CHAPTER 9

FUNCTORS

> *"It should be observed first that the whole concept of a category is essentially an auxiliary one; our basic concepts are essentially those of a functor and of a natural transformation."*
>
> S. Eilenberg and S. MacLane

9.1. The concept of functor

A functor is a transformation from one category into another that "preserves" the categorial structure of its source. As the quotation from the founders of the subject indicates, the notion of functor is of the very essence of category theory. The original perspective has changed somewhat, and as far at least as this book is concerned functors are not more important than categories themselves. Indeed the viability of the topos concept as a foundation for mathematics pivots on the fact that it can be *defined* without reference to functors. However we have now reached the stage where we can ignore them no longer. They provide the necessary language for describing the relationship between topoi and Kripke models, and between topoi and models of set theory.

A *functor F* from category $\mathscr{C}$ to category $\mathscr{D}$ is a function that assigns

(i) to each $\mathscr{C}$-object a, a $\mathscr{D}$-object $F(a)$;

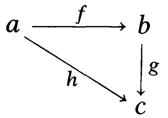
(ii) to each $\mathscr{C}$-arrow $f: a \to b$ a $\mathscr{D}$-arrow $F(f): F(a) \to F(b)$,

such that

(a) $F(1_a) = 1_{F(a)}$, all $\mathscr{C}$-objects a, i.e. the identity arrow on a is assigned the identity on $F(a)$,

(b) $F(g \circ f) = F(g) \circ F(f)$, whenever $g \circ f$ is defined.

This last condition states that the F-image of a composite of two arrows is the composite of their F-images, i.e. whenever



commutes in $\mathscr{C}$ $(h = g \circ f)$, then

$$\begin{array}{ccc} F(a) & \xrightarrow{F(f)} & F(b) \\ & \searrow^{F(h)} & \downarrow F(g) \\ & & F(c) \end{array}$$

commutes in $\mathscr{D}$. We write $F:\mathscr{C} \to \mathscr{D}$ or $\mathscr{C} \xrightarrow{F} \mathscr{D}$ to indicate that F is a functor from $\mathscr{C}$ to $\mathscr{D}$. Briefly then a functor is a transformation that "preserves" dom's, cod's, identities and composites.

EXAMPLE 1. The *identity functor* $\mathbf{1}_{\mathscr{C}}:\mathscr{C} \to \mathscr{C}$ has $\mathbf{1}_{\mathscr{C}}(a) = a$, $\mathbf{1}_{\mathscr{C}}(f) = f$. The same rule provides an *inclusion* functor $\mathscr{C} \hookrightarrow \mathscr{D}$ when $\mathscr{C}$ is a subcategory of $\mathscr{D}$.

EXAMPLE 2. *Forgetful functors:* Let $\mathscr{C}$ be any of the categories in the original list of §2.3, say $\mathscr{C} = \mathbf{Top}$. Then a $\mathscr{C}$-object is a set carrying some additional structure. The forgetful functor $U:\mathscr{C} \to \mathbf{Set}$ takes each $\mathscr{C}$-object to its underlying set, and each $\mathscr{C}$-arrow to itself. Thus U "forgets" the structure on $\mathscr{C}$-objects and remembers only that $\mathscr{C}$-arrows are set functions.

EXAMPLE 3. *Power set Functor:* $\mathscr{P}:\mathbf{Set} \to \mathbf{Set}$ maps each set A to its powerset $\mathscr{P}(A)$, and each function $f:A \to B$ to the function $\mathscr{P}(f):\mathscr{P}(A) \to \mathscr{P}(B)$ from $\mathscr{P}(A)$ to $\mathscr{P}(B)$ that assigns to each $X \subseteq A$ its f-image $f(X) \subseteq B$.

EXAMPLE 4. If $\mathbf{P}$ and $\mathbf{Q}$ are posets, then a functor $F:\mathbf{P} \to \mathbf{Q}$ is simply a function $F:P \to Q$ that is *monotonic*, i.e. whenever $p \sqsubseteq q$ in P then $F(p) \sqsubseteq F(q)$ in Q. As a special case of this consider the powerset as a poset $(\mathscr{P}(A), \subseteq)$. Given $f:A \to B$ and X, Y subsets of A, then $X \subseteq Y$ only if $f(X) \subseteq f(Y)$. Thus the function $\mathscr{P}(f):\mathscr{P}(A) \to \mathscr{P}(B)$ is itself a functor between (poset) categories.

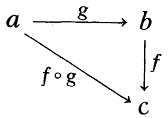
EXAMPLE 5. M*onoid homomorphisms:* A functor between monoids $(M, *, e)$ and $(N, \square, e')$, when these are construed as one-object categories, is essentially a monoid *homomorphism*, i.e. a function $F:M \to N$ that has

$$F(e) = e'$$

$$F(x * y) = F(x) \square F(y).$$

EXAMPLE 6. If $\mathscr{C}$ has products, each $\mathscr{C}$-object a determines a functor $-\times a:\mathscr{C}\to\mathscr{C}$ which takes each object b to the object $b\times a$, and each arrow $f:b\to c$ to the arrow $f\times 1_a:b\times a\to c\times a$.

