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SEARCH FOR THE GEOMETRODYNAMICAL GAUGE GROUP. HYPOTHESES AND SOME RESULTS

JAN J. SŁAWIANOWSKI and VASYL KOVALCHUK

Institute of Fundamental Technological Research, Polish Academy of Sciences
21, Świętokrzyska Str., 00–049 Warsaw, Poland

Abstract. Discussed is the problem of the mutual interaction between spinor and gravitational fields. The special stress is laid on the problem of the proper choice of the gauge group responsible for the spinorial geometrodynamics. According to some standard views this is to be the local, i.e., x-dependent, group $\mathrm{SL}(2,\mathbb{C})$, the covering group of the Lorentz group which rules the internal degrees of freedom of gravitational cotetrad. Our idea is that this group should be replaced by $\mathrm{SU}(2,2)$, i.e., the covering group of the Lorentz group in four dimensions. This leads to the idea of Klein-Gordon-Dirac equation which in a slightly different context was discovered by Barut and coworkers. The idea seems to explain the strange phenomenon of appearing leptons and quarks in characteristic, mysterious doublets in the electroweak interaction.

1. Introductory Remarks. Four-Component versus Two-Component Spinors in Special Relativity

Even now the concept of spinor is still rather mysterious. Let us begin with what is clean, doubtless and experimentally confirmed. Historically the first thing was the discovery by G. Uhlenbeck and S. Goudsmit that to understand the spectral lines of atoms one had to admit the existence of spin — internal angular momentum of electrons of the surprising magnitude $1/2$ in $\hbar$-units. The idea seemed so surprising and speculative that even prominent physicists like Lorentz and Fermi were strongly if not aggressively against it. Fortunately Ehrenfest and Bohr supported the hypothesis [15]. And the strongest support was experimental one, from atomic spectroscopy. The mathematical understanding came later on from group theory. An essential point is that the group $\mathrm{SU}(2)$ may be identified with the universal covering group of $\mathrm{SO}(3,\mathbb{R})$, orthogonal group in three real dimensions, isomorphic with the group of rotations around some fixed point in the physical Euclidean space. It is projective unitary representations rather than vector ones that is relevant for quantum mechanics. And when studying quantum projective representations it is natural to start from discussing the universal covering group. There is no direct nor commonly accepted interpretation of spin in terms of quantized gyroscopic degrees of freedom, although in spite of certain current views such an idea is not a priori meaningless. When relativistic quantum mechanics and field theory emerged, the half-integer internal angular momentum was interpreted in terms of the complex special linear group $\mathrm{SL}(2,\mathbb{C})$ as the universal covering group of the restricted Lorentz group $\mathrm{SO}^{\uparrow}(1,3)$. On this basis Wigner and Bargmann developed the systematic theory of relativistic linear wave equations. This theory was in a sense too general, formally predicting an infinity of particles and fields which do not seem to exist on the fundamental elementary level. Some new impact came from Dirac and his attempts of creating relativistic quantum mechanics based on first-order differential equations. The second-order Klein-Gordon equation did not seem to be satisfactory as a relativistic quantum-mechanical equation both because of its incompatibility with Born statistical interpretation (the non-existence of positively-definite probabilistic density) and because of its predictions incompatible with experimental data of atomic spectroscopy. This was the reason that the Klein-Gordon was rejected by Schrödinger who, by the way, was the first to formulate it. It turned out that the non-relativistic equation commonly referred to as Schrödinger equation gave much more satisfactory predictions, especially when combined phenomenologically with the spin idea into what is now known as two-component Pauli equation. It is well known that as a consequence of Dirac analysis the old XIX-th century idea of hypercomplex numbers and Clifford algebras revived. Namely, if the desired first-order equation

$$\mathrm{i}\gamma^{\mu}\partial_{\mu}\Psi = m\Psi \tag{1}$$

is to imply the **Klein-Gordon equation**

$$g^{\mu\nu}\partial_{\mu}\partial_{\nu}\Psi = -m^2\Psi, \qquad g^{\mu\nu}\left(\mathrm{i}\partial_{\mu}\right)\left(\mathrm{i}\partial_{\mu}\right)\Psi = m^2\Psi \tag{2}$$

where g denotes the specially-relativistic metric tensor of Minkowskian space-time ((2) is just the relativistic energy-momentum for free particles), then the "vector components" γ^{μ} have to satisfy

$$\gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu} = 2g^{\mu\nu} \tag{3}$$

i.e., they must be non-commutative algebraic entities, and certainly not numbers. Incidentally, this is obvious even from the very form of equation (1), because if γ is a usual vector, the equation would not be relativistically invariant. So certainly besides of the index μ, γ^{μ} must have certain additional indices and their interplay may result in the invariance under Poincaré group. The objects γ^{μ} commonly referred to as **Dirac matrices** are expected to be linear mappings of some linear space $\mathcal{D}$ into itself, so more rigorously, one should write (3) as

$$\gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu} = 2g^{\mu\nu} I_{\mathcal{D}} \tag{4}$$

where $I_{\mathcal{D}}$ denotes the identity operator in $\mathcal{D}$.

Alternatively, one can use **Dirac covectors** with components

$$\gamma_{\mu} := g_{\mu\nu}\gamma^{\nu} \tag{5}$$

satisfying

$$\gamma_{\mu}\gamma_{\nu} + \gamma_{\nu}\gamma_{\mu} = 2g_{\mu\nu} I_{\mathcal{D}}. \tag{6}$$

The above formulae tell us simply that the scalar quadrats of covectors and vectors are literally represented as squares of something

$$(\gamma^{\mu} p_{\mu})^2 = g^{\mu\nu} p_{\mu} p_{\nu}, \qquad (\gamma_{\mu} x^{\mu})^2 = g_{\mu\nu} x^{\mu} x^{\nu} \tag{7}$$

or, more precisely

$$(\gamma^{\mu} p_{\mu})^2 = g^{\mu\nu} p_{\mu} p_{\nu} I_{\mathcal{D}}, \qquad (\gamma_{\mu} x^{\mu})^2 = g_{\mu\nu} x^{\mu} x^{\nu} I_{\mathcal{D}}. \tag{8}$$

On the quantum level, when “momenta” p_{μ} are replaced by operators $\mathrm{i}\partial_{\mu}$, this is the “square-rootization” of the **d’Alembert operator**

$$(\gamma^{\mu} \mathrm{i}\partial_{\mu})^2 \; = \; I_{\mathcal{D}}\Box = -I_{\mathcal{D}} g^{\mu\nu} \left(\mathrm{i}\partial_{\mu}\right)\left(\mathrm{i}\partial_{\nu}\right) \tag{9}$$

$$(\gamma^{\mu} \partial_{\mu})^2 \; = \; I_{\mathcal{D}}\Box = I_{\mathcal{D}} g^{\mu\nu} \partial_{\mu}\partial_{\nu}. \tag{10}$$

To avoid the crowd of characters, in literature one usually omits the symbol $I_{\mathcal{D}}$, although literally incorrect, this does not lead to misunderstandings.

This linear realisation in terms of linear mappings $\gamma^{\mu}, \gamma_{\mu} \in \mathrm{L}\left(\mathcal{D}\right) \simeq \mathcal{D} \otimes \mathcal{D}^{*}$ is necessary in physics, both on the fundamental and computational level. Nevertheless, from the more abstract and formal point of view, the expressions above were a physical rediscovery (by Dirac) of Clifford algebras. This concept is certainly more general than physical problems appearing in four-dimensional Minkowski space-time or three-dimensional Euclidean space.

Let (V, g) be a pseudo-Euclidean space, so V is a finite-dimensional vector space and $g \in V^{*} \otimes V^{*}$ is a symmetric non-degenerate metric tensor in V. It needs not be definite; it is positive in the three-dimensional Euclidean space but has the normal-hyperbolic signature $(+, -, -, -)$ (or $(-, +, +, +)$) in Minkowskian space-time of special relativity. Obviously, in those examples V is a linear space of translation vectors, respectively in space and space-time. Let $\mathcal{T}_0(V)$ denote the associative algebra of all contravariant tensors in V

$$\mathcal{T}_0(V) = \left(\mathbb{R} \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \cdots\right) \tag{11}$$

i.e., the set of infinite sequences of contravariant tensors of all possible orders with the obvious multiplication rule. Although it is literally incorrect, nevertheless technically convenient to write those sequences as formal sums:

$$(c, v, t, s, \dots) = c + v + t + s + \cdots . \tag{12}$$

This is an abbreviation for

$$(c, 0, 0, 0, \dots) + (0, v, 0, 0, \dots) + (0, 0, t, 0, \dots) + (0, 0, 0, s, \dots) + \cdots \tag{13}$$

where $c \in \mathbb{R}$, $v \in V$, $t \in V \otimes V$, $s \in V \otimes V \otimes V$, etc. The notation (12) together with the reduction procedure enables one to perform the tensor multiplication in $\mathcal{T}_0(V)$ in a simple, automatic way.

Let us take the elements of $\mathcal{T}_0(V)$ of the form

$$u \otimes v + v \otimes u - 2g(u, v) \tag{14}$$

or, more precisely,

$$(-2g(u, v), 0, u \otimes v + v \otimes u, 0, \dots) \tag{15}$$

where the vectors u, v run over all of the space V.

Let $\mathcal{J}(V, g) \subset \mathcal{T}_0(V)$ denote the ideal of the associative algebra $\mathcal{T}_0(V)$, generated by elements of the form (15). Both $\mathcal{T}_0(V)$ and $\mathcal{J}(V, g)$ are infinite-dimensional, however the quotient space

$$\mathrm{Cl}(V, g) := \mathcal{T}_0(V)/\mathcal{J}(V, g)$$

has a finite dimension. This is just the Clifford algebra of (V, g). The associative product in $\mathrm{Cl}(V, g)$ is induced from that in $\mathcal{T}_0(V)$ as usual in the quotient space of an associative algebra with respect to its ideal. If $(\dots, e_i, \dots)$ is a basis in V, then the corresponding induced basis in $\mathcal{T}_0(V)$ consists of the elements

$$(1, e_i, e_i \otimes e_j, e_i \otimes e_j \otimes e_k, \dots) \tag{16}$$

where the labels run over all possible values $i = 1, \dots, \dim V$. The identification of $e_i \otimes e_j + e_j \otimes e_i$ with $2g_{ij}$, more precisely, the identification of

$$(-2g_{ij}, 0, e_i \otimes e_j + e_j \otimes e_i, 0, \dots) \tag{17}$$

with the null element when the quotient procedure is performed, tells us that the basis of $\mathrm{Cl}(V, g)$ consists of elements which for brevity will be denoted as follows:

$$1,\ e_i,\ e_i e_j,\ e_i e_j e_k,\ \dots,\ e_1 e_2 \cdots e_n, \qquad i < j, \quad i < j < k,\ \dots . \tag{18}$$

They are canonical projections (under the quotient procedure) of

$$\begin{aligned} &(1, 0, 0, 0, \dots),\ (0, e_i, 0, 0, \dots),\ (0, 0, e_i \otimes e_j, 0, \dots) \\ &(0, 0, 0, e_i \otimes e_j \otimes e_k, \dots),\ \dots,\ (0, \dots, 0, e_1 \otimes e_2 \otimes \cdots \otimes e_n) . \end{aligned} \tag{19}$$

The quotient-projections of other elements of (16), in particular, higher-order ones, may be expressed through (19), for example, if the basis e is g-orthogonal

$$e_j e_i = -e_i e_j + 2g_{ij}, \qquad e_1 e_2 \cdots e_n e_1 = (-1)^{n-1} g_{11} e_2 \cdots e_n . \tag{20}$$

Orthogonality means obviously

$$g_{ij} = g(e_i, e_j) = 0, \qquad \text{if} \quad i \neq j. \tag{21}$$

Usually, although not necessarily, we use orthonormal bases when besides of (21) the following holds

$$g_{ii} = g(e_i, e_i) = \pm 1. \tag{22}$$

We are dealing here only with real linear spaces (the ones over $\mathbb{R}$) when the concept of signature does exist and the number of diagonal $\pm$ signs is well defined and invariant.

In linear realisations, when the elements of $\mathrm{Cl}(V, g)$ are isomorphically represented by linear mappings of some linear complex space $\mathcal{D}$ into itself, the representants of basic elements e_i will be denoted by Dirac symbols γ_i.

