g$" is interpreted to mean that $f(i)$ and $g(i)$ are both defined (i.e. i is a member of the domains of both f and g) and they are the same element of A. In particular we must have

$$[\![f \approx g]\!] \subseteq \operatorname{dom} f \cap \operatorname{dom} g$$

and so by the analysis of 11.8 we can put

$$[\![f \approx g]\!] \subseteq [\![\mathbf{E}(f)]\!] \cap [\![\mathbf{E}(g)]\!]$$

which accords with (ii) above.

Notice that

$$[\![f \approx f]\!] = \{i : f(i) = f(i)\} = \operatorname{dom} f = [\![\mathbf{E}(f)]\!]$$

and so $[\![f \approx f]\!]$ is a measure of the degree of existence of f.

For the weaker concept of sameness, we regard the local sections f and g as equivalent if they agree whenever they are defined. Thus as a measure of the extent of their equivalence we take those i where neither is defined, together with those where they are both defined and agree. Thus

$$\begin{aligned} [\![f \approxeq g]\!] &= -(\operatorname{dom} f \cup \operatorname{dom} g) \cup [\![f \approx g]\!] \\ &= -([\![\mathbf{E}(f)]\!] \cup [\![\mathbf{E}(g)]\!]) \cup [\![f \approx g]\!] \end{aligned}$$

which corresponds to (iii), since $-B \cup C = B \Rightarrow C$ in $\mathcal{P}(I)$.

Analogously, in $\mathbf{Top}(I)$ we define a measure of the degree of equality of partial elements (continuous local sections) of a sheaf of germs by putting

$$[\![f \approx g]\!] = \{i : f(i) = g(i)\}^{0},$$

applying the interior operator $(\ )^{0}$ to ensure that $[\![f \approx g]\!]$ is an open set, i.e. a truth-value. $[\![\mathbf{E}(f)]\!] = [\![f \approx f]\!]$ remains as $\operatorname{dom} f$, since local sections always have open domains. For equivalence we put

$$[\![f \approxeq g]\!] = [\![\mathbf{E}(f)]\!] \cup [\![\mathbf{E}(g)]\!] \Rightarrow [\![f \approx g]\!],$$

where $B \Rightarrow C = (-B \cup C)^{0}$ is the relative pseudo-complementation of open sets in I. Notice that whereas $[\![f \approx f]\!]$ may be a proper subset of I ("$f = f$" is not totally true) we always have $[\![f \approxeq f]\!] = I$.

Emerging from this discussion is a generalised concept of a "set" as consisting of a collection of (partial) elements, with some Heyting-algebra-valued measure of the degree of equality of these elements. This notion admits of an abstract axiomatic development in the following way:

Let $(\Omega, \sqsubseteq)$ be a *complete* Heyting algebra (**CHA**), i.e. an **HA** in which *every* subset $A \subseteq \Omega$ has a least upper bound, denoted $\bigsqcup A$, and a greatest lower bound, denoted $\bigsqcap A$, in Ω. (Recall the definitions of l.u.b. and g.l.b. given in §8.3). An *Ω-valued set* (*Ω-set*) is defined to be an entity $\mathbf{A}$ comprising a set A and a function $A \times A \to \Omega$ assigning to each ordered pair $\langle x, y\rangle$ of elements of A an element $[\![x \approx y]\!]_{\mathbf{A}}$ of Ω, satisfying

$$[\![x \approx y]\!]_{\mathbf{A}} \sqsubseteq [\![y \approx x]\!]_{\mathbf{A}}$$

and

$$[\![x \approx y]\!]_{\mathbf{A}} \sqcap [\![y \approx z]\!]_{\mathbf{A}} \sqsubseteq [\![x \approx z]\!]_{\mathbf{A}}$$

for all $x, y, z \in A$. These two conditions give the Ω-validity of the formulae

$$(x \approx y) \supset (y \approx x)$$
$$(x \approx y) \wedge (y \approx z) \supset (x \approx z)$$

that express the symmetry and transitivity of the equality relation. The element $[\![x \approx x]\!]_{\mathbf{A}}$ will often be denoted $[\![Ex]\!]_{\mathbf{A}}$. We introduce the definition

$$[\![x \approxeq y]\!]_{\mathbf{A}} = ([\![Ex]\!]_{\mathbf{A}} \sqcup [\![Ey]\!]_{\mathbf{A}}) \Rightarrow [\![x \approx y]\!]_{\mathbf{A}}$$

The $\mathbf{A}$-subscripts in these expressions will be deleted whenever the meaning is clear without them.