EXAMPLE 7. *Hom-functors:* Given a $\mathscr{C}$-object a, then $\mathscr{C}(a,-):\mathscr{C}\to\mathbf{Set}$ takes each $\mathscr{C}$-object b to the set $\mathscr{C}(a,b)$ of $\mathscr{C}$-arrows from a to b and each $\mathscr{C}$-arrow $f:b\to c$ to the function $\mathscr{C}(a,f):\mathscr{C}(a,b)\to\mathscr{C}(a,c)$ that outputs $f\circ g$ for input g



$\mathscr{C}(a,-)$ is called a *hom-functor* because of the use of the word "homomorphism" in some contexts for "arrow". $\mathscr{C}(a,b)=\hom_{\mathscr{C}}(a,b)$ is known as a *hom-set.* There is a restriction as to when this hom-functor is defined. The hom-sets of $\mathscr{C}$ have to be *small,* i.e. actual sets, and not proper classes. □

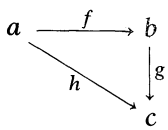
Contravariant functors

The above examples are all what are known as *covariant* functors. They preserve the "direction" of arrows, in that the domain of an arrow is assigned the domain of the image arrow, and similarly for codomains. A *contravariant* functor is one that *reverses* direction by mapping domains to codomains and vice versa.

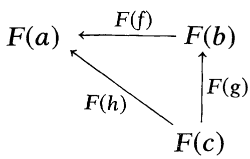
Thus $F:\mathscr{C}\to\mathscr{D}$ is a contravariant functor if it assigns to $f:a\to b$ an arrow $F(f):F(b)\to F(a)$, so that $F(1_a)=1_{F(a)}$ as before, but now

$$F(g\circ f)=F(f)\circ F(g),$$

i.e. commuting



goes to commuting



EXAMPLE 8. A contravariant functor between posets is a function $F:P \to Q$ that is *antitone*, i.e.

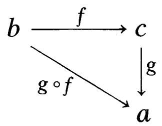
$$\text{if} \quad p \sqsubseteq q \quad \text{in } \mathbf{P}, \text{then} \quad F(q) \sqsubseteq F(p) \quad \text{in } \mathbf{Q}.$$

EXAMPLE 9. *Contravariant powerset functor:*

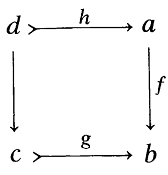
$$\bar{\mathcal{P}}: \mathbf{Set} \to \mathbf{Set}$$

takes each set A to its powerset $\mathcal{P}(A)$, and each $f: A \to B$ to the function $\bar{\mathcal{P}}(f): \mathcal{P}(B) \to \mathcal{P}(A)$ that assigns to $X \subseteq B$ its *inverse image* $f^{-1}(X) \subseteq A$.

EXAMPLE 10. *Contravariant hom-functor:* $\mathscr{C}(-, a): \mathscr{C} \to \mathbf{Set}$, for fixed object a, takes object b to $\mathscr{C}(b, a)$, and $\mathscr{C}$-arrow $f: b \to c$ to function $\mathscr{C}(f, a): \mathscr{C}(c, a) \to \mathscr{C}(b, a)$ that outputs $g \circ f$ for input g



EXAMPLE 11. $\text{Sub}: \mathscr{C} \to \mathbf{Set}$ is the functor taking each $\mathscr{C}$-object a to its collection $\text{Sub}(a)$ of subobjects in $\mathscr{C}$, and each $\mathscr{C}$-arrow $f: a \to b$ to the function $\text{Sub}(f): \text{Sub}(b) \to \text{Sub}(a)$, assigning to $g: c \rightarrowtail b$ the pullback $h: d \rightarrowtail a$ of g along f. Of course this construction is only possible if $\mathscr{C}$ has



pullbacks. It generalises Example 9. □

EXERCISE Verify that (1)–(11) really are functors. □

The word "functor" used by itself will always mean "covariant functor". In principle contravariant $F: \mathscr{C} \to \mathscr{D}$ can be replaced by covariant $\bar{F}: \mathscr{C}^{\text{op}} \to \mathscr{D}$, where $\bar{F}(a) = F(a)$, and for $f^{\text{op}}: b \to a$ in $\mathscr{C}^{\text{op}}$ (where $f: a \to b$ in $\mathscr{C}$), $\bar{F}(f^{\text{op}}) = F(f): F(b) \to F(a)$. We will not consider contravariant functors again until Chapter 14.

Now given functors $F:\mathscr{C}\to\mathscr{D}$, $G:\mathscr{D}\to\mathscr{F}$, *functional* composition of F and G yields a functor $G\circ F:\mathscr{C}\to\mathscr{F}$, and this operation is associative,

$$H\circ(G\circ F)=(H\circ G)\circ F.$$

We can thus consider functors as arrows between categories. We intuitively envisage a category **Cat**, the category of categories, whose objects are the categories, and arrows the functors. The identity arrows are the identity functors $1_{\mathscr{C}}$ of Example 1.

The notion of **Cat** leads us to some foundational problems. **Set** could not be an element of the class of **Cat**-objects (if we regard these as forming a class), since **Set** as a collection of things is a proper class, and not a member of any collection. Moreover contemplation of the question "is **Cat** a **Cat**-object?" leads us to the brink of Russell's paradox. Generally **Cat** is understood to be the category of *small* categories, i.e. ones whose collection of arrows is a set. Further discussion of these questions may be found in Hatcher [68] Chapter 8, (cf. also a paper by Lawvere [66] on **Cat** as a foundation for mathematics).

9.2. Natural transformations

Having originally defined categories as collections of objects with arrows between them, by introducing functors we took a step up the ladder of abstraction to consider categories as objects, with functors as arrows between them. Readers are now invited to fasten their mental safety-belts as we climb even higher, to regard functors themselves as objects!