The elements of $\mathrm{Cl}(V, g)$ for which the multiplicative inverse exists form the group $\mathrm{GCl}(V, g)$ under the associative product which is referred to as **Clifford group**. This group acts in $\mathrm{Cl}(V, g)$ through the similarity transformations

$$A \in \mathrm{GCl}(V, g): \qquad \mathrm{Cl}(V, g) \ni X \mapsto AXA^{-1}. \tag{23}$$

Let us distinguish the subgroup $\widetilde{\mathrm{O}}(V, g) \subset \mathrm{GCl}(V, g)$ which acting in this way does preserve the subspace V of $\mathrm{Cl}(V, g)$, or, to be more precise, the subspace $(0, V, 0, \ldots, 0)$

$$A \in \widetilde{\mathrm{O}}(V, g): \qquad A(0, V, 0, \ldots, 0)A^{-1} = (0, V, 0, \ldots, 0). \tag{24}$$

This action induces the action of the pseudo-orthogonal group $\mathrm{O}(V, g)$ on V

$$A(0, v, 0, \ldots, 0)A^{-1} = (0, L[A]v, 0, \ldots, 0) \tag{25}$$

where, obviously, the assignment

$$A \in \widetilde{\mathrm{O}}(V, g) \mapsto L[A] \in \mathrm{O}(V, g) \tag{26}$$

is a group homomorphism. Obviously, it is seen that A, $-A$ give rise to the same pseudo-orthogonal mappings

$$L[-A] = L[A]. \tag{27}$$

Moreover, $\widetilde{\mathrm{O}}(V, g)$ is the universal covering group of $\mathrm{O}(V, g)$. In the special case of three-dimensional Euclidean space or four-dimensional Minkowski space, the $2:1$ universal covering groups of the connected components of unity $\mathrm{SO}(3, \mathbb{R})$, $\mathrm{SO}^{\uparrow}(1, 3)$ may be identified respectively with $\mathrm{SU}(2)$ and $\mathrm{SL}(2, \mathbb{R})$, according to the well-known analytical procedure.

Linear realisation of all those objects is necessary for physical purposes. There is an infinity of possible dimensions of the space $\mathcal{D}$ of Dirac objects (Dirac spinors). In physics the special stress is laid on irreducible minimal realisation. It is well known that if the real dimension of V equals $n = 2m$, m being a natural number, then the lowest possible dimension of $\mathcal{D}$ equals $2^m = 2^{n/2}$ and this is the complex dimension ($\mathcal{D}$ is a linear space over the field $\mathbb{C}$). As usual in fundamental physics, field equations are self-adjoint, i.e., derivable from variational principles. To construct Lagrangians for the $\mathcal{D}$-valued Dirac field, we must have at disposal some sesquilinear Hermitian form G on $\mathcal{D}$, $G \in \overline{\mathcal{D}}^* \otimes \mathcal{D}$, such that the "Dirac matrices" $\gamma \in \mathrm{L}(\mathcal{D}) \simeq \mathcal{D} \otimes \mathcal{D}^*$ are Hermitian with respect to G. In any case, it is so if we wish to construct Lagrangian for the Dirac equation. Some comments are necessary here, because usually the literature devoted to the subject is either very mathematically abstract, one can say esoteric, or, much more often, purely analytical and full of misunderstandings. Those misunderstandings come from the analytical misuse of the matrix concept, without any attention paid to the essential problem, what are geometric objects represented by matrices. Let us stress a few important points. The so called "Dirac matrices" provide an analytical description of some mixed tensors, i.e., linear mappings in $\mathcal{D}$, $\gamma^\mu \in \mathrm{L}(\mathcal{D}) \simeq \mathcal{D} \otimes \mathcal{D}^*$, so their analytical representation reads $\gamma^{\mu r}{}_s$ in which the indices r, s refer to the space $\mathcal{D}$. The above-mentioned Hermitian form $\Gamma \in \overline{\mathcal{D}}^* \otimes \mathcal{D}$ is a twice covariant tensor in $\mathcal{D}$, "complex in the first index". The corresponding analytical expression is $G_{\bar{r}s}$. Evaluation of G on the pair of objects $\Psi, \varphi \in \mathcal{D}$ is analytically given by

$$G(\Psi, \varphi) = G_{\bar{r}s} \bar{\Psi}^{\bar{r}} \varphi^s = \overline{G(\varphi, \Psi)}. \tag{28}$$

And similarly, the action of γ^μ is analytically given by

$$(\gamma^\mu \Psi)^r = \gamma^{\mu r}{}_s \Psi^s. \tag{29}$$

The inverse form of G, $G^{-1} \in \mathcal{D} \otimes \overline{\mathcal{D}}$ is a twice contravariant tensor "complex in the second index". To avoid the crowd of symbols, in analytical representation we omit the symbol of inverting and use simply the analytical expression $G^{r\bar{s}}$, where

$$G_{\bar{r}z} G^{z\bar{s}} = \delta_{\bar{r}}{}^{\bar{s}}, \qquad G^{r\bar{z}} G_{\bar{z}s} = \delta^r{}_s. \tag{30}$$

The corresponding "deltas" represent, respectively, identity mappings of $\overline{\mathcal{D}}^*$ and $\mathcal{D}$. The choice of G must be compatible with γ^μ in the sense, that "gammas" must be Hermitian with respect to G. Namely, let us introduce sesquilinear forms Γ^μ, Γ_μ on $\mathcal{D}$, $\Gamma^\mu \in \overline{\mathcal{D}}^* \otimes \mathcal{D}$, $\Gamma_\mu \in \mathcal{D}^* \otimes \mathcal{D}$ by the G-shifting of spinor indices

$$\Gamma^\mu{}_{\bar{r}s} = G_{\bar{r}z} \gamma^{\mu z}{}_s, \qquad \Gamma_{\mu \bar{r}s} = g_{\mu\nu} \Gamma^\nu{}_{\bar{r}s} = G_{\bar{r}z} \gamma^z{}_{s\mu}. \tag{31}$$

It might be perhaps suggestive to use the symbols $\gamma^\mu{}_{\bar{r}s}$, $\gamma_{\mu\bar{r}s}$, however, this would be also confusing. The sesquilinear forms Γ^μ, Γ_μ must be Hermitian,

$$\Gamma^\mu(\Psi, \varphi) = \overline{\Gamma^\mu(\varphi, \Psi)}, \qquad \Gamma_\mu(\Psi, \varphi) = \overline{\Gamma_\mu(\varphi, \Psi)} \tag{32}$$

i.e., analytically,

$$\Gamma^\mu{}_{\bar{r}s} = \overline{\Gamma^\mu{}_{\bar{s}r}}, \qquad \Gamma_{\mu\bar{r}s} = \overline{\Gamma_{\mu\bar{s}r}} \tag{33}$$

where, as usual, the coefficients of Γ^μ are defined by

$$\Gamma^\mu(\Psi,\varphi) = \Gamma^\mu{}_{\bar{r}s}\overline{\Psi}^{\bar{r}}\varphi^s. \tag{34}$$

When one deals with Minkowski space of signature $(+,-,-,-)$ or $(-,+,+,+)$, G must have the neutral signature $(+,+,-,-)$.

Let us notice that G gives rise to the antilinear mappings

$$\mathcal{D} \ni \Psi \mapsto \widetilde{\Psi} \in \mathcal{D}^* \tag{35}$$

where

$$\widetilde{\Psi}_r := \overline{\Psi}^{\bar{s}} G_{\bar{s}r}. \tag{36}$$

This is the so-called **Dirac conjugation** (the "Dirac bar operation").

Let us stress that the particular matrix realisation of γ^μ and G is a matter of convenience and it is only their mutual relationships system quoted above that matters. In commonly used representation the matrix $[G_{\bar{r}s}]$ coincides numerically with $[\gamma^{0r}{}_s]$. This is at least one of infinitely many representations, perhaps computationally the most convenient one. If the machine producing $\widetilde{\Psi}^r$ from Ψ^r was essentially given by γ^0, this would be a drastic violation of the relativistic invariance.

Everything formulated according to the Clifford paradigm may be done in arbitrary dimension. But our physical space-time is just four-dimensional. And the higher-dimensional Universes in the Kaluza style are still rather hypothetical what concerns their fundamental existence. And some special features of dimension four lead to another paradigm. Namely, the Hermitian geometry of the Dirac space has the neutral signature $(+,+,-,-)$, so the group of pseudounitary transformations $\mathrm{U}(\mathcal{D},G) \simeq \mathrm{U}(2,2)$ preserving G seems to be something fundamental. But its special subgroup $\mathrm{SU}(\mathcal{D},G) \simeq \mathrm{SU}(2,2)$ consisting of transformations with determinants equal to unity is the universal covering group of the 15-dimensional conformal group $\mathrm{CO}(V,g) \simeq \mathrm{CO}(1,3)$ of Minkowskian space. Perhaps it is just here where another paradigm should be sought? In other dimensions this coincidence of the group of symmetries of Hermitian scalar product of spinors and the space-time conformal group breaks down. But our space-time at least in certain its aspects is just four-dimensional. So it is difficult to decide a priori which paradigm should be accepted. And in a sense they seem to suggest different dynamical models.

There is also another point of the special dimension four, which has to do with certain ideas formulated by Weizsäcker, Finkelstein and Penrose. They were also a basis towards reconciliation of quanta and gravitation (general relativity). The two theories seem to be historically incompatible. Everything has to do with the Weizsäcker idea of "urs".

The starting point is that every physical experiment may be finally decomposed into a sequence of yes-no experiments, i.e., in a sense the Universe is something like the giant computer device. So in the beginning there is a dichotomy – two-element set $\mathbb{Z}_2$, one can consider it as the $\{0,1\}$-set, non-excited and excited (active). But we know that physical phenomena are ruled by quantum mechanics with its superposition principle and wave-particle dualism. Therefore, the next step is to take the linear shell of $\mathbb{Z}_2$ over the complex field $\mathbb{C}$, i.e., the complex linear space $\mathbb{C}^2$. And it is also known that usually there is no physically fixed basis, so instead one should start with the $\mathbb{C}$-two-dimensional complex linear space W. As yet we do not assume any fixed geometric structure in W.

Let us make a small digression concerning the complex linear geometry. Any complex linear space W of arbitrary dimension n gives rise to the natural quadruple of mutually related linear complex spaces. Those are: W itself, its complex conjugate $\overline{W}$, the dual W^* (we mean dual over $\mathbb{C}$) and the antidual $\overline{W^*} = \overline{W}^*$. Obviously, as in every linear space, W^* is the space of linear (over $\mathbb{C}$) functionals on W. The antidual $\overline{W^*} = \overline{W}^*$ consists of antilinear (half-linear) functions on W. Its elements may be simply defined as argument-wise complex conjugates of linear functions, so $\overline{f} \in \overline{W^*}$ operates on W according to $\overline{f}(u) := \overline{f(u)}$. The assignment $W^* \ni f \mapsto \overline{f} \in \overline{W^*}$ is an antilinear (half-linear) isomorphism of W^* onto $\overline{W^*}$. In finite dimension, by analogy to the canonical isomorphism between W and W^{**}, we can define $\overline{W}$ as the space of antilinear functions on W^*. So, there exists an antilinear isomorphism of W onto $\overline{W}$, $W \ni u \mapsto \overline{u} \in \overline{W}$, such that $\overline{u}$ as a functional on W^* acts as follows: $\overline{u}(f) := \overline{f(u)}$. If $(\ldots, e_i, \ldots)$ is some basis in W, then the corresponding bases in W^*, $\overline{W}$, $\overline{W^*}$ will be denoted respectively by $(\ldots, e^i, \ldots)$, $(\ldots, \overline{e}_{\bar{i}}, \ldots)$ and $(\ldots, \overline{e}^{\bar{i}}, \ldots)$. It must be stressed that there is no canonical complex conjugate of vectors in a given linear space W and that the antilinear complex conjugate operation acts between different linear spaces, e.g., W and $\overline{W}$ are W^* and $\overline{W}^*$. The complex conjugate of vectors in a given linear space is possible only when W itself is endowed with an additional structure which is neither assumed here nor would be physically interpretable. Of course one could remain on the level of $\mathbb{C}$, but then the crowd of apparently natural but neither mathematically nor physically motivated objects like $\sum_{a=1}^{n} \overline{u^a} v^a$ appear. No such artefacts when working in an abstract W.