EXERCISE 1. Prove that the following conditions hold for any Ω-valued set:

$[\![x \approx y]\!] \sqsubseteq [\![Ex]\!]$

$[\![x \approx y]\!] = [\![x \approxeq y]\!] \sqcap [\![Ex]\!] \sqcap [\![Ey]\!]$

$[\![Ex]\!] \sqcap [\![x \approxeq y]\!] \sqsubseteq [\![Ey]\!]$

$[\![x \approxeq x]\!]$ is the unit (greatest element) of Ω

$[\![x \approxeq y]\!] \sqsubseteq [\![y \approxeq x]\!]$

$[\![x \approxeq y]\!] \sqcap [\![y \approxeq z]\!] \sqsubseteq [\![x \approxeq z]\!]$

$p \sqsubseteq [\![x \approxeq y]\!]$ iff $p \sqcap [\![Ex]\!] \sqsubseteq [\![x \approxeq y]\!]$ and $p \sqcap [\![Ey]\!] \sqsubseteq [\![x \approxeq y]\!]$. □

The justification for using the subobject-classifier symbol for our **CHA** is that the Ω-sets form the objects of a category, denoted Ω-**Set**, which is a topos, and in which the object of truth-values is Ω itself! More precisely, this object of truth-values is the Ω-set $\mathbf{\Omega}$ obtained by putting

$$[\![p \approx q]\!]_{\mathbf{\Omega}} = (p \Leftrightarrow q)$$

for each p, $q \in \Omega$, where

$$(p \Leftrightarrow q) = (p \Rightarrow q) \sqcap (q \Rightarrow p)$$

is the Ω-operation that interprets the biconditional connective $\equiv$. Since the members of Ω are going to serve as truth-values we will use the symbols $\perp$ and $\top$ to denote the least (zero) and greatest (unit) elements of Ω respectively.

EXERCISE 2. $[\![p \approx q]\!]_{\mathbf{\Omega}} = \top$ iff $p = q$.

EXERCISE 3. $[\![Ep]\!]_{\mathbf{\Omega}} = \top$.

EXERCISE 4. $[\![p \approx \top]\!]_{\mathbf{\Omega}} = p$.

EXERCISE 5. $[\![p \approx \perp]\!]_{\mathbf{\Omega}} = \neg p$ □

An arrow from **A** to **B** in Ω-**Set** may be thought of in the first instance as a function $f : A \to B$. Its graph would then be a subobject of $A \times B$ and so should correspond to a function of the form $A \times B \to \Omega$. We interpret the latter as assigning to $\langle x, y\rangle$ the truth-value $[\![f(x) \approx y]\!]$, giving the degree of equality of $f(x)$ and y, i.e. a measure of the extent to which y is the f-image of x. With this idea in mind we turn to the formal definition.

An arrow from **A** to **B** in Ω-**Set** is a function $f : A \times B \to \Omega$ satisfying

(iv) $[\![x \approx x']\!]_{\mathbf{A}} \sqcap f(\langle x, y\rangle) \sqsubseteq f(\langle x', y\rangle)$

(v) $f(\langle x, y\rangle) \sqcap [\![y \approx y']\!]_{\mathbf{B}} \sqsubseteq f(\langle x, y'\rangle)$

(vi) $f(\langle x, y\rangle) \sqcap f(\langle x, y'\rangle) \sqsubseteq [\![y \approx y']\!]_{\mathbf{B}}$

(vii) $[\![x \approx x]\!]_{\mathbf{A}} = \bigsqcup\{f(\langle x, y\rangle) : y \in B\}$

The first two conditions are laws of extensionality (indistinguishability of equals) and assert the Ω-validity of the formulae

$$(x \approx x') \wedge (f(x) \approx y) \supset (f(x') \approx y)$$
$$(f(x) \approx y) \wedge (y \approx y') \supset (f(x) \approx y')$$

(which are instances of the axiom $I2$ of §11.3). Condition (vi) gives the validity of the "unique output" property for the arrow f. It can be read "partial elements y and y' are each the f-image of x only to the extent that they are equal". To understand condition (vii) we note that the completeness of the **HA** Ω can be used to interpret an existential quantifier, by construing the latter as a (possibly infinite) disjunction (l.u.b.). That is, the sentence "there exists a $y \in B$ such that $\varphi(y)$" is construed as "$\varphi(y_1)$ or $\varphi(y_2)$ or $\varphi(y_3)$ or ..." where $y_1, y_2, \ldots$ run through all the members of B, and hence is given the truth-value

$$\bigsqcup\{[\![\varphi(y)]\!]: y \in B\}, \quad \text{or} \quad \bigsqcup_{y \in B} [\![\varphi(y)]\!].$$

(Dually, construing a universal quantifier as a conjunction, the sentence "for all $y \in B$, $\varphi(y)$" would be interpreted by

$$\bigsqcap\{[\![\varphi(y)]\!]: y \in B\}, \quad \text{or} \quad \bigsqcap_{y \in B} [\![\varphi(y)]\!].)$$

Thus we see that (vii) gives the validity of the statement that each $x \in A$ has some f-image $y \in B$, i.e. f is a total function. By giving an equation of the form $[\![Ex]\!] = [\![\varphi]\!]$ the suggestive reading "x exists to the extent that φ", we may read (vii) as "each element of **A** exists to the extent that it has an image in **B**".