Given two categories $\mathscr{C}$ and $\mathscr{D}$ we are going to construct a category, denoted Funct$(\mathscr{C},\mathscr{D})$, or $\mathscr{D}^{\mathscr{C}}$, whose objects are the functors from $\mathscr{C}$ to $\mathscr{D}$. We need a definition of arrow from one functor to another. Taking $F:\mathscr{C}\to\mathscr{D}$ and $G:\mathscr{C}\to\mathscr{D}$, we think of the functors F and G as providing different "pictures" of $\mathscr{C}$ inside $\mathscr{D}$. A reasonably intuitive idea of "transformation" from F to G comes if we image ourselves trying to superimpose or "slide" the F-picture onto the G-picture, i.e. we use the structure of $\mathscr{D}$ to translate the former into the latter. This could be done by assigning to each $\mathscr{C}$-object a an arrow in $\mathscr{D}$ from the F-image of a to the G-image of a. Denoting this arrow by τ_a, we have $\tau_a:F(a)\to G(a)$. In order for this process to be "structure-preserving" we require that each $\mathscr{C}$-*arrow* $f:a \to b$ gives rise to a diagram

$$\begin{array}{ccc} a & \qquad & F(a) \xrightarrow{\ \tau_a\ } G(a) \\ \Big\downarrow f & & F(f)\Big\downarrow \qquad\qquad \Big\downarrow G(f) \\ b & & F(b) \xrightarrow{\ \tau_b\ } G(b) \end{array}$$

that commutes. Thus τ_a and τ_b provide a categorial way of turning the F-picture of $f:a \to b$ into its G-picture.

In summary then, a *natural transformation* from functor $F:\mathscr{C} \to \mathscr{D}$ to functor $G:\mathscr{C} \to \mathscr{D}$ is an assignment τ that provides, for each $\mathscr{C}$-object a, a $\mathscr{D}$-arrow $\tau_a : F(a) \to G(a)$, such that for any $\mathscr{C}$-arrow $f:a \to b$, the above diagram commutes in $\mathscr{D}$, i.e. $\tau_b \circ F(f) = G(f) \circ \tau_a$. We use the symbolism $\tau : F \dot{\to} G$, or $F \xrightarrow{\tau} G$, to denote that τ is a natural transformation from F to G. The arrows τ_a are called the *components* of τ.

Now if each component τ_a of τ is an iso arrow in $\mathscr{D}$ then we can interpret this as meaning that the F-picture and the G-picture of $\mathscr{C}$ look the same in $\mathscr{D}$, and in this case we call τ a *natural isomorphism*. Each $\tau_a : F(a) \to G(a)$ then has an inverse $\tau_a^{-1} : G(a) \to F(a)$, and these τ_a^{-1}'s form the components of a natural isomorphism $\tau^{-1} : G \dot{\to} F$. We denote natural isomorphism by $\tau : F \cong G$.

EXAMPLE 1. The identity natural transformation $1_F : F \dot{\to} F$ assigns to each object a, the identity arrow $1_{F(a)} : F(a) \to F(a)$. This is clearly a natural isomorphism.

EXAMPLE 2. In **Set**, as noted in §3.4, we have $A \cong A \times 1$, for each set A. This isomorphism is a natural one, as we can see by using the functor $- \times 1 : \mathbf{Set} \to \mathbf{Set}$, as described in Example 6 of the last section. Given $f:A \to B$ then the diagram

$$\begin{array}{ccc} A & \qquad & A \xrightarrow{\ \tau_A\ } A \times 1 \\ \Big\downarrow f & & \Big\downarrow f \qquad\qquad \Big\downarrow f \times \mathrm{id}_1 \\ B & & B \xrightarrow{\ \tau_B\ } B \times 1 \end{array}$$

commutes, where $\tau_A(x) = \langle x, 0 \rangle$, and similarly for τ_B. (i.e. $\tau_A = \langle \mathrm{id}_A, !_A \rangle$). The left side of the square is the image of f under the identity functor. Thus the bijections τ_A are the components of a natural isomorphism τ from $1_{\mathbf{Set}}$ to $- \times 1$.

EXAMPLE 3. Again in **Set**, we have $A \times B \cong B \times A$ by the "twist" map $tw_B : A \times B \to B \times A$ given by the rule $tw_B(\langle x, y\rangle) = \langle y, x\rangle$. Now for given object A, as well as the "right product" functor $- \times A : \mathbf{Set} \to \mathbf{Set}$ we have a left-product functor $A \times - : \mathbf{Set} \to \mathbf{Set}$, taking B to $A \times B$, and $f : B \to C$ to $1_A \times f : A \times B \to A \times C$. Now for any $f : B \to C$, the diagram

$$\begin{array}{ccccc} B & \quad & A \times B & \xrightarrow{tw_B} & B \times A \\ {\scriptstyle f}\downarrow & & {\scriptstyle 1_A \times f}\downarrow & & \downarrow{\scriptstyle f \times 1_A} \\ C & & A \times C & \xrightarrow{tw_C} & C \times A \end{array}$$

commutes, showing that the bijections tw_B are the components of a natural isomorphism from $A \times -$ to $- \times A$. □

Equivalence of categories

When do two categories look the same? One possible answer is when they are isomorphic as objects in **Cat**. We say that functor $F : \mathscr{C} \to \mathscr{D}$ is *iso* if it has an inverse, i.e. a functor $G : \mathscr{D} \to \mathscr{C}$ such that $G \circ F = 1_{\mathscr{C}}$ and $F \circ G = 1_{\mathscr{D}}$. We then say that $\mathscr{C}$ and $\mathscr{D}$ are isomorphic, $\mathscr{C} \cong \mathscr{D}$, if there is an iso functor $F : \mathscr{C} \to \mathscr{D}$.