The next step, both mathematically and physically is the tower of tensor byproducts over W. The most important objects are hermitian forms on W and W^*. They are respectively sesquilinear forms on W and W^*

$$p : W \times W \to \mathbb{C}, \qquad p \in \overline{W}^* \otimes W^*$$

and

$$x : W^* \times W^* \to \mathbb{C}, \qquad x \in W \otimes \overline{W}$$

satisfying respectively the hermiticity conditions

$$p_{\bar{a}b} = \overline{p_{\bar{b}a}}, \qquad x^{a\bar{b}} = \overline{x^{b\bar{a}}} \tag{37}$$

i.e., more geometrically

$$p(w_1, w_2) = \overline{p(w_2, w_1)}, \qquad x(f_1, f_2) = \overline{x(f_2, f_1)}. \tag{38}$$

These four-dimensional spaces, denoted respectively as

$$\mathrm{Herm}\left(\overline{W}^* \otimes W^*\right), \qquad \mathrm{Herm}\left(W \otimes \overline{W}\right)$$

are evidently dual in a canonical form to each other in the sense of pairing

$$\langle p, x\rangle = \mathrm{Tr}(p, x) = p_{\bar{a}b}x^{b\bar{a}} = \mathrm{Tr}(xp) \in \mathbb{C}. \tag{39}$$

The natural bases of $W \otimes \overline{W}$, $\overline{W}^* \otimes W^*$, corresponding to some choice of basis $(e_1, \ldots, e_n)$ in W is obviously, the system of

$$e_i \otimes \overline{e}_{\bar{j}}, \qquad \overline{e}^{\,\bar{i}} \otimes e^j.$$

The subspaces Herm $\left(\overline{W}^* \otimes W^*\right)$ and Herm $\left(W \otimes \overline{W}\right)$ are spanned on some basic Hermitian forms on W and W^*. The most convenient possibility is to choose as coefficients some numerical Hermitian matrices. The traditional historical convention in field theory of fundamental two-component spinors are Pauli matrices and the corresponding bases in Herm $\left(\overline{W}^* \otimes W^*\right)$, Herm $\left(W \otimes \overline{W}\right)$

$$\sigma[e]^A = \frac{1}{\sqrt{2}}\sigma^A{}_{\bar{a}b}\overline{e}^{\,\bar{a}} \otimes e^b, \qquad \sigma[e]_A = \frac{1}{\sqrt{2}}\sigma_A{}^{b\bar{a}}e_b \otimes \overline{e}_{\bar{a}}. \tag{40}$$

Some remarks are necessary here. Obviously, we mean here the "relativistic" quadruplet of sigma-matrices, so $\sigma^0 = \sigma_0 = I_2$ is the 2×2 identity matrix. The remaining ones, σ_R, $R = 1, 2, 3$, are a usual triplet of Pauli matrices. But, of course, unlike in the non-relativistic Pauli theory of spinning electron, they are not the spin operators (multiplied by $2/\hbar$) acting in the two-dimensional internal Hilbert space. They are Hermitian forms, so twice covariant and twice contravariant (once complex), certainly they are not Hermitian operators acting in a two-dimensional Hilbert space. Incidentally, it is very essential that in the internal spaces of Weyl fields W, $\overline{W}^*$ there is no fixed Hermitian scalar product with respect to which sigmas would be linear Hermitian operators, i.e., mixed tensors. This has to do with the structure of Weyl equations, their self-adjoint structure and their noninvariance under spatial reflections.

A very important point is the status of the internal "relativistic" index A. The lower and upper cases of A have nothing to do with the metrical shifting of indices with the help of some internal Minkowski metric η_{AB}. The point is important because the alternative linear bases in Herm $\left(\overline{W}^* \otimes W^*\right)$, Herm $\left(W \otimes \overline{W}\right)$

$$\widetilde{\sigma}[e]_A := \eta_{AB}\sigma[e]^B, \qquad \widetilde{\sigma}[e]^A := \eta^{AB}\sigma[e]_B \tag{41}$$

are also used for certain purposes. The level of writing the capital indices in (40) and separately in (41) has only to do with the pairs of dual bases. Namely, $\mathrm{Herm}\left(\overline{W}^{*} \otimes W^{*}\right)$ and $\mathrm{Herm}\left(W \otimes \overline{W}\right)$ are mutually dual in the canonical way and the corresponding bases are also dual

$$\left\langle \sigma[e]^{A}, \sigma[e]_{B} \right\rangle = \frac{1}{2}\sigma[e]^{A}{}_{\bar{a}b}\sigma[e]_{B}{}^{b\bar{a}} = \frac{1}{2}\,\mathrm{Tr}\left(\sigma^{A}\sigma_{B}\right) = \delta^{A}{}_{B} \tag{42}$$

independently on the choice of the base e. When this choice is fixed, we do not distinguish graphically between $\sigma[e]^{A}$, $\sigma[e]_{A}$ and σ^{A}, σ_{A}. And similarly for $\widetilde{\sigma}[e]^{A}$, $\widetilde{\sigma}[e]_{A}$. But unlike this

$$\left\langle \widetilde{\sigma}[e]^{A}, \widetilde{\sigma}[e]^{B} \right\rangle = 2\eta^{AB}, \qquad \langle \widetilde{\sigma}[e]_{A}, \widetilde{\sigma}[e]_{B} \rangle = 2\eta_{AB}. \tag{43}$$

Numerically the matrices σ^{A}, σ_{A} coincide and equal the "relativistic" quadruplet of "sigmas". Similarly, $\widetilde{\sigma}^{A}$ coincide with $\widetilde{\sigma}_{A}$ and equal the quadruplet of "sigmas" with relativistically η-corrected signs.

We were dealing here (and are so all over in analytical manipulations of spinors) with few of infinity possibility of mistakes appearing when one does not distinguish between bi(sesqui)linear forms, linear mappings and their matrices.

Let us follow the idea of two-component spinors as something primary and its impact on Dirac theory and its conformal modifications.

First, let us remind that if $\dim W = 2$, then the subspaces of Hermitian tensors $\mathrm{H}(W) \subset W \otimes \overline{W}$, $\mathrm{H}(W)^{*} \subset \overline{W}^{*} \otimes W^{*}$ are endowed with a natural conformal-Minkowskian geometry, i.e., Minkowski tensor defined up to a constant multiplier. Indeed, the peculiarity of dimension two is that for any $x \in \mathrm{H}(W)$, $p \in \mathrm{H}(W)^{*}$, the determinants

$$\det\left[x^{b\bar{a}}\right], \qquad \det\left[p_{\bar{a}b}\right] \tag{44}$$

are quadratic forms and one can easily see they have normal-hyperbolic signature. It is still a mystery if there is something deep in this fact and the underlying reasoning or this is a strange accident. There is an idea that starting from this one can reconciliate quanta and gravitation (more generally – quanta and gravitation). As both indices have the same valence, the determinants are not scalars in $\mathrm{H}(W)$, but respectively scalar densities of weight -2 and 2. Changing the basis in W multiplies them by the appropriate power of the transformation matrix.

When some among infinity of conformally equivalent metrics $\eta \in \mathrm{H}(W)^{*} \otimes \mathrm{H}(W)^{*}$ is fixed once for all, i.e., the standard of scale is chosen, then we can always choose the basis (e_1, e_2) in W in such a way that, e.g.,

$$[\eta_{AB}] = \mathrm{diag}\,(1, -1, -1, -1) \tag{45}$$

i.e.,

$$\eta = \eta_{AB}\sigma[e]^{A} \otimes \sigma[e]^{B} \tag{46}$$

the bases $\sigma[e]^A$ and $\sigma[e]_A$, $A = 0, 1, 2, 3$, are η-orthonormal.

Another fixation of scale is based on the choice of symplectic structure ε on W. Being two-dimensional, it has only one such up to a complex multiplier. So, in a fixed basis (e_1, e_2) we can take

$$[\varepsilon_{ab}] = [\overline{\varepsilon}_{\bar{a}\bar{b}}] = -\left[\varepsilon^{ab}\right] = -\left[\overline{\varepsilon}^{\bar{a}\bar{b}}\right] = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \tag{47}$$

And then for $x = x^A\sigma[e]_A$, $y = y^A\sigma[e]_A$

$$\eta(x, y) = \eta_{AB}x^Ay^B, \qquad \eta_{AB} = \frac{1}{2}\sigma_A{}^{b\bar{a}}\sigma_B{}^{d\bar{c}}\varepsilon[e]_{bd}\overline{\varepsilon}[e]_{\bar{a}\bar{c}}. \tag{48}$$

Obviously, the unimodular complex multiplier $\exp(\mathrm{i}\varphi)$, $\varphi \in \mathbb{R}$, does not influence η and it is only the absolute value of the multiplier that modifies the scale. Obviously, the inverse objects in (47) are meant in the usual sense

$$\varepsilon[e]^{ac}\varepsilon[e]_{cb} = \delta^a{}_b, \qquad \overline{\varepsilon}[e]^{\bar{a}\bar{c}}\overline{\varepsilon}[e]_{\bar{c}\bar{b}} = \delta^{\bar{a}}{}_{\bar{b}}. \tag{49}$$

Another similar, but in a sense intrinsic object in W is the tensor density of weight one E_{ab} defined by the condition that in all possible bases in W

$$[E_{ab}] = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \tag{50}$$

Obviously the inverse E^{ab} given in all coordinates by

$$\left[E^{ab}\right] = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \qquad E^{ac}E_{cb} = \delta^a{}_b \tag{51}$$

is the tensor density of weight minus one. Those Ricci objects enable one to construct in $\mathrm{H}(W)$ the symmetric tensor density of weight two, using just the second of the formulae (48)

$$N_{AB} := \frac{1}{2}\sigma_A{}^{b\bar{a}}\sigma_B{}^{d\bar{c}}E_{bd}E_{\bar{a}\bar{c}}$$

but it is hard to decide if some physical meaning may attributed to this object and to its contravariant inverse of weight minus two.

No doubt, the idea of deriving specially-relativistic geometry from two-component complex objects (spinors), especially in the context of Weizsäcker "urs" is interesting, although not yet proven (if provable at all) in a very convincing way, just one of hypothetical paradigms. It is very interesting that the non-definite Hermitian tensors, i.e., elements of $\mathrm{H}(W)$, $\mathrm{H}(W^*) \simeq \mathrm{H}(W)^*$ are space-like in the sense of above Minkowski metric η, degenerate ones correspond to the "light cones" of isotropic vectors and covectors, whereas the definite ones are time-like.

The positively definite ones may be assumed to define future, whereas the negative ones are by definition past-oriented. The degenerate forms, i.e., light-cone elements are future- or past oriented depending on whether they adhere respectively to the bulk of positively- or negatively-definite hermitian tensors. There is nothing like such a basis for defining canonically future and past when Minkowskian space is primary one, not derived as a byproduct of the Weyl space W.

There is another interesting link between ideas of two-dimensional "quantum amplitudes" and "specially-relativistic" geometry. Namely, W as a two-dimensional linear space over $\mathbb{C}$ is completely amorphous. No particular geometric object is fixed in W as an absolute one; in particular, none of infinity of Hermitian forms is distinguished in it. So, there is no fixed positive scalar product in W, it is not a Hilbert space and there is no probabilistic interpretation in the usual sense. However, if we once fix some positive sesquilinear form $\kappa \in \mathrm{H}(W^*) = \mathrm{Herm}\left(\overline{W}^* \otimes W^*\right)$, i.e., some positive scalar product, then (W, κ) becomes the Hilbert space admitting a true quantum-mechanical interpretation. Let us remind the idea, controversial but interesting one, expressed may years ago in the book by Marshak and Sudarshan [7] that the quantum-mechanical formalism becomes operationally interpretable always with respect to some reference frame. And as said above, any positively definite, thus time-like and future oriented element κ of $\mathrm{H}(W^*)$ is a reference frame in the "space time" $(\mathrm{H}(W), \mathbb{R}^+\eta)$.

Let us continue with byproducts of the Weyl paradigm of two-component spinors. The target space W of Weyl spinor fields is completely amorphous as no absolute objects are fixed in it. Unlike this, its byproducts like $\mathrm{H}(W^*)$, $\mathrm{H}(W)$, and the space of Dirac bispinors

$$\mathcal{D} := W \times \overline{W}^* \tag{52}$$

are full of byproducts structures. By analogy to linear spaces $W \times W^*$ which carry canonical symplectic structures (and in the real W case – the neutral-signature pseudo-Euclidean structures), any complex space of the form $W \times \overline{W^*}$, it does not matter of what dimension, is endowed with two natural Hermitian structures of neutral signature. Let us quote them

$$G\left(\left(w_1, f_1\right), \left(w_2, f_2\right)\right) := \overline{f_1(w_2)} + f_2(w_1) \tag{53}$$

$$\mathrm{i}F\left(\left(w_1, f_1\right), \left(w_2, f_2\right)\right) := \mathrm{i}\left(\overline{f_1(w_2)} - f_2(w_1)\right). \tag{54}$$

The sesquilinear forms G, F are respectively Hermitian and anti-Hermitian

$$G\left(\Psi_1, \Psi_2\right) = \overline{G\left(\Psi_2, \Psi_1\right)}, \qquad F\left(\Psi_1, \Psi_2\right) = -\overline{F\left(\Psi_2, \Psi_1\right)}. \tag{55}$$

If we use adapted coordinates in the physical dimension four, we obtain

$$[G_{\bar{r}s}] = \begin{bmatrix} 0 & I_2 \\ I_2 & 0 \end{bmatrix}, \qquad [F_{\bar{r}s}] = \begin{bmatrix} 0 & -I_2 \\ I_2 & 0 \end{bmatrix} \tag{56}$$

where I_2 denotes the 2×2 identity matrix, 0 is the zero matrix.