In summary then, an arrow from **A** to **B** is represented, via its graph, as an extensional, functional and total Ω-valued relation from **A** to **B**. But then it is not hard to see that the equality relation on **A** satisfies these properties, i.e. the function $\langle x, y\rangle \mapsto [\![x \approx y]\!]_{\mathbf{A}}$ is an arrow $\mathbf{A} \to \mathbf{A}$ according to (iv)–(vii). And indeed it will be the identity arrow for **A**, with the truth-value of "$\mathrm{id}(x) = y$" thus being precisely that of "$x = y$", as it should be.

The composite of arrows $f: \mathbf{A} \to \mathbf{B}$ and $g: \mathbf{B} \to \mathbf{C}$ is the function $g \circ f: A \times C \to \Omega$ given by

$$g \circ f(\langle x, z\rangle) = \bigsqcup_{y \in B} (f(\langle x, y\rangle) \sqcap g(\langle y, z\rangle))$$

(compare this to the statement "for some $y \in B$, $f(x) = y$ and $g(y) = z$").

These definitions complete the description of Ω-**Set** as a category. In order to describe its topos structure we will from now on use the notations $f(\langle x, y\rangle)$ and $[\![f(x) \approx y]\!]$ interchangeably in reference to an arrow $f: \mathbf{A} \to \mathbf{B}$.

Terminal Object: This is the Ω-set **1** comprising the ordinary set $\{0\}$ with

$[\![0 \approx 0]\!] = \top$. The unique arrow $f : \mathbf{A} \to \mathbf{1}$ is given by

$$[\![f(x) \approx 0]\!] = [\![Ex]\!]$$

i.e. "$f(x)$ equals 0 to the extent that x exists".

Products: $\mathbf{A} \times \mathbf{B}$ is the product set $A \times B$ with the Ω-valued equality

$$[\![\langle x, y\rangle \approx \langle x', y'\rangle]\!] = [\![x \approx x']\!]_{\mathbf{A}} \sqcap [\![y \approx y']\!]_{\mathbf{B}}$$

The projection arrow $pr_{\mathbf{A}} : \mathbf{A} \times \mathbf{B} \to \mathbf{A}$ has

$$[\![pr_{\mathbf{A}}(\langle x, y\rangle) \approx z]\!] = [\![x \approx z]\!] \sqcap [\![Ex]\!] \sqcap [\![Ey]\!]$$

i.e. "the $\mathbf{A}$-projection of $\langle x, y\rangle$ equals z to the extent that x and y exist and x equals z".

Pullbacks: To realise the diagram

$$\begin{tikzcd} \mathbf{D} \arrow[r, "g'"] \arrow[d, "f'"'] & \mathbf{B} \arrow[d, "g"] \\ \mathbf{A} \arrow[r, "f"'] & \mathbf{C} \end{tikzcd}$$

as a pullback we define, for $x \in A$ and $y \in B$

$$E_D(\langle x, y\rangle) = \bigsqcup_{c \in C} ([\![f(x) \approx c]\!] \sqcap [\![g(y) \approx c]\!])$$

(cf. "there exists $c \in C$ with $f(x) = c$ and $g(y) = c$", i.e. "$f(x) = g(y)$").

Then $\mathbf{D}$ is the product set $A \times B$, with

$$[\![\langle x, y\rangle \approx \langle x', y'\rangle]\!]_{\mathbf{D}} = E_D(\langle x, y\rangle) \sqcap E_D(\langle x', y'\rangle) \sqcap [\![x \approx x']\!]_{\mathbf{A}} \sqcap [\![y \approx y']\!]_{\mathbf{B}}$$

Then in fact,

$$[\![E\langle x, y\rangle]\!]_{\mathbf{D}} = E_D(\langle x, y\rangle)$$

i.e. "$\langle x, y\rangle$ exists in D to the extent that $f(x) = g(y)$".

The "projection" f' is given by

$$[\![f'(\langle x, y\rangle) \approx z]\!] = E_D(\langle x, y\rangle) \sqcap [\![x \approx z]\!]_{\mathbf{A}}$$

and similarly for g'.

Subobjects: In **Set**, the pullback is a subset D of $A \times B$ specified by the condition "$f(x)=g(y)$". We have just seen that in Ω-**Set**, **D** is a kind of subobject of $\mathbf{A}\times\mathbf{B}$ that has the same partial elements as the latter but with degrees of existence determined by the pullback condition. This sort of phenomenon is typical of the description of subobjects in Ω-**Set**.