This notion of "sameness" is stricter than it need be. If F has inverse G then for given $\mathscr{C}$-object a we have $a = G(F(a))$, and for $\mathscr{D}$-object b, $b = F(G(b))$. In view of the basic categorial principle of indistinguishability of isomorphic entities we might still regard $\mathscr{C}$ and $\mathscr{D}$ as "essentially the same" if we just had $a \cong G(F(a))$ in $\mathscr{C}$ and $b \cong F(G(b))$ in $\mathscr{D}$. In other words $\mathscr{C}$ and $\mathscr{D}$ are to be categorially equivalent if they are "isomorphic up to isomorphism". This will occur when the isomorphisms $a \to G(F(a))$ and $b \to F(G(b))$ are natural.

Thus a functor $F : \mathscr{C} \to \mathscr{D}$ is called an *equivalence of categories* if there is a functor $G : \mathscr{D} \to \mathscr{C}$ such that there are natural isomorphisms $\tau : 1_{\mathscr{C}} \cong G \circ F$, and $\sigma : 1_{\mathscr{D}} \cong F \circ G$, from the identity functor on $\mathscr{C}$ to $G \circ F$, and from the identity functor on $\mathscr{D}$ to $F \circ G$.

Categories $\mathscr{C}$ and $\mathscr{D}$ are *equivalent*, $\mathscr{C} \simeq \mathscr{D}$, when there exists an equivalence $F : \mathscr{C} \to \mathscr{D}$.

EXAMPLE. **Finord** $\simeq$ **Finset**. Let $F : \mathbf{Finord} \hookrightarrow \mathbf{Finset}$ be the inclusion functor. For each finite set X, let $G(X) = n$, where n is the number of elements in X. For each X, let τ_X be a bijection from X to $G(X)$, with τ_X

being the identity when X is an ordinal. Given $f:X \to Y$, put $G(f) = \tau_Y \circ f \circ \tau_X^{-1}$. Then G is a functor from **Finset** to **Finord**.

Since

$$\begin{array}{ccc} X & \xrightarrow{\tau_X} & F(G(X)) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle F(G(f))} \\ Y & \xrightarrow{\tau_Y} & F(G(Y)) \end{array}$$

commutes, by definition of $G(f) = F(G(f))$, the τ_X's are the components of a natural isomorphism $\tau: 1 \to F \circ G$. But also $G \circ F$ is the identity functor on **Finord**. □

The notion of equivalence of categories can be clarified by considering *skeletal* categories. Recall from §3.4 that these are categories in which isomorphic objects are identical, $a \cong b$ only if $a = b$. **Finord** is skeletal, since isomorphic *finite* sets have the same number of elements. A *skeleton* of a category $\mathscr{C}$ is a full subcategory $\mathscr{C}_0$ of $\mathscr{C}$ that is skeletal, and such that each $\mathscr{C}$-object is isomorphic to one (and only one) $\mathscr{C}_0$-object. **Finord** is a skeleton of **Finset**. In general a skeleton $\mathscr{C}_0$ of $\mathscr{C}$ exhibits the essential categorial structure of $\mathscr{C}$. $\mathscr{C}_0$ is equivalent to $\mathscr{C}$, and the equivalence is provided by the inclusion functor $\mathscr{C}_0 \hookrightarrow \mathscr{C}$, as may be shown by the method of the last Example.

Any category $\mathscr{C}$ has a skeleton. The relation of isomorphism partitions the collection of $\mathscr{C}$-objects into equivalence classes. Choose one object from each equivalence class and let $\mathscr{C}_0$ be the full subcategory of $\mathscr{C}$ based on this collection of choices. $\mathscr{C}_0$ is a skeleton of $\mathscr{C}$ (cf. Chapter 12 for a discussion of the legitimacy of such a selection process in set-theory). Equivalence of categories is described in these terms by:

> *categories $\mathscr{C}$ and $\mathscr{D}$ are equivalent iff they have isomorphic skeletons ($\mathscr{C} \simeq \mathscr{D}$ iff $\mathscr{C}_0 \cong \mathscr{D}_0$),*

and in this sense equivalent categories are categorially "essentially the same". Note however that they need not be in bijective correspondence, indeed need not be comparable in size at all. The collection of finite ordinals is small, i.e. a set, identifiable with the set of natural numbers, whereas the objects of **Finset** form a proper class (e.g. it includes $\{x\}$, for each set x).

EXERCISE 1. Any two skeletons of a given category are isomorphic.

EXERCISE 2. In a topos $\mathscr{E}$, for each object d there is a bijection $\mathrm{Sub}(d) \cong \mathscr{E}(d, \Omega)$ (§4.2). Show that these bijections form a natural isomorphism between the functors $\mathrm{Sub}: \mathscr{E} \to \mathbf{Set}$ and $\mathscr{E}(-, \Omega): \mathscr{E} \to \mathbf{Set}$ (this is a functorial statement of the Ω-axiom).