If G is interpreted as the bispinor scalar product, then the G-raising of the bar-index of F leads to that is usually interpreted as the γ^5-Dirac "matrix"

$$\gamma^5 = -\gamma^0\gamma^1\gamma^2\gamma^3 = \mathrm{i}\begin{bmatrix} I_2 & 0 \\ 0 & -I_2 \end{bmatrix}. \tag{57}$$

But as yet the Dirac matrices were not introduced in any particular analytical representation, in particular, in the one compatible with (56) and (57). From the point of view of the Weyl paradigm of two-component spinors as primary entities, when the Minkowskian target metric η_{AB} is fixed in its particular standard form, the most natural is the Weyl-van-der Waerden-Infeld representation

$$\gamma^A = \begin{bmatrix} 0 & \widetilde{\sigma}^A \\ \sigma^A & 0 \end{bmatrix}. \tag{58}$$

More precisely, this analytical matrix representation is to be understood in such a way that γ^A are linear mappings from $\mathcal{D} = W \times \overline{W}^*$ with matrices

$$\left[\gamma^{Ar}{}_s\right] = \begin{bmatrix} 0 & \widetilde{\sigma}^{Aa\bar{b}} \\ \sigma^A{}_{\bar{a}b} & 0 \end{bmatrix} \tag{59}$$

where the action on bispinors $[\Psi^r]^T = [u^a, v_{\bar{a}}]^T$ is analytically meant as follows

$$\begin{bmatrix} 0 & \widetilde{\sigma}^{a\bar{b}} \\ \sigma_{\bar{a}b} & 0 \end{bmatrix}\begin{bmatrix} u^b \\ v_{\bar{b}} \end{bmatrix} = \begin{bmatrix} \widetilde{\sigma}^{a\bar{b}}v_{\bar{b}} \\ \sigma_{\bar{a}b}u^b \end{bmatrix}. \tag{60}$$

Obviously, the summation convention is used here and the first Latin indices run over the range $(1, 2)$, whereas the bispinor ones have the range $(1, 2, 3, 4)$.

Roughly speaking, the Weyl two-component spinors (W) are transformed into anti-Weyl ones $\left(\overline{W}^*\right)$ and conversely. It is clear that the anticommutation rules (4) and the Hermitian compatibility conditions (31), (33) are satisfied. This bispinor representation based on Weyl spinors is particularly suggestive and is very convenient when describing the action of improper Lorentz group. For example, spatial rotations are not only very simple in analytical sense, but roughly speaking they consist in a sense in the mutual interchanging of weyl and anti-Weyl spinors. As is well-known, the particular matrix realisation $[\gamma^r{}_s]$, $[G_{\bar{r}\bar{s}}]$ does not matter. It is only the system of algebraic relationships between them, that is essential. Nevertheless, for historical reasons let us mention also Dirac representation. We have then

$$\gamma^0_{\mathrm{Dir}} = \begin{bmatrix} I_2 & 0 \\ 0 & -I_2 \end{bmatrix} = \begin{bmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{bmatrix}, \qquad \gamma^R_{\mathrm{Dir}} = \begin{bmatrix} 0 & \sigma^R \\ -\sigma^R & 0 \end{bmatrix} \tag{61}$$

$$[G_{\bar{r}s}]_{\mathrm{Dir}} = \begin{bmatrix} I_2 & 0 \\ 0 & -I_2 \end{bmatrix}, \qquad \gamma^5_{\mathrm{Dir}} = \mathrm{i}\begin{bmatrix} 0 & I_2 \\ I_2 & 0 \end{bmatrix}. \tag{62}$$

Analytically the both representation are interrelated via the change of coordinates described by the matrix

$$B = B^{-1} = B^T = B^+ = \frac{1}{\sqrt{2}} \begin{bmatrix} I_2 & I_2 \\ I_2 & -I_2 \end{bmatrix}. \tag{63}$$

The last chain of equalities implies that "accidentally" $\gamma^{Ar}{}_s$ and $G_{\bar{r}s}$ transform according to the same rules in spite of their different geometric nature. Again the accident not to be repeated generally! Spinor representation based on the space W is geometrically more natural but there are physical problems in which Dirac representation is more convenient. For example, nonrelativistic approximation is more visible then; one obtains the two-component Pauli equation for spinning electron almost automatically.

Very important geometric problems appear when one injects Lie groups and their Lie algebras of mappings acting in W into $\mathrm{L}(\mathcal{D})$, the set of linear mappings of $\mathcal{D}$ into itself and into $\mathrm{L}(\mathrm{H})$, $\mathrm{L}(\mathrm{H}^*) \simeq \mathrm{L}(\mathrm{H})^*$-real spaces of Hermitian tensors on H.

Any $A \in \mathrm{GL}(W)$ gives rise to $U(A) \subset \mathrm{GL}(\mathcal{D})$, namely

$$U(A) := A \times \overline{A}^{*-1} \tag{64}$$

acting as follows on bispinors

$$(U(A)\Psi) = U(A) \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} Au \\ v \circ \overline{A}^{-1} \end{bmatrix} = \begin{bmatrix} u' \\ v' \end{bmatrix} \tag{65}$$

where, analytically

$$u'^a = (Au)^a = A^a{}_b u^b, \qquad v'_{\bar{a}} = \left(v \circ \overline{A}^{-1}\right)_{\bar{a}} = v_{\bar{b}} \overline{A}^{-1\bar{b}}{}_{\bar{a}}. \tag{66}$$

This is evidently a faithful representation (injection) of $\mathrm{GL}(W)$ into $\mathrm{GL}(\mathcal{D})$. And moreover, this is an injection into the pseudounitary subgroup $\mathrm{U}(\mathcal{D}, G) \subset \mathrm{GL}(\mathcal{D})$, isomorphic (non-canonically) with $\mathrm{U}(2,2) \subset \mathrm{GL}(4, \mathbb{C})$, namely the subgroup of $\mathrm{GL}(\mathcal{D})$ preserving the scalar product G

$$G_{\bar{r}s} \overline{U(A)}^{\bar{r}}{}_{\bar{z}} U(A)^s{}_w = G_{\bar{z}w} \tag{67}$$

or briefly

$$U(A)^* G = G. \tag{68}$$

As mentioned, the unimodular subgroup $\mathrm{SU}(\mathcal{D}, G)$ isomorphic (non-canonically) with $\mathrm{SU}(2,2)$ is isomorphic with something very important, namely, with the universal covering group of the full conformal group $\mathrm{SO}^{\uparrow}(\mathrm{H}(W))$ isomorphic (non-canonically) with the Lorentz group $\mathrm{SO}^{\uparrow}(1,3)$. So we again return to the fundamental question of our four-dimensional conformal paradigm: Perhaps the Clifford structure is something accidental which in the special case of the four-dimensional space time is related to the conformal group, but perhaps the latter one is just the proper physical way? The deep physical meaning of the Minkowskian conformal group seems to work in support of this hypothesis. This is the group which preserves the set of uniformly accelerated motion (the uniform inertial motions form the very special subset of this set. This group preserves the light cones. It is semisimple and finite 15-dimensional. Moreover, it is the smallest semisimple group containing the (non-semisimple) Poincaré group and every larger diffeomorphism group of this property must be infinite-dimensional. Perhaps the admitting of $\mathrm{U}(\mathcal{D}, G)$ instead its subgroup given by (64), (66) is justified as an extension of the group of extended point transformations in cotangent bundles to the group of canonical transformations as there is a complete analogy.

The one-parameter subgroups of $\mathrm{GL}(W)$ may be (at least locally) written in exponential form

$$A(\tau) = \exp(a\tau), \qquad a \in \mathrm{L}(W) \simeq \mathfrak{gl}(W). \tag{69}$$

They give rise to one-parameter subgroups of (64), (66)

$$U\left(A(\tau)\right) = \exp\left(u(a)\tau\right) \tag{70}$$

where the generators $u(a)$ act on $\mathcal{D}$ as the following elements of $\mathrm{L}(\mathcal{D})$

$$\begin{bmatrix} u \\ v \end{bmatrix} \mapsto \begin{bmatrix} u' \\ v' \end{bmatrix}, \qquad u'^{b} = a^{b}{}_{c} u^{c}, \qquad v'_{\bar{b}} = -v_{\bar{c}} \overline{a}{}^{\bar{c}}{}_{\bar{b}}. \tag{71}$$

Let us notice that when the transformations A are restricted to the proper linear group $\mathrm{SL}(W)$, so that

$$\operatorname{Tr} a = 0 \tag{72}$$

then the transformations (66) acting on the u- and v-components are exactly what in the standard literature is referred to as the $D^{(1/2,0)}$ and $D^{(0,1/2)}$ representations of $\mathrm{SL}(W) \simeq \mathrm{SL}(2, \mathbb{C})$, i.e., the corresponding two-valued representations of $\mathrm{SO}(\mathrm{H}, \eta) \simeq \mathrm{SO}^{\uparrow}(1,3)$. Then the total representation (66) is reducible one, equivalent to

$$D^{(1/2,0)} \otimes D^{(0,1/2)} \tag{73}$$

unless we admit spatial reflection which destroy the reducibility. Those reflections are always meant with respect to some reference frames in $\mathrm{H}(W)$, $\mathrm{H}(W^{*}) \simeq \mathrm{H}(W)^{*}$, i.e., with respect to some positively definite Hermitian form $\kappa_{\bar{a}b}$ or its inverse $\kappa^{b\bar{a}}$

$$\kappa_{\bar{a}c}\kappa^{c\bar{b}} = \delta_{\bar{a}}{}^{\bar{b}}, \qquad \kappa^{a\bar{c}}\kappa_{\bar{c}b} = \delta^{a}{}_{b}. \tag{74}$$

It is assumed to be η-normalised to unity, i.e., if

$$\kappa^{a\bar{b}} = \kappa^{A}\frac{1}{\sqrt{2}}\sigma_{A}{}^{a\bar{b}}, \qquad \kappa_{\bar{a}b} = \kappa_{A}\frac{1}{\sqrt{2}}\sigma^{A}{}_{\bar{a}b} \tag{75}$$

$$\eta_{AB}\kappa^{A}\kappa^{B} = 1, \qquad \eta^{AB}\kappa_{A}\kappa_{B} = 1 \tag{76}$$

then the corresponding spatial reflection interchanging u, v is analytically given by

$$P\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 & \kappa^{a\bar{b}} \\ \kappa_{\bar{a}b} & 0 \end{bmatrix}\begin{bmatrix} u^b \\ v_{\bar{b}} \end{bmatrix} = \begin{bmatrix} \kappa^{a\bar{b}}v_{\bar{b}} \\ \kappa_{\bar{a}b}u^b \end{bmatrix}. \tag{77}$$

The natural question: if once to admit the mixing of $W, \overline{W}^*$ to introduce the spatial reflection, then why not to admit its total pseudounitary mixing by $\mathrm{U}(\mathcal{D}, G) \simeq \mathrm{U}(2,2)$ – the covering group of conformals!

The next problem is the relationship between linear mappings in W and those in conformal-Minkowski spaces $\mathrm{H}(W)$, $\mathrm{H}(W^*) \simeq \mathrm{H}(W)^*$. It is clear that any $A \in \mathrm{GL}(W)$ acts on Hermitian tensors according to the rules A_*, A^*. Analytically

$$(A^*x)^{a\bar{b}} = A^a{}_c\overline{A}^{\bar{b}}{}_{\bar{d}}x^{c\bar{d}}, \qquad (A^*x)_{\bar{a}b} = p_{\bar{c}d}\overline{A}^{-1\bar{c}}{}_{\bar{a}}A^{-1d}{}_b. \tag{78}$$

Obviously, this transformation preserves Hermicity, i.e., $\mathrm{H}(W)$ is mapped onto $\mathrm{H}(W)$, $\mathrm{H}(W^*) \simeq \mathrm{H}(W)^*$ is also mapped onto itself. And it is again clear that replacing A by $\exp(\mathrm{i}\varphi)A$, $\varphi \in \mathbb{R}$, we do not modify the transformation rule for Hermitian tensors. If we use the "sigma-basis" in $\mathrm{H}(W)$, $\mathrm{H}(W^*)$, then the matrix

$$\left[(A_{\mathrm{H}})^A{}_B\right], \qquad (A_{\mathrm{H}})\,\sigma_L = A_*\sigma_L = \sigma_K\,(A_{\mathrm{H}})^K{}_L \tag{79}$$

is given by

$$(A_{\mathrm{H}})^K{}_L = \frac{1}{2}\sigma^K{}_{\bar{b}a}A^a{}_c\overline{A}^{\bar{b}}{}_{\bar{d}}\sigma_L{}^{c\bar{d}} \tag{80}$$

and of course in the second formula of (78) is based on the matrix contragradient to (80)

$$A_*\sigma^L = \left(A_{\mathrm{H}}^{-1}\right)^L{}_K\sigma^K. \tag{81}$$

It is explicitly seen that A, $-A$, or more generally, $\exp(\mathrm{i}\varphi)A$, $\varphi \in \mathbb{R}$, lead to the same transformation rule A_{H} in H. If A-s are restricted to $\mathrm{SL}(W) \simeq \mathrm{SL}(2,\mathbb{C})$, the assignment $\mathrm{GL}(W) \ni A \mapsto A_{\mathrm{H}} \in \mathrm{GL}(\mathrm{H})$ is a universal $2:1$ covering of the restricted Lorentz group. Obviously, for any unimodular transformation, i.e., for any element of the subgroup

$$\mathrm{UL}(W) := \{A \in \mathrm{GL}(W);\, |\det A| = 1\} \tag{82}$$

the corresponding A_* does preserve separately any of the natural conformally invariant metrics on $\mathrm{H}(W)$. But obviously, any real multiplier at A different from one does violate this isometry properly and the corresponding (A_{H}) becomes the Weyl transformation of H, multiplying any of possible η-s by the real dilatation factor

$$\eta \mapsto |\det A|^{-2}\,\eta \tag{83}$$

more precisely

$$\eta_{KL}\left(A_{\mathrm{H}}^{-1}\right)^K{}_M\left(A_{\mathrm{H}}^{-1}\right)^L{}_N = |\det A|^{-2}\,\eta_{MN} \tag{84}$$

for any of mutually proportional η. This agrees with the mentioned nature of η as a tensor Weyl density of weight two rather than tensor. And similarly ε behaves like the skew-symmetric tensor density of weight one in W.