Intuitively, a subset of **A** may be represented by a function of the form $s: A \to \Omega$. Such a function assigns to each $x \in A$ an element $s(x)$ of Ω, which we think of as the truth-value of "$x \in s$", or as a measure of the extent to which x belongs to the "set" s. Thus we also denote $s(x)$ by $[\![x \in s]\!]$. Formally, a subset of an Ω-set **A** is a function $s: A \to \Omega$ that has

(viii) $\qquad [\![x \in s]\!] \sqcap [\![x \approx y]\!] \sqsubseteq [\![y \in s]\!]$ $\qquad$ (extensional)

and

(ix) $\qquad [\![x \in s]\!] \sqsubseteq [\![Ex]\!]$ $\qquad$ (strict)

EXAMPLE. Let $E: A \to \Omega$ be given by

$$E(x)=[\![x \approx x]\!]=[\![Ex]\!].$$

E represents the set of *existing* elements of **A**. Since

$$[\![Ex]\!]=[\![x \in E]\!]$$

we have that "x exists to the extent that it belongs to the set of existing elements of **A**". □

Now an arrow $f: \mathbf{A} \to \mathbf{B}$ can be shown to be monic just in case it satisfies

$$[\![f(x) \approx z]\!] \sqcap [\![f(y) \approx z]\!] \sqsubseteq [\![x \approx y]\!]$$

for all $x, y \in A$ and $z \in B$. Such an arrow corresponds to a subset of **B** (the "f-image" of **A**), and hence to a function $s_f: B \to \Omega$. This is given by

$$s_f(y)=\bigsqcup_{x \in A} [\![f(x) \approx y]\!]$$

i.e. "y belongs to s_f to the extent that it is the f-image of some $x \in A$". Thus $s_f(y)$ is the truth-value of "$y \in f(A)$".

Conversely, a subset $s: B \to \Omega$ of **B** determines a monic arrow $f_s: \mathbf{A}_s \rightarrowtail \mathbf{B}$. $\mathbf{A}_s$ has the same collection B of elements as **B**, but with equality given by

$$[\![x \approx y]\!]_{\mathbf{A}_s}=[\![x \in s]\!] \sqcap [\![y \in s]\!] \sqcap [\![x \approx y]\!]_{\mathbf{B}}$$

i.e. "x and y are equal in $\mathbf{A}_s$ to the extent that they are equal in $\mathbf{B}$ and belong to s". The "inclusion" arrow f_s has

$$[\![f_s(x) \approx y]\!] = [\![x \approx y]\!]_{\mathbf{A}_s}.$$

EXERCISE 6. (I) Prove that $s_{f_s} = s$.

(ii) Let $f_{s_f} : \mathbf{A}_{s_f} \rightarrowtail \mathbf{B}$ be constructed from the set s_f corresponding to a monic $f : \mathbf{A} \rightarrowtail \mathbf{B}$ as above. Then $\mathbf{A}_{s_f}$ has the same collection of elements as $\mathbf{B}$. Define $g : \mathbf{A} \to \mathbf{A}_s$ by

$$[\![g(x) \approx y]\!] = [\![f(x) \approx y]\!].$$

Show that g is iso in Ω-**Set** and that

$$\begin{tikzcd} \mathbf{A} \arrow[dr, tail, "f"] \arrow[dd, "g"'] & \\ & \mathbf{B} \\ \mathbf{A}_{s_f} \arrow[ur, tail, "f_{s_f}"'] & \end{tikzcd}$$

commutes. □

The import of this exercise is that subobjects $\mathbf{A} \rightarrowtail \mathbf{B}$ of $\mathbf{B}$ are uniquely determined by subsets $B \to \Omega$ of $\mathbf{B}$. The latter in fact form the power object $\mathcal{P}(\mathbf{B})$ of $\mathbf{B}$. To define this, let $S(\mathbf{B})$ be the collection of all subsets $s : B \to \Omega$ of $\mathbf{B}$. Then $\mathcal{P}(\mathbf{B})$ comprises $S(\mathbf{B})$ with the equality

$$[\![s \approx t]\!]_{\mathcal{P}(\mathbf{B})} = \bigsqcap_{x \in B} (s(x) \Leftrightarrow t(x))$$

(cf. "for all $x \in B$, $x \in S$ iff $x \in t$").

EXERCISE 7. $[\![s \approx t]\!]_{\mathcal{P}(\mathbf{B})} = \top$ iff $s = t$ (i.e. s and t are the same function).

EXERCISE 8. $[\![Es]\!]_{\mathcal{P}(\mathbf{B})} = \top$

EXERCISE 9. $[\![x \in s]\!] \sqcap [\![s \approx t]\!] \sqsubseteq [\![x \in t]\!]$ □

Now the function $e : A \times S(\mathbf{A}) \to \Omega$ having $e(\langle x, s\rangle) = s(x)$ satisfies (viii) and (ix), and so is a subset of the Ω-set $\mathbf{A} \times \mathcal{P}(\mathbf{A})$. The corresponding subobject f_e is precisely the membership relation $\in_{\mathbf{A}} \rightarrowtail \mathbf{A} \times \mathcal{P}(\mathbf{A})$ on $\mathbf{A}$. The definition of e thus gives that "$\langle x, s\rangle$ belongs to $\in_{\mathbf{A}}$ to the same extent that x belongs to s".