9.3. Functor categories

We return now to the intention stated at the beginning of §9.2 – to define the functor category $\mathscr{D}^{\mathscr{C}}$ of all functors from $\mathscr{C}$ to $\mathscr{D}$. Let F, G, H be such functors, with natural transformations $\tau: F \dot{\to} G$, $\sigma: G \dot{\to} H$. Then for any $\mathscr{C}$-arrow $f: a \to b$ we get a diagram

$$
\begin{array}{ccccccc}
a & \quad & F(a) & \xrightarrow{\tau_a} & G(a) & \xrightarrow{\sigma_a} & H(a) \\
{\scriptstyle f}\downarrow & & {\scriptstyle F(f)}\downarrow & & {\scriptstyle G(f)}\downarrow & & {\scriptstyle H(f)}\downarrow \\
b & & F(b) & \xrightarrow{\tau_b} & G(b) & \xrightarrow{\sigma_b} & H(b)
\end{array}
$$

We wish to define the composite $\sigma \circ \tau$ of τ and σ, and have it as a natural transformation. The diagram indicates what to do. For each a, put $(\sigma \circ \tau)_a = \sigma_a \circ \tau_a$. Now each of the two squares in the diagram commutes, so the outer rectangle commutes, giving $(\sigma \circ \tau)_b \circ F(f) = H(f) \circ (\sigma \circ \tau)_a$, and thus the $(\sigma \circ \tau)_a$'s are the components of a natural transformation $\sigma \circ \tau: F \dot{\to} H$. This then provides the operation of composition in the functor category $\mathscr{D}^{\mathscr{C}}$. For each functor $F: \mathscr{C} \to \mathscr{D}$ the identity transformation $1_F: F \dot{\to} F$ (Example 1, §9.2) is the identity arrow on the $\mathscr{D}^{\mathscr{C}}$-object F.

EXERCISE 1. The natural isomorphisms are precisely the iso arrows in $\mathscr{D}^{\mathscr{C}}$.

EXERCISE 2. Let C and D be sets, construed as discrete categories with only identity arrows. Show that for F, $G: C \to D$ there is a transformation $F \dot{\to} G$ iff $F = G$, and that the functor category D^C is the set of functions $C \to D$.

EXERCISE 3. $\tau: F \dot{\to} G$ is monic in $\mathscr{D}^{\mathscr{C}}$ if τ_a is monic in $\mathscr{D}$ for all a. □

A number of the topoi described in Chapter 4 can be construed as "set-valued functor" categories, as follows.

(1) **Set**^{2}. The set $2 = \{0, 1\}$ is a discrete category. A functor $F: 2 \to \mathbf{Set}$ assigns a set F_0 to 0 and a set F_1 to 1. Since F as a functor is required to preserve identity arrows, and 2 only has identities, we can suppress all mention of arrows, and identify F with the pair $\langle F_0, F_1 \rangle$. Thus functors $2 \to \mathbf{Set}$ are essentially objects in the category $\mathbf{Set}^2$ of pairs of sets. Now given two such functors F and G, identified with $\langle F_0, F_1 \rangle$ and $\langle G_0, G_1 \rangle$, a natural transformation $\tau: F \dot{\to} G$ has components $\tau_0: F_0 \to G_0$, $\tau_1: F_1 \to G_1$. We may thus identify τ with the pair $\langle \tau_0, \tau_1 \rangle$, which is none other than a $\mathbf{Set}^2$-arrow from $\langle F_0, F_1 \rangle$ to $\langle G_0, G_1 \rangle$.

(2) **Set**$^{\to}$. Consider the poset category $\mathbf{2} = \{0, 1\}$ with non-identity arrow $0 \to 1$. A functor $F: \mathbf{2} \to \mathbf{Set}$ comprises two sets F_0, F_1, and a function $f: F_0 \to F_1$. Thus F is "essentially" an arrow f in **Set**, i.e. an object in $\mathbf{Set}^{\to}$. Now given another such functor G, construed as $g: G_0 \to G_1$, then a $\tau: F \dot{\to} G$ has components τ_0, τ_1 that make

$$\begin{array}{ccc} 0 & \qquad & F_0 \xrightarrow{\tau_0} G_0 \\ \downarrow & & f\downarrow \qquad \downarrow g \\ 1 & & F_1 \xrightarrow{\tau_1} G_1 \end{array}$$

commute. We see then that τ, identified with $\langle \tau_0, \tau_1 \rangle$ becomes an arrow from f to g in $\mathbf{Set}^{\to}$, and so the latter "is" the category $\mathbf{Set}^{\mathbf{2}}$ of functors from **2** to **Set**.

(3) **M-Set**. Let $\mathbf{M} = (M, *, e)$ be a monoid. An **M**-set is a pair (X, λ) where X is a set and λ assigns to each $m \in M$ a function $\lambda_m: X \to X$, so that

(i) $\lambda_e = \mathrm{id}_X$, and

(ii) $\lambda_m \circ \lambda_p = \lambda_{m*p}$.

Now **M** is a category with one object, say M, arrows the members m of M, $*$ as a composition, and $e = \mathrm{id}_M$. Then λ becomes a functor $\lambda: \mathbf{M} \to \mathbf{Set}$, with $\lambda(M) = X$ for the one object, and $\lambda(m) = \lambda_m$, each arrow m. Indeed (i), (ii) are precisely the conditions for λ to be a functor. Now given any other functor $\mu: \mathbf{M} \to \mathbf{Set}$, with $\mu(M) = Y$, then a $\tau: \lambda \dot{\to} \mu$ assigns to M a function $f: X \to Y$ so that

$$\begin{array}{ccc} M & \qquad & X \xrightarrow{f} Y \\ m\downarrow & & \lambda_m\downarrow \qquad \downarrow \mu_m \\ M & & X \xrightarrow{f} Y \end{array}$$

commutes for each $m \in M$. But this says precisely that f is an equivariant map from (X, λ) to (Y, μ). Thus **M-Set** is the category $\mathbf{Set}^{\mathbf{M}}$ of functors from **M** to **Set**.