Let us now do some comments concerning the action of $A \in \mathrm{GL}(W)$ through $U(A) \in \mathrm{U}(\mathcal{D}, G) \subset \mathrm{GL}(\mathcal{D})$. Transformations $U(A)$ act as similarities on the associative algebra $\mathrm{L}(\mathcal{D})$. In particular, they transform "Dirac matrices" as follows

$$\gamma^K \mapsto U(A)\gamma^K U(A)^{-1}. \tag{85}$$

According to (67), the bispinor scalar product G is invariant under the action of pseudounitary group. However, if $|\det A| \neq 1$, the similarities (85) do not preserve the Clifford anticommutation rules, because the metric η is not conserved then. Instead, Clifford rules are then transformed into ones with the modified metric (83). The point is that (58) are explicitly built of η. Therefore, the conformal paradigm is not compatible with the Clifford one, and to reconciliate them, one would have to start with introducing additional dynamical variable, namely, the one-dimensional scalar factor in η and, henceforth, in $\widetilde{\sigma}^A$ and γ^A. Then the resulting scheme would become scale-free, i.e., invariant under the Weyl group, although still not under the total conformal group or its covering $\mathrm{SU}(\mathcal{D}, G) \simeq \mathrm{SU}(2,2)$.

For the sake of further developments, let us complete those comments by remarks in the spirit of (69)–(71) in application to (79)–(80). Again, for any $a \in \mathrm{L}(W) \simeq \mathfrak{gl}(W)$ we shall consider the one-parameter group

$$\{A(\tau) = \exp(a\tau) \in \mathrm{GL}(W)\,;\ \tau \in \mathbb{R}\} \tag{86}$$

and the corresponding induced action on H-spaces, which in "sigma-basis" is given by (80)

$$\left(A_{\mathrm{H}}\right)(\tau)^K{}_L = \frac{1}{2}\sigma^K{}_{\bar{b}e}\exp(a\tau)^e{}_c \exp(\overline{a}\tau)^{\bar{b}}{}_{\bar{d}}\sigma_L{}^{c\bar{d}}. \tag{87}$$

By analogy to (69)–(71) let us represent it as follows

$$A_{\mathrm{H}}(\tau) = \exp\left(\alpha_{(\mathrm{H})}\tau\right). \tag{88}$$

After some calculations one can show that

$$\alpha_{(\mathrm{H})}{}^K{}_L = \frac{1}{2}\sigma^K{}_{\bar{b}e}a^e{}_c\sigma_L{}^{c\bar{b}} + \frac{1}{2}\sigma^K{}_{\bar{b}e}\overline{a}{}^{\bar{b}}{}_{\bar{c}}\sigma_L{}^{e\bar{c}} \tag{89}$$

or more precisely

$$\alpha_{(\mathrm{H})} = \mathrm{Re}\,\mathrm{Tr}\left(\sigma^K a\sigma_L\right). \tag{90}$$

As expected, there is a direct relationship between traces, i.e., generators of dilatations

$$\mathrm{Tr}\,\alpha_{(\mathrm{H})} = 4\mathrm{Re}\,\mathrm{Tr}\,(a)\,. \tag{91}$$

The inverse formula of (90) is not unique, because, obviously, the purely imaginary part of the trace of a does not contribute to anything in geometry of H and with a given $\alpha_{(\mathrm{H})}$ it is completely arbitrary

$$a^{e}{}_{f} = \frac{1}{4}\left(\alpha^{K}_{(\mathrm{H})L}\sigma_{K}{}^{e\bar{c}}\sigma^{L}{}_{\bar{c}f} - \frac{1}{2}\alpha^{K}_{(\mathrm{H})K}\delta^{e}{}_{f}\right) + \frac{\mathrm{i}}{2}\mathrm{Im}\left(a^{c}{}_{c}\right)\delta^{e}{}_{f} \tag{92}$$

the last term, as mentioned, is completely arbitrary and generates the phase transformations:

$$w \mapsto \exp(\mathrm{i}\varphi)w, \qquad \varphi \in \mathbb{R}. \tag{93}$$

Many misunderstandings result when one uses without a sufficient care the analytical language, identifying simply the target spaces of Dirac and Weyl spinors respectively with $\mathbb{C}^4$ or $\mathbb{C}^2$ (some artefact structure of those spaces). Nevertheless, this language is commonly used ($\mathbb{C}^4$ and $\mathbb{C}^2$ are identified with some standard fibres of the corresponding bundles). So, to finish with, let us quote some popular analytical formulae. For any $a \in \mathfrak{gl}(2,\mathbb{C}) \simeq \mathrm{L}(2,\mathbb{C})$ the corresponding injections into pseudounitary Lie algebra $\mathfrak{u}(4,G)$ are given by

$$u(a) = \begin{bmatrix} a & 0 \\ 0 & -a^{+} \end{bmatrix}, \qquad u(a) = \frac{1}{2}\begin{bmatrix} a-a^{+} & a+a^{+} \\ a+a^{+} & a-a^{+} \end{bmatrix} \tag{94}$$

respectively in the van der Waerden-Infeld-Weyl spinor representation and Dirac representation.

The covering projection $P : \mathrm{SL}(2,\mathbb{C}) \to \mathrm{SO}^{\uparrow}(1,3)$ and the corresponding isomorphism $p : \mathrm{SL}(2,\mathbb{C})' \to \mathfrak{so}(1,3)$ are respectively given by

$$U(A)\gamma_{K}U(A)^{-1} = \gamma_{L}P(A)^{L}{}_{K}, \qquad [u(a),\gamma_{K}] = \gamma_{L}p(a)^{L}{}_{K} \tag{95}$$

$$A\sigma_{K}A^{+} = \sigma_{L}A^{L}{}_{K}, \qquad a\sigma_{K} + \sigma_{K}a^{+} = \sigma_{L}p(a)^{L}{}_{K} \tag{96}$$

where also

$$P(A)^{L}{}_{K} = \frac{1}{2}\,\mathrm{Tr}\left(\sigma^{L}A\sigma_{K}A^{+}\right) = \frac{1}{4}\,\mathrm{Tr}\left(\gamma^{L}U(A)\gamma_{K}U(A)^{-1}\right) \tag{97}$$

$$p(a)^{L}{}_{K} = \frac{1}{2}\,\mathrm{Tr}\left(\sigma^{L}a\sigma_{K}\right) + \frac{1}{2}\,\mathrm{Tr}\left(\sigma_{K}a^{+}\sigma^{L}\right) = \frac{1}{2}\,\mathrm{Tr}\left(\gamma^{L}u(a)\gamma_{K}\right) \tag{98}$$

and respectively, in the spinor Weyl-van der Waerden and Dirac representations we have

$$U(A) = \begin{bmatrix} A & 0 \\ 0 & A^{-1+} \end{bmatrix}, \qquad U(A) = \frac{1}{2}\begin{bmatrix} A+A^{-1+} & A-A^{-1+} \\ A-A^{-1+} & A+A^{-1+} \end{bmatrix}. \tag{99}$$

But an important *warning*: The hermitian conjugations a^{+}, A^{+} in formulae (94)-(99) are *analytical artefacts* – just the formal hermitian conjugate of matrices meant as tables of numbers. There is no scalar product with respect to which they would be *true*, *geometric hermitian conjugates*.

2. Spinors, Fermions and Four-Dimensional Einstein-Cartan Gravitation. Some Standard Ideas, Doubts and Questions

As mentioned, we usually base on the analytical language. For majority of non-prepared audience the premature use of fibre bundle concepts more obscures than elucidates. Nevertheless, from the principal point of view the fibre bundle language is a proper one. Thus, all over in this paper, in particular in this section, our treatment will rely on some compromise: the basic expressions are formulated analytically, but certain fibre bundle comments are also included.

Let M be a four-dimensional space-time. It is inhabited by two realities: matter and geometry, i.e., gravitation. According to the known figurative statement: "Matter tells to space how to curve, and space tells to matter how to move". This is a mutual interaction. According to contemporary ideas, the fundamental heavy matter like leptons and quarks has the fermionic nature, i.e., it is described by spinor fields. Higgs bosons, if they really exist, are an exception. Fundamental interactions are transferred by gauge fields and it is natural to expect that gravitation, the oldest known and very important interaction is not an exception. So we remind the basic ideas of the dynamics of Dirac-Einstein-Cartan system, starting from the analytical concepts, e.g., $\mathbb{R}^4$ as the bispinor target space.

Analytically, **bispinor fields** are described by mappings

$$\Psi : M \to \mathbb{C}^4 \tag{100}$$

i.e., four-component complex fields-amplitudes on the space-time manifold. At this stage we are interested only in bispinors as such, so we do not take into account the existence of other, more specific quantum numbers (internal indices) at Ψ. If x^μ are some local space-time coordinates in M, then Ψ is analytically represented by the system of symbols

$$\Psi^r\left(x^\mu\right). \tag{101}$$

This is the material sector. Degrees of freedom of the geometric-gravitational sector are described by two objects: gravitational cotetrad e and some $\mathrm{SO}(1,3)$-ruled abstract connection Γ, explicitly

$$M \ni x \;\mapsto\; e_x \in \mathrm{L}\left(T_x M, \mathbb{R}^4\right) \simeq \mathbb{R}^4 \otimes T_x^* M \tag{102}$$

$$M \ni x \;\mapsto\; \Gamma_x \in \mathrm{L}\left(T_x M, \mathfrak{so}(1,3)\right). \tag{103}$$

Obviously, $T_x M$, $T_x^* M$ denote respectively the tangent and cotangent spaces at $x \in M$, $\mathrm{SO}(1,3)$ denotes the restricted Lorentz group in Minkowskian space $\mathbb{R}^4$ meant with the signature $(+,-,-,-)$, and $\mathfrak{so}(1,3)$ is the Lie algebra of the Lorentz group. To be more precise, we must use also the total non-connected group $\mathrm{O}(1,3)$ consisting of four connected components and its subgroups like $\mathrm{O}^\uparrow(1,3)$ (orthochronous one), $\mathrm{SO}(1,3)$ (preserving the total orientation of $\mathbb{R}^4$ as a Minkowski space) and $\mathrm{SO}^{\uparrow}(1,3)$ (preserving separately the temporal and spatial orientations). Obviously, Γ as a vector-valued differential form takes values in the Lie algebra of the connected component of unity $\mathrm{SO}^{\uparrow}(1,3)$. Analytically, the objects e, Γ are represented by systems of their components

$$e^{A}{}_{\mu}(x), \qquad \Gamma^{A}{}_{B\mu}(x) \tag{104}$$

where the Latin capitals, just like small Greek indices, run the standard range $(0,1,2,3)$. Obviously, no confusion of the manifold M and the arithmetic space $\mathbb{R}^4$ meant with the standard Minkowskian $(+,-,-,-)$ metric is admissible. M is an amorphous differential manifold with no fixed geometry, whereas $\mathbb{R}^4$ with its Minkowski metric η is one of target spaces. Obviously, the cotetrad e is algebraically equivalent to its dual contravariant tetrad $\widetilde{e}$ with components $e^{\mu}{}_{A}(x)$, where

$$e^{A}{}_{\mu}e^{\mu}{}_{B} = \delta^{A}{}_{B}, \qquad e^{\mu}{}_{A}e^{A}{}_{\nu} = \delta^{\mu}{}_{\nu}. \tag{105}$$

To be pedantic and complete with notation let us remind that the elements $L \in \mathrm{O}(4,\eta) \simeq \mathrm{O}(1,3)$ are defined analytically by

$$\eta_{AB} = \eta_{CD}L^{C}{}_{A}L^{D}{}_{B}, \qquad \eta = L^{*}\eta. \tag{106}$$

The contravariant inverse η^{AB} is obviously given by

$$\eta^{AC}\eta_{CB} = \delta^{A}{}_{B} \tag{107}$$

and the elements of Lie algebra, $\ell \in \mathfrak{so}(4,\eta) \simeq \mathfrak{so}(1,3)$ are η-skew-symmetric

$$\ell^{A}{}_{B} = -\eta^{AC}\eta_{BD}\ell^{D}{}_{C} = -\ell_{B}{}^{A}. \tag{108}$$

Let us stress that the above connection $\Gamma^{A}{}_{B\mu}$ is not an affine connection, it is as yet some abstract connection ruled by the Lorentz group and operating (e.g., parallel-shifting) on objects with the capital $\mathbb{R}^4$-indices. Of course, as expected, the pair $\left(e^{A}{}_{\mu}, \Gamma^{A}{}_{B\mu}\right)$ gives rise to some affine connection, cf. (135) below. However, for many reasons it is more convenient (although apparently less natural) to use just $\Gamma^{A}{}_{B\mu}$ as a primary quantity. Taking values in the Lie algebra $\mathfrak{so}\left(\mathbb{R}^4,\eta\right) = \mathfrak{so}(1,3)$, $\Gamma^{A}{}_{B\mu}$ is η-skew-symmetric, i.e.,

$$\Gamma^{A}{}_{B\mu} = -\Gamma_{B}{}^{A}{}_{\mu} = -\eta^{AC}\eta_{BD}\Gamma^{D}{}_{C\mu}. \tag{109}$$

In other words, the primary object is some lower-case-index skew-symmetric quantity

$$\Gamma_{AB\mu} = -\Gamma_{BA\mu} \tag{110}$$

and later on we define its byproduct

$$\Gamma^{A}{}_{B\mu} = \eta^{AC}\Gamma_{CB\mu} \tag{111}$$

the latter one automatically satisfies (109).