SUBOBJECT CLASSIFIER: The arrow $true : \mathbf{1} \to \mathbf{\Omega}$ has

$$[\![true(0) \approx p]\!] = [\![p \approx \top]\!]_{\mathbf{\Omega}}$$

("p is *true* to the extent that p equals $\top$") and so

$$[\![true(0) \approx p]\!] = (p \Leftrightarrow \top) = p.$$

Now let $f : \mathbf{A} \to \mathbf{D}$ be a monic, with corresponding subset $s_f : \mathbf{D} \to \mathbf{\Omega}$ of $\mathbf{D}$. The character $\chi_f : \mathbf{D} \to \mathbf{\Omega}$ of f has

$$[\![\chi_f(d) \approx p]\!] = [\![Ed]\!]_{\mathbf{D}} \sqcap [\![s_f(d) \approx p]\!]_{\mathbf{\Omega}}$$

i.e. "$\chi_f(d)$ equals p to the extent that d exists and p is the truth-value of "$d \in f(A)$" ".

EXERCISE 10. Show that this construction satisfies the Ω-axiom.

EXERCISE 11. $[\![false(0) \approx p]\!] = [\![p \approx \bot]\!]_{\mathbf{\Omega}} = (p \Leftrightarrow \bot) = \neg p$

EXERCISE 12. The truth arrows $\cap$, $\cup$ have

$$[\![p \cap q \approx r]\!] = [\![(p \sqcap q) \approx r]\!]_{\mathbf{\Omega}}$$

EXERCISE 13. $[\![p \cup q \approx r]\!] = [\![(p \sqcup q) \approx r]\!]_{\mathbf{\Omega}}$

EXERCISE 14. Show that the r.p.c. operation $\Rightarrow : \Omega \times \Omega \to \Omega$ on the **HA** Ω is a subset of $\mathbf{\Omega} \times \mathbf{\Omega}$ in the sense of (viii) and (ix) and that the corresponding subobject is $\circledS \rightarrowtail \mathbf{\Omega} \times \mathbf{\Omega}$. Show that the character of the latter, i.e. the implication arrow $\Rightarrow' : \mathbf{\Omega} \times \mathbf{\Omega} \to \mathbf{\Omega}$ has

$$[\![p \Rightarrow' q \approx r]\!] = (p \Rightarrow q) \Leftrightarrow r = [\![(p \Rightarrow q) \approx r]\!]_{\mathbf{\Omega}}. \qquad \square$$

Object of partial elements

In **Set**, a "singleton" is a set with exactly one member. In the present context of partial elements we are more interested in sets with *at most* one member. Formally a subset (extensional, strict function) $s : A \to \Omega$ of $\mathbf{A}$ is a *singleton* if it satisfies

(x) $\qquad [\![x \in s]\!] \sqcap [\![y \in s]\!] \sqsubseteq [\![x \approx y]\!]$

i.e. "elements of A belong to s only to the extent that they are equal".

EXAMPLE 1. If $a \in A$, then the map $\{\mathbf{a}\} : A \to \Omega$ that assigns to $x \in A$ the degree $[\![x \approx a]\!]$ of its equality with a is a singleton in this sense, with $[\![x \in \{\mathbf{a}\}]\!] = [\![x \approx a]\!]$.

EXAMPLE 2. Suppose $\mathbf{A}$ is the Ω-set (with $\Omega = \mathcal{P}(I)$) of all local sections of some bundle over I, as considered earlier. Included in $\mathbf{A}$ is the empty section $\emptyset_{\mathbf{A}}$, the unique section whose domain is the empty subset of I. For any other section x, we have $[\![x \approx \emptyset_{\mathbf{A}}]\!] = \emptyset$. Generalising to an arbitrary Ω and arbitrary Ω-set $\mathbf{A}$, the map $\{\emptyset_{\mathbf{A}}\}: A \to \Omega$ assigning $\bot$ to each $x \in A$ is a singleton, with $[\![x \in \{\emptyset_{\mathbf{A}}\}]\!] = \bot$. □

EXERCISE 15. If s is a singleton

$$[\![x \in s]\!] \sqsubseteq ([\![y \in s]\!] \Leftrightarrow [\![y \approx x]\!])$$

EXERCISE 16. $\{\mathbf{a}\} = \{\mathbf{b}\}$ iff $[\![a \approx b]\!] = [\![Ea]\!] = [\![Eb]\!]$.

EXERCISE 17. Let $s \in S(\mathbf{A})$ and $p \in \Omega$. The restriction of s to p is the function $s \restriction p: A \to \Omega$ assigning $s(x) \sqcap p$ to x. Show that $s \restriction p \in S(\mathbf{A})$ and that $s \restriction p$ is a singleton if s is. □

Now the object $\tilde{\mathbf{A}}$ is to be regarded as the Ω-set of all subsets of $\mathbf{A}$ that are singletons in the present sense. Thus $\tilde{\mathbf{A}}$ is to be thought of as itself being a subobject of $\mathcal{P}(\mathbf{A})$ and hence corresponds to a function $sing: S(\mathbf{A}) \to \Omega$. The formal definition, for $s \in S(\mathbf{A})$, is

$$[\![s \in sing]\!] = \bigsqcap_{x,y \in A} ([\![x \in s]\!] \sqcap [\![y \in s]\!] \Rightarrow [\![x \approx y]\!])$$

(cf. "for all x, $y \in A$, if x and y belong to s then $x = y$".)