(4) $\mathbf{Bn}(I)$. Taking the set I as a discrete category, a functor $F: I \to \mathbf{Set}$ assigns to each $i \in I$ a set F_i. So we can identify such functors with collections $\{F_i : i \in I\}$ of sets indexed by I.

An object (X, f) in $\mathbf{Bn}(I)$ (i.e. a function $f: X \to I$) gives a functor $\bar{f}: I \to \mathbf{Set}$, with $\bar{f}(i) = f^{-1}(\{i\})$, the stalk of f over i.

An arrow $h: (X, f) \to (Y, g)$ is a function that maps the f-stalk over i to the g-stalk over i, hence determines a function $h_i : \bar{f}(i) \to \bar{g}(i)$. These h_i's are the components for $\bar{h}: \bar{f} \rightarrowtriangle \bar{g}$. Thus each bundle can be turned into a functor from I to **Set**. The converse will only work if the F_i's are pairwise disjoint. So given $F: I \to \mathbf{Set}$ we define a new functor $\bar{F}: I \to \mathbf{Set}$ by putting $\bar{F}(i) = F(i) \times \{i\}$ and then turn $\{\bar{F}(i) : i \in I\}$ into a bundle over I. Since $F(i) \cong F(i) \times \{i\}$, the functors F and $\bar{F}$ are naturally isomorphic. What this all boils down to is that the passage from (X, f) to $\bar{f}$ is an equivalence of categories. The category $\mathbf{Bn}(I)$ of bundles over I is equivalent to the category $\mathbf{Set}^I$ of set-valued functors defined on I. □

These last four examples illustrate a construction that provides us with many topoi. We have:

> *for any "small" category $\mathscr{C}$, the functor category* $\mathbf{Set}^{\mathscr{C}}$ *is a topos!*

We devote the rest of this chapter to describing the topos structure of $\mathbf{Set}^{\mathscr{C}}$.

Terminal object

In $\mathbf{Set}^{\mathscr{C}}$ this is the constant functor $1: \mathscr{C} \to \mathbf{Set}$ that takes every $\mathscr{C}$-object to the one-element set $\{0\}$, and every $\mathscr{C}$-arrow to the identity on $\{0\}$. For any $F: \mathscr{C} \to \mathbf{Set}$ the unique arrow $F \rightarrowtriangle 1$ *in* $\mathbf{Set}^{\mathscr{C}}$ is the natural transformation whose components are the unique functions $!: F(a) \to \{0\}$ for each $\mathscr{C}$-object a.

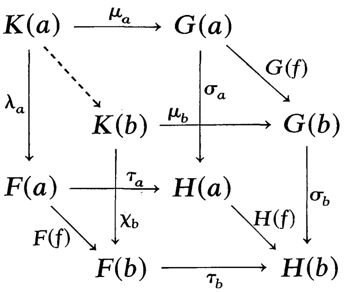
Pullback

This is defined "componentwise", as indeed are all limits and colimits in $\mathbf{Set}^{\mathscr{C}}$.

Given $\tau : F \dot{\rightarrow} H$ and $\sigma : G \dot{\rightarrow} H$, then for each $\mathscr{C}$-object a, form the pullback

$$\begin{array}{ccc} K(a) & \xrightarrow{\mu_a} & G(a) \\ {\scriptstyle\lambda_a}\downarrow & & \downarrow{\scriptstyle\sigma_a} \\ F(a) & \xrightarrow{\tau_a} & H(a) \end{array}$$

in **Set** of the components τ_a and σ_a. The assignment of $K(a)$ to a establishes a functor $K : \mathscr{C} \rightarrow \mathbf{Set}$. Given $\mathscr{C}$-arrow $f : a \rightarrow b$, $K(f)$ is the unique arrow $K(a) \rightarrow K(b)$ in the "cube"



given by the universal property of the front face as pullback. The λ_a's and μ_a's are components for $\lambda : K \dot{\rightarrow} F$ and $\mu : K \dot{\rightarrow} G$ that make

$$\begin{array}{ccc} K & \xrightarrow{\mu} & G \\ {\scriptstyle\lambda}\downarrow & & \downarrow{\scriptstyle\sigma} \\ F & \xrightarrow{\tau} & H \end{array}$$

a pullback in $\mathbf{Set}^{\mathscr{C}}$.

EXERCISE 4. Define the product $F \times G : \mathscr{C} \rightarrow \mathbf{Set}$ of two objects in $\mathbf{Set}^{\mathscr{C}}$. □

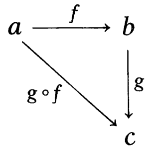
Subobject classifier

To define this we introduce a new notion. For a given $\mathscr{C}$-object a, let S_a be the collection of all $\mathscr{C}$-arrows with domain a,

$$S_a = \left\{ f \text{: for some } b,\ a \xrightarrow{f} b \text{ in } \mathscr{C} \right\}$$

(S_a is the class of objects for the category $\mathscr{C} \uparrow a$ of "objects under a" described in Chapter 3).

We note that S_a is "closed under left composition", i.e. if $f \in S_a$, then for any $\mathscr{C}$-arrow $g: b \rightarrow c$, $g \circ f \in S_a$



We define a *sieve on a*, or an *a-sieve* to be a subset S of S_a that is itself closed under left composition, i.e. has $g \circ f \in S$ whenever $f \in S$. For any object a there are always at least two a-sieves S_a and $\emptyset$ (the empty sieve).