The space-time manifold M is amorphous and no absolute objects are assumed in it, except, of course, the very differential structure. Unlike this, our target spaces are endowed with some fixed, absolute geometries. Let us quote them detailly, starting with the analytical description of target spaces like $\mathbb{R}^4$, $\mathbb{R}^2$, $\mathbb{C}^4$, $\mathbb{C}^2$, etc., in spite of certain possibilities of doing mistakes.

i) $\mathbb{R}^4$ as a target space of the cotetrad field e is endowed with some fixed Minkowskian structure, i.e., normal-hyperbolic metric η of the signature $(+,-,-,-)$, usually we simply put

$$[\eta_{AB}] = \mathrm{diag}(1,-1,-1,-1). \tag{112}$$

ii) $\mathbb{C}^4$ as a target space of the Dirac bispinor field is endowed with some sesquilinear Hermitian form G of the neutral signature $(+,+,-,-)$. It is only signature what matters here; the particular numerical shape of G is a merely choice of basis. We assume G to be antilinear/linear in the first/second argument. There are two most popular choices, as we reminded in the previous section are as follows:

$$[G_{\bar{r}s}] = \begin{bmatrix} 0 & I_2 \\ I_2 & 0 \end{bmatrix} \tag{113}$$

$$[G_{\bar{r}s}]_D = \begin{bmatrix} I_2 & 0 \\ 0 & -I_2 \end{bmatrix}. \tag{114}$$

The first one occurs when so-called spinor representation (Weyl-van der Waerden-Infeld representation) of bispinor objects is used. The second choice is used in the Dirac representation. The choice (113) is well suited to the use of two-component Weyl spinors as elementary entities. Representation (114) is convenient when one discusses the non-relativistic limit and the Pauli equation. Representations (113) and (114) are related to each other through the covariant rule transformation

$$B^{+}\begin{bmatrix} 0 & I_2 \\ I_2 & 0 \end{bmatrix} B = \begin{bmatrix} I_2 & 0 \\ 0 & -I_2 \end{bmatrix} \tag{115}$$

where

$$B = B^{-1} = B^{+} = B^{T} = \frac{1}{\sqrt{2}}\begin{bmatrix} I_2 & I_2 \\ I_2 & -I_2 \end{bmatrix}.$$

Bilinear form G is preserved by the pseudo-unitary group

$$\mathrm{U}(4,G) \simeq \mathrm{U}(2,2) \subset \mathrm{GL}(4,\mathbb{C}).$$

Its Lie algebra $\mathfrak{u}(4,G) \simeq \mathfrak{u}(2,2)$ consists of matrices u which are anti-hermitian in the G-sense

$$G_{\bar{r}z}u^{z}{}_{s} + \overline{G_{\bar{s}z}u^{z}{}_{r}} = 0. \tag{116}$$

Obviously, the elements of $\mathrm{U}(4,G) \simeq \mathrm{U}(2,2)$ satisfy by definition

$$G_{\bar{z}s}\overline{U}^{\bar{z}}{}_{\bar{r}}U^{s}{}_{t} = G_{\bar{r}t}. \tag{117}$$

The contravariant inverse of G will be denoted by $[G^{r\bar{s}}]$ and obviously

$$G^{r\bar{s}}G_{\bar{s}t} = \delta^{r}{}_{t}, \qquad G_{\bar{r}t}G^{t\bar{s}} = \delta_{\bar{r}}{}^{\bar{s}}. \tag{118}$$

The space of G-Hermitian elements of $\mathrm{L}(4,\mathbb{C})$ will be denoted by $\mathrm{H}(4,G)$, and the space of sesquilinear Hermitian forms simply by $\mathrm{H}(4)$. Obviously, they are real vector spaces (ones over $\mathbb{R}$) and their elements are related to each other by

$$h^{r}{}_{s} = G^{r\bar{z}}h_{\bar{z}s}. \tag{119}$$

Lie algebra $\mathfrak{u}(4,G) \simeq \mathfrak{u}(2,2)$ is the imaginary unit multiple of $\mathrm{H}(4,G) \simeq \mathrm{H}(2,2)$

$$\mathfrak{u}(4,G) = \mathrm{i}\mathrm{H}(4,G), \qquad \mathfrak{u}(2,2) = \mathrm{i}\mathrm{H}(2,2). \tag{120}$$

iii) Another element of the target geometry is some fixed Clifford injection, i.e., a linear monomorphism

$$\gamma : \mathbb{R}^4 \hookrightarrow \mathrm{H}(4,G) = \mathrm{i}\mathfrak{u}(4,G) \subset \mathrm{L}(4,\mathbb{C}) \tag{121}$$

or, when the Dirac convention (114) is used,

$$\gamma : \mathbb{R}^4 \hookrightarrow \mathrm{i}\mathfrak{u}(2,2) \subset \mathrm{L}(4,\mathbb{C}).$$

This is to be Clifford injection, so, if ε_A are elements of the standard zero-one basis of $\mathbb{R}^4$, then their γ-images

$$\gamma_A := \gamma\varepsilon_A \tag{122}$$

satisfy the anticommutator rule

$$\{\gamma_A, \gamma_B\} = \gamma_A\gamma_B + \gamma_B\gamma_A \ = \ 2\eta_{AB}I_4 \tag{123}$$

$$\{\gamma^A, \gamma^B\} = \gamma^A\gamma^B + \gamma^B\gamma^A \ = \ 2\eta^{AB}I_4 \tag{124}$$

where, obviously

$$\gamma^A := \eta^{AB}\gamma_B \tag{125}$$

and the two conditions (123) and (124) are equivalent.
These conditions are but the special case of the general rules of Clifford algebra. The fundamental idea of Clifford paradigm is to represent the scalar square of vectors and convectors just as the usual square of something

$$\left(\gamma_A u^A\right)^2 = \eta_{AB}u^A u^B I, \qquad \left(\gamma^A f_A\right)^2 = \eta^{AB}f_A f_B I \tag{126}$$

where I is the identity operator. From the tensorial point of view, the objects γ^A, γ_A commonly referred to as Dirac matrices, are mixed tensors, i.e., linear mappings of the target space of the Ψ-objects into itself. Here, in the physical four-dimensional case, $\gamma^A, \gamma_A \in \mathrm{L}(4,G)$. Obviously, the coincidence of the real dimension of M and the complex dimension of $\mathbb{C}^4$ is accidental and does not occur for the general $n = \dim M$. For physical reasons (real-valuedness of the Lagrangian for Ψ), linear mappings γ^A, γ_A must be G-hermitian

$$\gamma^A, \gamma_A \in \mathrm{H}(4, G) = \mathrm{iu}(4, G) \simeq \mathrm{H}(2, 2) = \mathrm{iu}(2, 2). \tag{127}$$

So, they must be imaginary-unit-multiples of elements of the Lie algebra of pseudounitary operators in $\mathbb{C}^4$ preserving G. The linear shell of γ^A-s over reals will be denoted by V

$$V := \gamma\left(\mathbb{R}^4\right) \subset \mathrm{iu}(4, G) \simeq \mathrm{iu}(2, 2). \tag{128}$$

Obviously, V is a real linear subspace of the real linear space $\mathrm{iu}(4, G) = \mathrm{H}(4, G)$. Lowering the first indices of γ^A-s, γ_A-s we obtain sesquilinear hermitian forms on $\mathbb{C}^4$

$$\Gamma^A{}_{\bar{r}s} = G_{\bar{r}z}\gamma^{Az}{}_s, \qquad \Gamma_{A\bar{r}s} = G_{\bar{r}z}\gamma_A{}^z{}_s \tag{129}$$

cf. (119). Their linear shell will be denoted by $\widetilde{V}$. It is a real four-dimensional subspace of the real space $\mathrm{H}(4)$ of all sesquilinear hermitian forms on $\mathbb{C}^4$

$$\widetilde{V} \subset \mathrm{H}(4). \tag{130}$$

Obviously, $\mathrm{H}(4, G) \subset \mathrm{L}(4, G)$, $\mathrm{H}(4)$ are 16-dimensional over reals

$$\dim_{\mathbb{R}} \mathrm{H}(4, G) = \dim_{\mathbb{R}} \mathrm{U}(4, G) = \dim_{\mathbb{R}} \mathrm{H}(4) = 16. \tag{131}$$

Let us stress that the concept of hermitian linear mappings as elements of $\mathrm{L}(4, G)$ is always related to some hermitian scalar product G in $\mathbb{C}^4$. Unlike this, the concept of sesquilinear hermitian form is metric-independent and does not assume any G.

This was the list of absolute geometric objects in target spaces and in some their byproducts. There are some important points concerning those objects. The bispinor metric G which appears explicitly in Lagrangian for the Dirac field Ψ seems to suggest that the pseudounitary group $\mathrm{U}(4, G) \simeq \mathrm{U}(2, 2)$ may be expected to describe some fundamental symmetries of spinorial geometrodynamics. At the same time, it is well-known that the subgroup $\mathrm{SU}(4, G) \simeq \mathrm{SU}(2, 2)$ consisting of pseudounitary mappings with determinants equal to unity (the modulus is always unity) is the covering group of the Minkowskian conformal group, just like $\mathrm{SL}(2, \mathbb{C})$ is the universal covering group of $\mathrm{SO}^{\uparrow}(1, 3)$. But this is the peculiarity of the space-time dimension four.

In other dimensions there is no link between conformal and Cliffordian paradigms. As our space-time seems to be just four-dimensional (no convincing evidence for Kaluza philosophy), it is not clear which paradigm is to be accepted as a proper foundation.

As mentioned, the particular matrix (coordinate) realisation of objects G, γ^A, Γ^A does not matter. What matters are relationships between them, i.e., (123), (124), (127), (129). In other words, no particular choice of subspaces V, $\widetilde{V}$ of the above properties is essential. Let us stress however an important point. Although in the above sense arbitrary, when once fixed, the subspaces V, $\widetilde{V}$ are globally fixed all over M, independently on the choice of $x \in M$. This is not necessary for the Clifford "square-rootization" (126) of the η-scalar product. In a sense this is some global "action-at-distance" element of the description. If the theory is formulated in fibre bundle terms, where Ψ are cross-sections of some complex fibre bundle over M and the cotetrad field e is a cross-section of the principal fibre bundle F^*M of linear co-frames in M, this means that some structure in M is in a sense flat.

The above target-space objects enable one to construct some family of important geometric byproducts of the fields Ψ^r, $e^A{}_\mu$, $\Gamma^A{}_{B\mu}$. Let us quote them.

i) Dirac-Einstein metric tensor field $g\,[e,\eta]$ on M

$$g\,[e,\eta] := \eta_{AB} e^A \otimes e^B \tag{132}$$

i.e., analytically,

$$g_{\mu\nu} = \eta_{AB} e^A{}_\mu e^B{}_\nu. \tag{133}$$

With respect to this metric, the frame e is automatically η-orthonormal, i.e., Lorentzian

$$g_{\mu\nu} e^\mu{}_A e^\nu{}_B = \eta_{AB}, \qquad e^A{}_\mu e^B{}_\nu g^{\mu\nu} = \eta^{AB}. \tag{134}$$

ii) Affine connection on M. Its holonomic coefficients are given by

$$\Gamma^\alpha{}_{\beta\mu} = e^\alpha{}_A \Gamma^A{}_{B\mu} e^B{}_\beta + e^\alpha{}_A e^A{}_{\beta,\mu} \tag{135}$$

where, as usual, comma denotes the partial differentiation. The η-skew-symmetry of Γ (109), (110) implies that it is automatically Einstein-Cartan connection. It is metrical, i.e., the corresponding covariant derivative of the Einstein-Dirac metric vanishes

$$\nabla_{[\Gamma]} g = 0 \tag{136}$$

however, in general it is not symmetric. Its **torsion tensor** S, i.e., skew-symmetric part of Γ

$$S^\alpha{}_{\beta\mu} := \Gamma^\alpha{}_{[\beta\mu]} = \frac{1}{2}\left(\Gamma^\alpha{}_{\beta\mu} - \Gamma^\alpha{}_{\mu\beta}\right) \tag{137}$$

is in general non-vanishing and the following holds

$$\Gamma^\alpha{}_{\beta\mu} = \left\{ \begin{array}{c} \alpha \\ \beta\mu \end{array} \right\} + K^\alpha{}_{\beta\mu} = \left\{ \begin{array}{c} \alpha \\ \beta\mu \end{array} \right\} + S^\alpha{}_{\beta\mu} + S_{\beta\mu}{}^\alpha - S_\mu{}^\alpha{}_\beta. \tag{138}$$

In the literature $K^{\alpha}{}_{\beta\mu}$ is referred to as the contortion tensor. Raising and lowering of indices is understood in the sense of Dirac-Einstein metric $g_{\mu\nu}$. In the sense of this metric contorsion is skew-symmetric in its two first indices

$$K^{\alpha}{}_{\beta\mu} = -K_{\beta}{}^{\alpha}{}_{\mu} = -g_{\beta\lambda}g^{\alpha\kappa}K^{\lambda}{}_{\kappa\mu}. \tag{139}$$

These relatively complicated properties show that really it is

$$\Gamma^{A}{}_{B\mu} = -\Gamma_{B}{}^{A}{}_{\mu} = -\eta_{BC}\eta^{AD}\Gamma^{C}{}_{D\mu}$$

that is to be used as a primary object. It is constrained only by the simple algebraic condition of η-skew-symmetry. All Einstein-Cartan properties are then just direct consequences.