The inclusion arrow $\eta_{\mathbf{A}}: \mathbf{A} \rightarrowtail \tilde{\mathbf{A}}$ of $\mathbf{A}$ into $\tilde{\mathbf{A}}$ has

$$[\![\eta_{\mathbf{A}}(a) \approx s]\!] = [\![Ea]\!]_{\mathbf{A}} \sqcap [\![s \approx \{\mathbf{a}\}]\!]_{\mathcal{P}(\mathbf{A})}$$

("$\eta_{\mathbf{A}}(a)$ is s to the extent that a exists and s is $\{\mathbf{a}\}$").

EXERCISE 18. $[\![s \in sing]\!] = \top$ iff s is a singleton.

EXERCISE 19. $[\![\{\mathbf{a}\} \approx s]\!] \sqsubseteq [\![s \in sing]\!]$. □

Now we know that each bundle over I gives rise to an Ω-set, where $\Omega = \mathcal{P}(I)$, whose elements are the partial sections of the bundle. Conversely, given an arbitrary $\mathcal{P}(I)$-set $\mathbf{A}$, each $i \in I$ determines an equivalence relation $\sim_i$ on the set

$$A_i = \{x \in A : i \in [\![Ex]\!]\}$$

that is defined by

$$x \sim_i y \quad \text{iff} \quad i \in [\![x \approx y]\!].$$

We then obtain a bundle over I by taking the quotient set $A_i/\sim_i$ as the stalk over the point i. These constructions may be used to establish that the categories $\mathbf{Bn}(I)$ and $\mathcal{P}(I)$-**Set** are equivalent. They can also be adapted to the case of sheaves of sets of germs, showing that $\mathbf{Top}(I)$ is equivalent to Θ-**Set**, where Θ is the **CHA** of open subsets of a topological space I. These facts are a special case of a result of D. Higgs [73] to the effect that Ω-**Set**, for any **CHA** Ω, is equivalent to the category of "sheaves over Ω". Precisely what that means will be explained in Chapter 14, where we shall see also that Ω-**Set** is equivalent to a subcategory of itself in which arrows $\mathbf{A} \to \mathbf{B}$ may be identified with actual set-functions $A \to B$.

Elementary Logic in Ω-Set

We have been interpreting the operations $\bigsqcap$ and $\bigsqcup$ *informally* as universal and existential quantifiers in order to understand the constructions that define Ω-**Set**. When we come to interpret a first-order language in this topos, these same operations may serve to give meanings to the formal symbols $\forall$ and $\exists$. Moreover, instead of assigning a formula an arrow of the type $\tilde{\mathbf{A}} \to \mathbf{\Omega}$, we may work directly with functions of the form $A \to \Omega$, and take advantage of the presence of the extents $[\![Ea]\!]$ of individuals to formalize the principle that quantifiers are to range over *existing* individuals.

To illustrate this approach, suppose that our language $\mathcal{L}$ has a single two-place relation symbol $\mathbf{R}$. Our basic alphabet is presumed to include the existence predicate $\mathbf{E}$ and the identity (equality) symbol $\approx$. The symbol $\approxeq$ for equivalence is introduced according to clause (iii) at the beginning of this section. Alternatively, $\approx$ may be defined in terms of $\approxeq$ by (iiia).

For this language, a model in Ω-**Set** is a pair $\mathfrak{A} = \langle \mathbf{A}, r\rangle$ comprising an Ω-set $\mathbf{A}$ and a subset $r : A \times A \to \Omega$ of $\mathbf{A} \times \mathbf{A}$. (By Exercise 6, r corresponds to a unique subobject of $\mathbf{A} \times \mathbf{A}$, hence to a unique *arrow* $\mathbf{A} \times \mathbf{A} \to \mathbf{\Omega}$, and so this approach accords within the definition of "model" in §11.4). We then extend $\mathcal{L}$ by adjoining an individual constant $\mathbf{c}$ for each element $c \in A$. A truth-value $[\![\varphi]\!]_{\mathfrak{A}} \in \Omega$ can then be calculated for each sentence φ by induction as follows:

Atomic Sentences:

$$[\![\mathbf{c} \approx \mathbf{d}]\!]_{\mathfrak{A}} = [\![c \approx d]\!]_{\mathbf{A}}$$

$$[\![\mathbf{E}(\mathbf{c})]\!]_{\mathfrak{A}} = [\![Ec]\!]_{\mathbf{A}}$$

$$[\![\mathbf{cRd}]\!]_{\mathfrak{A}} = r(\langle c, d\rangle)$$

Propositional Connectives:

$\wedge$, $\vee$, $\supset$, $\sim$ are interpreted by $\sqcap$, $\sqcup$, $\Rightarrow$, $\neg$ in Ω.