EXAMPLE 1. In a discrete category, $S_a = \{1_a\}$, and so S_a and $\emptyset$ are the only a-sieves.

EXAMPLE 2. In **2**, with $f: 0 \rightarrow 1$ the unique non-identity arrow there are three 0-sieves, $\emptyset$, $S_0 = \{1_0, f\}$, and $\{f\}$.

EXAMPLE 3. In a one-object category (monoid) **M**, an M-sieve is a set $S \subseteq M$ of arrows closed under left composition = left multiplication. The sieves are just the left-ideals of **M**. □

Now we define $\Omega: \mathscr{C} \rightarrow \mathbf{Set}$ by

$$\Omega(a) = \{S : S \text{ is an } a\text{-sieve}\}$$

and for $\mathscr{C}$-arrow $f: a \rightarrow b$, let $\Omega(f): \Omega(a) \rightarrow \Omega(b)$ be the function that takes the a-sieve S to the b-sieve $\{b \xrightarrow{g} c : g \circ f \in S\}$ (why *is* this a sieve?)

Thus in $\mathbf{Set}^{\mathbf{M}}$, we find that $\Omega(M) = L_M$, the set of left ideals in **M**, and for arrow $m: M \rightarrow M$, $\Omega(m): L_M \rightarrow L_M$ takes S to $\{n : n * m \in S\} = \omega(m, S)$. So Ω becomes the action (L_M, ω) that is the codomain of the subobject classifier.

In $\mathbf{Set}^{\mathscr{C}}$ we define $\top: 1 \dot{\rightarrow} \Omega$ to be the natural transformation that has components $\top_a: \{0\} \rightarrow \Omega(a)$ given by $\top_a(0) = S_a$, the "largest" a-sieve. This arrow is the classifier for $\mathbf{Set}^{\mathscr{C}}$. To see how $\top$ works, suppose that $\tau: F \dot{\rightarrow} G$ is a monic arrow in $\mathbf{Set}^{\mathscr{C}}$. Then for each $\mathscr{C}$-object a, the component $\tau_a: F(a) \rightarrow G(a)$ is monic in **Set** (Exercise 3) and we will suppose it to be the inclusion $F(a) \hookrightarrow G(a)$. Now the character $\chi_\tau: G \dot{\rightarrow} \Omega$

of τ is to be a natural transformation with the component $(\chi_\tau)_a$ a set function from $G(a)$ to $\Omega(a)$. Thus $(\chi_\tau)_a$ assigns to each $x \in G(a)$, an a-sieve $(\chi_\tau)_a(x)$. The question then is to decide when an arrow $f: a \to b$ with domain a is in $(\chi_\tau)_a(x)$. For such an f, we have a commutative diagram

$$\begin{array}{ccc} F(a) & \overset{\tau_a}{\hookrightarrow} & G(a) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(b) & \overset{\tau_b}{\hookrightarrow} & G(b) \end{array}$$

so that $F(f)$ is the restriction of $G(f)$ to $F(a)$. We put f in $(\chi_\tau)_a(x)$ if and only if $G(f)$ maps x into $F(b)$. (Compare this with the picture for $\mathbf{Set}^{\to}$ in §4.4). Thus $(\chi_\tau)_a(x) = \{f: a \to b: G(f)(x) \in F(b)\}$.

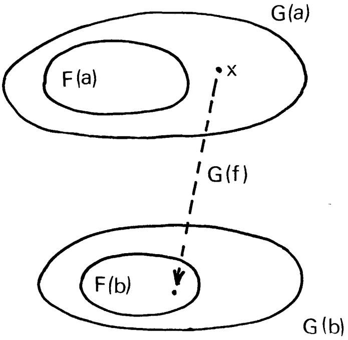


Fig. 9.1.

More generally, assuming only that τ_a is a function, perhaps not an inclusion, we put

$$(\chi_\tau)_a(x) = \left\{a \xrightarrow{f} b: G(f)(x) \in \tau_b(F(b))\right\}$$

$$= \left\{a \xrightarrow{f} b: \text{for some } y \in F(b),\ G(f)(x) = \tau_b(y)\right\}$$

Exercise 5. Verify that $(\chi_\tau)_a(x)$ is an a-sieve, and that this construction satisfies the Ω-axiom. (see §10.3)

Exercise 6. Show that it produces the classifiers for $\mathbf{Set}^2$, $\mathbf{Set}^{\to}$ and $\mathbf{Bn}(I)$.

Exercise 7. Let S be an a-sieve. Define $\bar{S}: \mathscr{C} \to \mathbf{Set}$ by $\bar{S}(b) = S \cap \mathscr{C}(a, b)$. Show that the inclusions $\bar{S}(b) \hookrightarrow \mathscr{C}(a, b)$ are the components of a monic $\mathbf{Set}^{\mathscr{C}}$-arrow $\bar{S} \rightarrowtail \mathscr{C}(a, -)$. Show that in fact the a-sieves are in bijective correspondence with the subobjects of the homfunctor $\mathscr{C}(a, -)$ in $\mathbf{Set}^{\mathscr{C}}$.