For the sake of completeness, let us mention that the second term on the right-hand side of (135) is in geometry referred to as the teleparallelism connection $\Gamma_{\text{tel}}[e]$ induced by the (co)frame e

$$\Gamma_{\text{tel}}[e]^{\alpha}{}_{\beta\mu} := e^{\alpha}{}_{A}e^{A}{}_{\beta,\mu}. \tag{140}$$

It is uniquely defined by the condition that e is parallel with respect to $\Gamma_{\text{tel}}[e]$

$$\nabla_{[\text{tel}]\beta}e^{\alpha}{}_{A} = 0, \qquad \nabla_{[\text{tel}]\beta}e^{A}{}_{\alpha} = 0. \tag{141}$$

As always the difference of affine connections Γ, $\Gamma_{\text{tel}}[e]$ is a tensor field of valence $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$$\Gamma - \Gamma_{\text{tel}}[e] = \Gamma^{A}{}_{B}e_{A} \otimes e^{B} \tag{142}$$

$$\Gamma^{\alpha}{}_{\beta\mu} - \Gamma_{\text{tel}}[e]^{\alpha}{}_{\beta\mu} = \Gamma^{A}{}_{B\mu}e^{\alpha}{}_{A}e^{B}{}_{\beta}. \tag{143}$$

The quantities $\Gamma^{A}{}_{B\mu}$ are referred to as non-holonomic components of Γ with respect to e. Obviously, $\Gamma_{\text{tel}}[e]$ has the vanishing curvature tensor and in general non-vanishing torsion, i.e., skew-symmetric part. This torsion vanishes if and only if e is holonomic. If one expresses the torsion S of Γ through its non-holonomic coefficients with respect to e

$$S[e] = \frac{1}{2}\Omega^{A}{}_{BC}e_{A} \otimes e^{B} \otimes e^{C} \tag{144}$$

one obtains the quantity Ω which in traditional geometric literature is known as the non-holonomy object of e [10]

$$\Omega^{A}{}_{BC} = 2\widehat{S}^{A}{}_{BC} = \left\langle e^{A}, [e_B, e_C]\right\rangle \tag{145}$$

$$[e_A, e_B] = \Omega^{C}{}_{AB}e_C, \qquad \mathrm{d}e^{A} = \frac{1}{2}\Omega^{A}{}_{BC}e^{C} \otimes e^{B}. \tag{146}$$

Let us introduce the system of scalars

$$\Gamma^{A}{}_{BC} := \Gamma^{A}{}_{B\mu}e^{\mu}{}_{C}. \tag{147}$$

These quantities by their very definition satisfy

$$\nabla_C e_B = \Gamma^A{}_{BC} e_A \tag{148}$$

$$\nabla_\mu e_\nu = \Gamma^\lambda{}_{\nu\mu} e_\lambda \tag{149}$$

where, obviously, ∇_C denotes the covariant differentiation along the vector field e_C, and $e_\nu = \partial/\partial x^\nu$ is the ν-th coordinate tangent vector of the coordinate system x^μ.

It is clear that non-holonomic coefficients of $\Gamma[e]$ do vanish

$$\Gamma_{\text{tel}}[e]^A{}_{BC} = 0. \tag{150}$$

iii) Bispinor connection form given analytically as $\omega^r{}_{s\mu}$. It operates with bispinor indices and parallel transport of objects with such indices. This quantity is G-antihermitian in internal indices r, s, i.e., ω is a differential form on M with values in the Lie algebra $\mathfrak{u}(4, G) \simeq \mathfrak{u}(2, 2)$. The matrices ω_μ are explicitly given by

$$\omega_\mu = \frac{1}{8}\Gamma_{KL\mu}\left(\gamma^K\gamma^L - \gamma^L\gamma^K\right) \tag{151}$$

cf. (111), and the inverse formula reads

$$\Gamma^K{}_{L\mu} = \frac{1}{2}\operatorname{Tr}\left(\gamma^K\omega_\mu\gamma_L\right) \tag{152}$$

just $u(a)$ as an element of Lie algebra differential connection one-form taking values in the Lie algebra. So, there is a one-to-one relationship between systems of differential forms $\left[\Gamma^K{}_L\right]$ and $[\omega^r{}_s]$. When the field of frames e is fixed, then any of vector-valued (Lie-algebra-valued) differential forms $\left[\Gamma^K{}_L\right]$ and $[\omega^r{}_s]$ determines uniquely some Einstein-Cartan connection $\Gamma^\alpha{}_{\beta\mu}$.

iv) Dirac conjugation. This is an antilinear operation from $\mathbb{C}^4$ to its dual space $\mathbb{C}^{4*}$. Being numerical spaces, $\mathbb{C}^4$ and $\mathbb{C}^{4*}$ might seem identical. But of course, this is misleading. Working more precisely, we should have used some abstract complex four-dimensional space $\mathcal{D}$ endowed with geometry based on some sesquilinear hermitian form G of the neutral signature $(+, +, -, -)$. Then $\mathcal{D}^*$, the dual of $\mathcal{D}$, is evidently something else. If Ψ is an element of $\mathcal{D}$, then $\widetilde{\Psi}$, its Dirac conjugate, is an element of $\mathcal{D}^*$. Analytical $\mathbb{C}^4$-representation is as follows

$$\widetilde{\Psi}_r := \overline{\Psi}^{\bar{s}} G_{\bar{s}r}. \tag{153}$$

As seen from the use of complex conjugation, the assignment $\Psi \mapsto \widetilde{\Psi}$ is an antilinear, or half-linear, isomorphism acting between complex linear spaces $\mathcal{D}$, $\mathcal{D}^*$.

An important remark-warning! In majority of papers and textbooks on field theory, one simply identifies G with γ^0 and uses the analytical formula

$$\widetilde{\Psi} = \Psi^+ \gamma^0 \tag{154}$$

where for Ψ analytically given by $[\Psi_1, \Psi_2, \Psi_3, \Psi_4]^T$, Ψ^+ is defined as the formal hermitian conjugation of matrices, $\Psi^+ = \left[\overline{\Psi}_1, \overline{\Psi}_2, \overline{\Psi}_3, \overline{\Psi}_4\right]$. Obviously, this is geometrically meaningless and follows from the automatic use of analytical $\mathbb{C}^4$-language. Indeed, Ψ, $\widetilde{\Psi}$ are in fact elements of different linear space, $\mathcal{D}$ and $\mathcal{D}^*$. Matrices $[G_{\bar{r}s}]$, $[\gamma^{0r}{}_s]$ represent completely different geometric objects. G is a sesquilinear hermitian form on $\mathcal{D}$, i.e., twice covariant tensor, $G \in \overline{\mathcal{D}}^* \otimes \mathcal{D}^*$ and $\overline{\mathcal{D}}$ denotes here the complex conjugate space of $\mathcal{D}$ and also $\mathcal{D}$, $\overline{\mathcal{D}}$ are different linear spaces. Dirac "matrices" γ^A are mixed tensors in $\mathcal{D}$, i.e., linear mappings of $\mathcal{D}$ into $\mathcal{D}^*$, $\gamma^A \in \mathrm{L}(\mathcal{D}) \simeq \mathcal{D} \otimes \mathcal{D}^*$. What is written in standard, common language is just that for the sake of convenience one uses special representations in which the matrices of G, γ^0 numerically coincide. But of course there exists an infinity of other representations where this is not true, the point is only that usually they are not used, being, claimly, less convenient in calculations.

v) Covariant differentiation. Affine connection $\Gamma^\alpha{}_{\beta\mu}$ gives rise to the usual covariant differentiation of tensor fields on M. Its e-non-holonomic coefficients $\Gamma^A{}_{B\mu}$ enable one to differentiate objects with non-holonomic indices with respect to e, for example

$$\nabla_{[\widehat{\Gamma}]\mu} t^A{}_B = \partial_\mu t^A{}_B + \Gamma^A{}_{C\mu} t^C{}_B - \Gamma^C{}_{B\mu} t^A{}_C. \tag{155}$$

The corresponding mixed tensor $t^\alpha{}_\beta$ in M

$$t^\alpha{}_\beta = e^\alpha{}_A t^A{}_B e^B{}_\beta \tag{156}$$

is then automatically differentiated in the usual way

$$\nabla_{[\Gamma]\mu} t^\alpha{}_\beta = \partial_\mu t^\alpha{}_\beta + \Gamma^\alpha{}_{\lambda\mu} t^\lambda{}_\beta - \Gamma^\lambda{}_{\beta\mu} t^\alpha{}_\lambda. \tag{157}$$

Objects with spinor indices are covariantly differentiated according to the rule generated by one for bispinors

$$\nabla_{[\omega]\mu}\Psi = \partial_\mu \Psi + \omega_\mu \Psi \tag{158}$$

i.e., analytically

$$\nabla_{[\omega]\mu}\Psi^r = \partial_\mu \Psi^r + \omega^r{}_{s\mu}\Psi^s. \tag{159}$$

This rule is extended to the tensor algebra over $\mathbb{C}^4$, or more precisely, over $\mathcal{D}$, e.g.,

$$\nabla_{[\omega]\mu}\varphi^r{}_s = \partial_\mu \varphi^r{}_s + \omega^r{}_{z\mu}\varphi^z{}_s - \omega^z{}_{s\mu}\varphi^r{}_z$$

etc., including also objects having indices related to the complex conjugate space $\overline{\mathcal{D}}$.

All those rules are combined in an obvious way when one deals with objects having various kinds of indices. Such a unified differentiation will be denoted simply by $\mathbf{D}_\mu$. Let us quote an example.

$$\begin{aligned}\mathbf{D}_\mu \varphi^{\alpha}{}_{\beta}{}^{A}{}_{B}{}^{r}{}_{s}{}^{\bar{z}}{}_{\bar{u}} = \partial_\mu \varphi^{\alpha}{}_{\beta}{}^{A}{}_{B}{}^{r}{}_{s}{}^{\bar{z}}{}_{\bar{u}} \\ + \Gamma^{\alpha}{}_{\lambda\mu}\varphi^{\lambda}{}_{\beta}{}^{A}{}_{B}{}^{r}{}_{s}{}^{\bar{z}}{}_{\bar{u}} - \Gamma^{\lambda}{}_{\beta\mu}\varphi^{\alpha}{}_{\lambda}{}^{A}{}_{B}{}^{r}{}_{s}{}^{\bar{z}}{}_{\bar{u}} \\ + \Gamma^{A}{}_{C\mu}\varphi^{\alpha}{}_{\beta}{}^{C}{}_{B}{}^{r}{}_{s}{}^{\bar{z}}{}_{\bar{u}} - \Gamma^{C}{}_{B\mu}\varphi^{\alpha}{}_{\beta}{}^{A}{}_{C}{}^{r}{}_{s}{}^{\bar{z}}{}_{\bar{u}} \\ + \omega^{r}{}_{t\mu}\varphi^{\alpha}{}_{\beta}{}^{A}{}_{B}{}^{t}{}_{s}{}^{\bar{z}}{}_{\bar{u}} - \omega^{t}{}_{s\mu}\varphi^{\alpha}{}_{\beta}{}^{A}{}_{B}{}^{r}{}_{t}{}^{\bar{z}}{}_{\bar{u}} \\ + \overline{\omega}^{\bar{z}}{}_{\bar{v}\mu}\varphi^{\alpha}{}_{\beta}{}^{A}{}_{B}{}^{r}{}_{s}{}^{\bar{v}}{}_{\bar{u}} - \overline{\omega}^{\bar{v}}{}_{\bar{u}\mu}\varphi^{\alpha}{}_{\beta}{}^{A}{}_{B}{}^{r}{}_{s}{}^{\bar{z}}{}_{\bar{v}}.\end{aligned} \tag{160}$$

All possibilities of indices are here exhausted. There may be more or less indices; everything is done according to the above rule which is automatically compatible with the Leibniz rule for tensor products.