Quantifiers:

$$[\![\forall v\varphi]\!]_{\mathfrak{A}} = \bigsqcap_{c \in A} ([\![\mathbf{E}(\mathbf{c}) \supset \varphi(v/\mathbf{c})]\!]_{\mathfrak{A}})$$

("$\varphi(\mathbf{c})$ holds for all existing c")

$$[\![\exists v\varphi]\!]_{\mathfrak{A}} = \bigsqcup_{c \in A} ([\![\mathbf{E}(\mathbf{c}) \wedge \varphi(v/\mathbf{c})]\!]_{\mathfrak{A}})$$

("$\varphi(\mathbf{c})$ holds for some existing c").

Satisfaction: For a formula $\varphi(v_1, \ldots, v_n)$ we define $\mathfrak{A} \models \varphi[c_1, \ldots, c_n]$, where $c_1, \ldots, c_n \in A$, to mean that $[\![\varphi(v_1/\mathbf{c}_1, \ldots, v_n/\mathbf{c}_n)]\!]_{\mathfrak{A}} = \top$. Then truth-$\mathfrak{A} \models \varphi$ - of φ in $\mathfrak{A}$ can then be defined as usual by

$$\mathfrak{A} \models \varphi[c_1, \ldots, c_n] \quad \text{for all} \quad c_1, \ldots, c_n \in A.$$

EXERCISE 20. Show that the following are true in $\mathfrak{A}$:

$$(t \approx u) \wedge \varphi(v/u) \supset \varphi(v/t)$$

$$\forall v_i((v_i \approx v_j) \equiv (v_i \approx v_k)) \supset (v_j \approx v_k)$$

$$\forall v\varphi \wedge \mathbf{E}(t) \supset \varphi(v/t)$$

$$\varphi(v/t) \wedge \mathbf{E}(t) \supset \exists v\varphi$$

$$\mathbf{E}(t) \equiv \exists v(v \approx t)$$

$$\exists v(v \approx \mathbf{t}) \equiv \exists v(v \approx \mathbf{t})$$

$$\forall v_i \forall v_j((v_i \approx v_j) \equiv (v_i \approx v_j))$$

$$\forall v\varphi \equiv \forall v(\mathbf{E}(v) \supset \varphi)$$

$$\exists v\varphi \equiv \exists v(\mathbf{E}(v) \wedge \varphi)$$

$$\forall v\mathbf{E}(v)$$

$$(\mathbf{E}(v_i) \vee \mathbf{E}(v_j) \supset (v_i \approx v_j)) \supset (v_i \approx v_j)$$

EXERCISE 21. Show that the following rules preserve truth in $\mathfrak{A}$:

$$\textit{From } \varphi \wedge \mathbf{E}(v) \supset \psi \quad \textit{infer} \quad \varphi \supset \forall v \psi$$

$$\textit{From } \psi \wedge \mathbf{E}(v) \supset \varphi \quad \textit{infer} \quad \exists v \psi \supset \varphi$$

provided in both cases that v is not free in φ. □

This semantical theory will be used in Chapter 14 to define number-systems in Ω-**Set**. We will find it convenient there to have available the following result, which simplifies the calculation of the truth-value of quantified formulae in some cases by allowing the range of quantification to be further restricted.

We say that a subset C of A *generates* the Ω-set $\mathbf{A}$ if for each $a \in A$,

$$[\![Ea]\!]_{\mathbf{A}} = \bigsqcup_{c \in C} [\![a \approx c]\!]_{\mathbf{A}}$$

EXERCISE 22. If C generates A then

$$[\![\forall v \varphi]\!]_{\mathfrak{A}} = \bigsqcap_{c \in C} ([\![\mathbf{E}(\mathbf{c}) \supset \varphi(v/\mathbf{c})]\!]_{\mathfrak{A}})$$

and

$$[\![\exists v \varphi]\!]_{\mathfrak{A}} = \bigsqcup_{c \in C} ([\![\mathbf{E}(\mathbf{c}) \wedge \varphi(v/\mathbf{c})]\!]_{\mathfrak{A}})$$ □

11.10. Higher-order logic

In closing this chapter on quantificational logic we mention briefly the study that has been made of the relationship between higher order logic and topoi.

Higher order logic has formulae of the form $(\forall X)\varphi$ and $(\exists X)\varphi$, where X may stand for a set, a relation, a set of sets, a set of relations, a set of sets of sets of ..., etc. So for a classical model $\mathfrak{A} = \langle A, \ldots \rangle$ the range of X may be any of $\mathcal{P}(A)$, $\mathcal{P}(A^n)$, $\mathcal{P}(\mathcal{P}(A^n))$, etc. Analogues of these exist in any topos, in the form of Ω^a, Ω^{a^n}, etc., and so higher order logic is interpretable in $\mathscr{E}$. In fact the whole topos becomes a model for a many sorted language, having one sort (infinite list) of individual variables for each $\mathscr{E}$-object. Given a theory Γ (i.e. a consistent set of sentences) in this language, a topos $\mathscr{E}_\Gamma$ can be constructed that is a model of Γ. Conversely given a topos $\mathscr{E}$ a theory $\Gamma_{\mathscr{E}}$ can be defined whose associated topos $\mathscr{E}_{\Gamma_{\mathscr{E}}}$ is categorially equivalent to $\mathscr{E}$. These results were obtained for the logic of partial elements by Fourman [74] and subsequently for the other approach to free logic by Boileau [75]. They amount to a demonstration that the concept of "elementary topos" is co-extensive with that of "model for many-sorted higher-order intuitionistic free logic", and hence provide a full explication of Lawvere's statement in [72] that "the notion of topos summarizes in objective categorical form the essence of 'higher-order logic'." The work of Fourman incorporates a number of interesting and unusual logical features, which we will outline briefly.