EXERCISE 8. Show that for each $\mathscr{C}$-object a, $(\Omega(a), \subseteq)$ is a Heyting algebra of subsets of S_a, with

$$\neg S = \left\{a \xrightarrow{f} b \text{: for any } b \xrightarrow{g} c,\ g \circ f \notin S\right\}$$

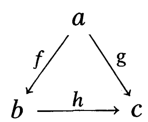
$$S \Rightarrow T = \{f\text{: whenever } g \circ f \in S, \text{ then } g \circ f \in T\}$$

Show that $\neg S$ is the largest (union) of all the a-sieves contained in $-S$, and $S \Rightarrow T$ is the largest a-sieve contained in $-S \cup T$. □

The dual to the notion of sieve is called an *a-crible*. This is a collection of arrows with codomain a that is closed under right-composition. Cribles are used to show that the category of *contravariant* functors from $\mathscr{C}$ to **Set** is a topos. This type of functor arises naturally in the study of sheaves, and the work of Grothendieck et al. [SGA4] is done in terms of cribles. We have used co-cribles because they are appropriate to the conventions of the Kripke semantics. Cribles themselves will be discussed in Chapter 14.

Exponentiation in $\mathbf{Set}^{\mathscr{C}}$

Let $F{:}\mathscr{C} \to \mathbf{Set}$. For each $\mathscr{C}$-object a, define a "forgetful" functor $F_a{:}\mathscr{C}{\uparrow}a \to \mathbf{Set}$ that takes $f{:}a \to b$ to $F(b)$, and $h{:}f \to g$ where



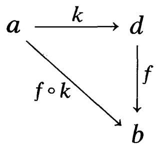
commutes, to $F(h)$.

Now given $F, G{:}\mathscr{C} \to \mathbf{Set}$, define $G^F{:}\mathscr{C} \to \mathbf{Set}$ by

$$G^F(a) = \mathrm{Nat}[F_a, G_a],$$

the collection of natural transformations from F_a to G_a.

Acting on arrows, G^F takes $k{:}a \to d$ to a function $G^F(k)$ from $\mathrm{Nat}[F_a, G_a]$ to $\mathrm{Nat}[F_d, G_d]$. This takes $\tau{:}F_a \dot{\to} G_a$ to $\tau'{:}F_d \dot{\to} G_d$ that has components $\tau'_f = \tau_{f \circ k}$, for



f an object in $\mathscr{C} \uparrow d$.

EXAMPLE. Let F and G be functors $\mathbf{2} \to \mathbf{Set}$, thought of as functions $f: A \to B$ and $g: C \to D$ (i.e. $\mathbf{Set}^{\to}$-objects). Now $\mathbf{2} \uparrow 1$ is the discrete one-object category. So F_1 is identifiable with $F(1) = B$, likewise G_1 "is" D, and

$$G^F(1) = D^B, \qquad \text{the set of functions } B \to D.$$

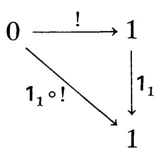
Now $\mathbf{2} \uparrow 0$ is isomorphic to $\mathbf{2}$ itself, so F_0 and G_0 can be taken as just F and G. Then

$$G^F(0) = \mathrm{Nat}[F, G] \text{ "=" } E,$$

where E is the set of $\mathbf{Set}^{\to}$-arrows from f to g. Finally G^F takes $!: 0 \to 1$ to

$$E \xrightarrow{g^f} D^B, \text{ as follows:}$$

Given $\tau: F \dot\to G$, corresponding to the $\mathbf{Set}^{\to}$-arrow $\langle \tau_0, \tau_1 \rangle$ from f to g, $G^F(\tau)$ is the transformation $F_1 \dot\to G_1$ whose sole component is τ_1, since 1 corresponds to the unique member 1_1 of $\mathbf{2} \uparrow 1$.



Thus $g^f(\langle \tau_0, \tau_1 \rangle) = \tau_1$, and this very complex construction has yielded the exponential object in $\mathbf{Set}^{\to}$. □

We have yet to define the evaluation arrow $ev: G^F \times F \dot\to G$ in $\mathbf{Set}^{\mathscr{C}}$. This has components $ev_a: G^F(a) \times F(a) \to G(a)$, where $ev_a(\langle \tau, x \rangle) = \tau_{1_a}(x)$ whenever $x \in F(a)$ and $\tau \in G^F(a)$, i.e. $\tau: F_a \dot\to G_a$ (note that the component τ_{1_a} of the $\mathscr{C} \uparrow a$-object 1_a is indeed a function from $F(a)$ to

$G(a)$. Now for a $\mathbf{Set}^{\mathscr{C}}$ arrow $\tau : H \times F \dot{\rightarrow} G$, the exponential adjoint $\hat{\tau} : H \dot{\rightarrow} G^F$ has components that are functions of the form

$$\hat{\tau}_a : H(a) \rightarrow G^F(a).$$

For each y in $H(a)$, $\hat{\tau}_a(y)$ is a natural transformation $F_a \dot{\rightarrow} G_a$. For each $\mathscr{C} \uparrow a$-object $f : a \rightarrow b$, $\hat{\tau}_a(y)$ assigns to f that function from $F(b)$ to $G(b)$ that for input $x \in F(b)$ gives output

$$\tau_b(\langle H(f)(y), x\rangle)$$

(note that τ_b: $H(b) \times F(b) \rightarrow G(b)$ and $H(f) : H(a) \rightarrow H(b)$).

The reader who has the head for such things may check out the details of this construction and relate it to exponentials in **M-Set**, **Bn**(I) etc. We shall need it only for the description of power objects in a special topos of Kripke models in Chapter 11. Our major concern will be with the subobject classifier of "set-valued" functor categories.