Firstly, as already noted in §11.8, variables are to be thought of as ranging over, and constants denoting, *potential* elements of an $\mathscr{E}$-object a. Thus a formula is interpreted by an arrow of the form $[\![\varphi]\!]:(\tilde{a})^n \to \Omega$, corresponding to the subobject of all n-tuples of potential elements that satisfy φ.

Next, the system includes a theory of *definite descriptions* as terms of the formal language. A definite description is an expression of the form $\mathbf{I}v\varphi$, which is read "the unique v such that φ". The expression serves as a name for this unique v whenever it exists. The basic axiom governing this descriptions-operator is

$$\forall u((u \approx \mathbf{I}v\varphi(v)) \equiv \forall v(\varphi(v) \equiv (v \approx u)))$$

which has the reading "an existing element u is equivalent to the element $\mathbf{I}v\varphi(v)$ iff u is the one and only existing element satisfying φ" (recall that quantifiers range over existing elements).

To interpret a definite description semantically in $\mathscr{E}$ suppose, by way of example, that the $\mathscr{E}$-arrow $[\![\varphi]\!]: \tilde{a} \to \Omega$ has been defined, where $\varphi(v)$ has index 1. Let $f:1 \to \Omega^a$ be the *name* of the arrow $[\![\varphi]\!] \circ \eta_a : a \to \Omega$ (cf. §4.1). (In **Set**, f corresponds to the element

$$|\varphi| = \{x \in a : \varphi(x)\}$$

of the powerset of a, i.e. the subset of a defined by φ).

Form in $\mathscr{E}$ the pullback

$$\begin{tikzcd} b \arrow[r, tail] \arrow[d, tail, "g"'] & 1 \arrow[d, "f"] \\ a \arrow[r, tail, "\{\cdot\}_a"] & \Omega^a \end{tikzcd}$$

of f along the "singleton arrow" $\{\cdot\}_a$, that was defined in §11.8. (In **Set** we may regard g as the inclusion $b \hookrightarrow a$, with $b = |\varphi|$ if $|\varphi|$ is a non-empty singleton, i.e. if $|\varphi| = \{x\}$ for some $x \in a$, and $b = \emptyset$ otherwise). Notice that $g:1 \rightsquigarrow a$, i.e. g is a partial element of a, and so corresponds to an arrow

$\tilde{g}:1 \to \tilde{a}$. We take this $\tilde{g}$ to be $[\![\mathbf{I}v\varphi]\!]$. (In **Set**, taking $\tilde{a}$ as $a \cup \{*\}$, $\tilde{g}$ corresponds to the element x of a if $|\varphi| = \{x\}$, and is the "null entity" $*$ otherwise.)

Of course the description operator and its semantic interpretation can be developed in the context of first-order logic. In higher order logic it becomes particularly useful, in that is provides simple and straightforward ways of expressing both the Comprehension Principle, and the operation of *functional abstraction*, the latter being the process of defining a term that denotes a function whose graph is specified by a formula.

To consider Comprehension, suppose by way of example that $\varphi(v)$ has a single free variable whose range is a collection of entities of a certain level, or type, in a higher-order structure comprising subsets, sets of subsets, sets of sets of subsets etc. In a higher-order language there will also be variables w that range over the subsets of the range of v. Then the sentence

$$\mathbf{EI}w\forall v(\varphi(v) \equiv w(v))$$

asserts the actual existence of the unique set whose elements are precisely those entities that satisfy φ.

If instead $\varphi(v, w)$ has two free variables, it defines a relation when interpreted. We denote by $\varphi'(v)$ *the term*

$$\mathbf{I}w\varphi(v, w).$$

If the interpretation of φ is a functional relation (one with the unique output property) then this term will provide a notation for function values. Functional abstraction may now be performed by forming the expression

$$\mathbf{I}u\forall v\forall w(u(v, w) \equiv \varphi'(v) \approx w)$$

(which is abbreviated to $\lambda v \cdot \varphi'(v)$), where u is a variable that ranges over the relations from the range of v to the range of w. The expression $\lambda v \cdot \varphi'(v)$ may be read "the function which for input v gives output $\varphi'(v)$".

The details of this higher-order language and its use in characterising topoi as models of higher-order theories may be found in Fourman's article "The Logic of Topoi" in Barwise [77]. This work is important for a broad understanding of the structural properties of topoi. It offers a different perspective to the one we are dealing with here. Our present concern is to develop the view of a topos as a universe of set-like objects and hence, qua foundation for mathematics, as a model of a first-order theory of set-membership. We take this up in earnest in the next chapter.