An important example: the cotetrad and tetrad fields. In the sense of the above unified connection the following holds

$$\mathbf{D}_\nu e^{A}{}_{\mu} = 0, \qquad \mathbf{D}_\nu e^{\mu}{}_{A} = 0. \tag{161}$$

This is equivalent to the relationship (135).

Obviously, when the differentiated object has only one kind of indices, then $\mathbf{D}$ reduces to some corresponding rule like $\nabla_{[\Gamma]}$, $\nabla_{[\widehat{\Gamma}]}$, $\nabla_{[\omega]}$, where $\widehat{\Gamma}_\mu$ is an abbreviation for $\Gamma^{A}{}_{B\mu}$. Let us observe that in certain situations objects with a few kinds of indices may be differentiated in a sense of only one of them. For example, we may be interested in affine Γ-differentiation of any of the "egs" ("Beine") e_A, A being fixed. Then, to avoid mistakes, in such situations the corresponding label like Γ, $\widehat{\Gamma}$, ω will be used in the ∇-symbol, so, for example

$$\nabla_{[\Gamma]\mu} e^{\nu}{}_{A} = \partial_\mu e^{\nu}{}_{A} + \Gamma^{\nu}{}_{\lambda\mu} e^{\lambda}{}_{A} \neq \mathbf{D}_\mu e^{\nu}{}_{A} = 0. \tag{162}$$

vi) The "world Dirac matrices", i.e., $\mathrm{H}(\mathcal{D}, G) \simeq \mathrm{H}(4, G) \simeq \mathrm{H}(2,2)$-valued differential form e on M. More precisely, it is V-valued and analytically defined as follows

$$e^{r}{}_{s\mu} := \gamma_A{}^{r}{}_{s} e^{A}{}_{\mu}. \tag{163}$$

In other words, this e is a differential form with values in the imaginary unit multiple of the Lie algebra of $\mathrm{U}(\mathcal{D}, G) \simeq \mathrm{U}(4, G) \simeq \mathrm{U}(2,2)$ when some representation is fixed, it is V-valued

$$M \ni x \mapsto e_x \in \mathrm{L}\left(T_x M, V\right) \subset \mathrm{L}\left(T_x M, \mathrm{iu}(\mathcal{D}, G)\right).$$

By the G-lowering of the first spinorial index of the "world Dirac matrices", we obtain a differential form with values in the space of Hermitian sesquilinear forms on $\mathcal{D} \simeq \mathbb{C}^4$. Analytically

$$e_{\bar{r}s\mu} = G_{\bar{r}z}e^{z}{}_{s\mu} = \Gamma_{A\bar{r}s}e^{A}{}_{\mu}. \tag{164}$$

Again, when some representation is fixed, $e_{\bar{r}s\mu}\mathrm{d}x^{\mu}$ is $\widetilde{V}$-valued

$$M \ni x \mapsto \widetilde{e}_x \in \mathrm{L}\left(T_xM, \widetilde{V}\right) \subset \mathrm{L}\left(T_xM, \mathrm{H}(\mathcal{D}, G)\right). \tag{165}$$

In analogy to (162), the quantities $e^{r}{}_{s\mu}$ and $e_{\bar{r}s\mu}$, i.e., the corresponding V- and $\overline{V}$-valued differential one-forms on M, are parallel with respect to the total $\mathbf{D}_{\mu}$-differentiation, i.e., total connection on M

$$\mathbf{D}_{\nu}e^{r}{}_{s\mu} = 0, \qquad \mathbf{D}_{\nu}e_{\bar{r}s\mu} = 0. \tag{166}$$

This is not true for the partial connection $\nabla_{[\Gamma]\mu}$.

Let us notice that the equivalence of two equations in (168) is due to the fact that

$$\mathbf{D}_{\nu}G_{\bar{r}s} = 0 \tag{167}$$

and the latter formula follows exactly from two properties of G: it is a multiplet of constant scalar fields on M (scalars in the sense of M as a manifold), and ω_{ν} is G-antihermitian (it takes values in the Lie algebra of pseudounitary group $\mathrm{U}(\mathcal{D}, G) \simeq \mathrm{U}(2,2)$, i.e., in the space of G-antihermitian mappings).

As mentioned previously, the use of analytical description and numerical spaces like $\mathbb{C}^2$, $\mathbb{C}^4$, $\mathbb{R}^4$, $\mathrm{L}(4,\mathbb{C})$, etc. as target spaces may be even essentially misleading. What is worse, it obscures the physical interpretation and may prevent us to find a proper model. That was the reason we suggested to use linear spaces W, $\mathcal{D}$, etc. as proper target (standard fibres).

Linear frames $e_x : T_xM \rightarrow V \subset \mathrm{iu}(\mathcal{D}, G) \simeq \mathrm{iu}(2,2) \simeq \mathrm{L}(4,\mathbb{C})$ extend naturally to isomorphisms of complexified Clifford algebras

$$\mathrm{CCl}\left(T_xM, g[e,\eta]_x\right) \rightarrow \mathrm{L}\left(\mathcal{D}\right) \simeq \mathrm{L}\left(4,\mathbb{C}\right) \simeq \mathrm{CCl}\left(\mathbb{R}^4, \eta\right). \tag{168}$$

Incidentally, it may look strange that the vector-valued differential forms $[e^{r}{}_{s}]$, $[e_{\bar{r}s}]$ take values in some fixed subspaces V and $\widetilde{V}$, not in the total, non-restricted linear spaces

$$\mathrm{H}(\mathcal{D}, G) = \mathrm{iu}(\mathcal{D}, G) \simeq \mathrm{H}(2,2) \simeq \mathrm{iu}(2,2). \tag{169}$$

This is a kind of rigid action at a distance structure. When translated into rigorous fibre bundle language this means that there is some kind of rigid "flatness" built up into the theory. In this respect the theory of two-component Weyl spinors is much more natural and free of such aprioric absolute objects. Analytically, Weyl spinors take values in $\mathbb{C}^2$ or, to be more precise, in a two-dimensional complex linear space W. Geometrically, they are cross-sections of some associated vector bundle with the typical fibre W (or, more analytically, $\mathbb{C}^2$). As we have seen, the bispinor space is then defined as the Cartesian product

$$\mathcal{D} = W \times \overline{W}^*. \tag{170}$$

The space W is completely amorphous, whereas its byproducts like $\mathcal{D}$ and $\mathrm{H}(W)$ are full of intrinsic, inherited structures. Strictly speaking, and this is theoretically important, not everything in Clifford geometry developed over $\mathcal{D}$ is completely intrinsic. An intrinsic element is the particular choice of normalisation of η. In geometry of single Weyl spaces this normalisation was not so essential, i.e., we have no Clifford concepts there. If one follows the Clifford paradigm, a particular, fixed normalisation of η must be assumed. Similarly, in the bispinor connection expression (151), (152) η is essential with a given fixed normalisation. This normalisation has to do with the normalisation of the symplectic structure in W. Let us remind the formula

$$\eta_{AB} = \frac{1}{2}\sigma_A{}^{b\bar{a}}\sigma_B{}^{d\bar{c}}\varepsilon_{bd}\overline{\varepsilon}_{\bar{a}\bar{c}} \tag{171}$$

$$\eta^{AB} = \frac{1}{2}\sigma^A{}_{b\bar{a}}\sigma^B{}_{d\bar{c}}\overline{\varepsilon}^{\,\bar{a}\bar{c}}\varepsilon^{bd} \tag{172}$$

where

$$\varepsilon^{ac}\varepsilon_{cb} = \delta^a{}_b, \qquad \overline{\varepsilon}^{\,\bar{a}\bar{c}}\,\overline{\varepsilon}_{\bar{c}\bar{e}} = \delta^{\bar{a}}{}_{\bar{e}}. \tag{173}$$

As mentioned, they are defined up to squared real multipliers at the complex symplectic form ε and its inverse $\widetilde{\varepsilon}$; the factors $\exp(\mathrm{i}\varphi)$, $\varphi \in \mathbb{R}$, do not influence them. The four-component bispinor fields $\Psi{:}M \to \mathcal{D} \simeq \mathbb{C}^4$ may be interpreted as pairs of two-component spinors: Weyl $u{:}M \to W$ and anti-Weyl $v{:}M \to \overline{W}^*$. Let us remind that physically the objects u, v may be used respectively for describing, e.g., the anti-neutrino and neutrino fields. One can introduce $\mathrm{H}(W)$-valued differential one-form given by analytically by

$$e^{a\bar{b}}{}_\mu := \sigma_A{}^{a\bar{b}}e^A{}_\mu \tag{174}$$

and similarly the vector-valued differential one-form with $\mathrm{H}(W^*) \simeq \mathrm{H}(W)^*$ as the target space

$$e_{\bar{a}b\mu} := \widetilde{\sigma}_{A\bar{a}b}e^A{}_\mu = \eta_{AB}\sigma^B{}_{\bar{a}b}e^A{}_\mu. \tag{175}$$

Those quantities are the "world Pauli matrices". What concerns the second object, perhaps from some point of view the reciprocal version contravariant in M would be more adequate

$$e^\mu{}_{a\bar{b}} = e^\mu{}_A\sigma^A{}_{a\bar{b}} \tag{176}$$

with the dual contravariant tetrad instead the cotetrad

$$e^A{}_\mu e^\mu{}_B = \delta^A{}_B, \qquad e^\mu{}_A e^A{}_\nu = \delta^\mu{}_\nu. \tag{177}$$

In any case (176) is free of the fixed normalisation of η.

Let us observe that unlike the world Dirac matrices $e^{r}{}_{s\mu}$, the Weyl quantities $e^{a\bar{b}}{}_{\mu}$, $e_{\bar{a}b\mu}$ (or rather $e^{\mu}{}_{\bar{a}b}$) are completely amorphous. They take values in the total spaces of all Hermitian tensors on the (two-dimensional) spinor target spaces. Nothing like the aforementioned restriction to the real four-dimensional subspaces V, $\widetilde{V}$, is assumed (in the fibre bundle language — no global "action at distance" "flatness" assumption). So again the two-component Weyl spinors seem to be most elementary objects. The bispinor joining of the Weyl-anti-Weyl pairs into Dirac objects and first of all its Clifford explaining might be perhaps an artificial peculiarity of the space-time dimension $\dim M = 4$. Physically this joining is necessary for introducing the mass terms to Lagrangians and for the invariance under "spatial reflections" in spinor degrees of freedom.

In connection with this two-component language let us mention another byproduct of fundamental Weyl quantities, namely the Weyl spinor connection and covariant differentiation. Basing on the local action of the group $\mathrm{GL}(W)$ on Weyl and anti-Weyl spinors we can introduce $\mathfrak{gl}(W) \simeq \mathrm{L}(W)$-valued Weyl connection forms ${}^{w}\omega^{a}{}_{b\mu}$ and the corresponding covariant differentiation. Later on those operations may be raised to Dirac $\mathcal{D}$-valued fields and to $\mathrm{H}(W)$- and $\mathrm{H}(W^{*}) \simeq \mathrm{H}(W)^{*}$-valued fields.

According to formulae (89), (90), we have for the corresponding connection components:

$$\Gamma_{\mu} = \mathrm{Re}\,\mathrm{Tr}\left(\sigma^{K}\left({}^{w}\omega_{\mu}\right)\sigma_{L}\right) \tag{178}$$

i.e., completely analytically

$$\Gamma^{K}{}_{L\mu} = \frac{1}{2}\sigma^{K}{}_{\bar{b}a}\left({}^{w}\omega^{a}{}_{c\mu}\right)\sigma_{L}{}^{c\bar{b}} + \frac{1}{2}\sigma^{K}{}_{\bar{b}a}\left({}^{w}\overline{\omega}^{\bar{b}}{}_{\bar{c}\mu}\right)\sigma_{L}{}^{a\bar{c}}. \tag{179}$$

It is thus clear that

$$\mathrm{Tr}\left(\Gamma_{\mu}\right) = \Gamma^{K}{}_{K\mu} = 4\mathrm{Re}\,\mathrm{Tr}\left({}^{w}\omega_{\mu}\right) = 4\mathrm{Re}\left({}^{w}\omega^{a}{}_{a\mu}\right). \tag{180}$$

The formula inverse to (179), by analogy to (92), reads

$${}^{w}\omega^{a}{}_{b\mu} = \frac{1}{4}\left(\Gamma^{K}{}_{L\mu}\sigma_{K}{}^{a\bar{c}}\sigma^{L}{}_{\bar{c}b} - \frac{1}{2}\Gamma^{K}{}_{K\mu}\delta^{a}{}_{b}\right) + \frac{\mathrm{i}}{2}\mathrm{Im}\left({}^{w}\omega^{c}{}_{c\mu}\right)\delta^{a}{}_{b} \tag{181}$$

where with a fixed $\Gamma^{K}{}_{L\mu}$ as a primary quantity, the last term, i.e., purely imaginary gauging is completely arbitrary. Its value does not influence $\Gamma^{K}{}_{L\mu}$.

The corresponding covariant differentiation of the Weyl spinors has the form

$$\nabla_{[{}^{w}\omega]\mu}u^{a} = \partial_{\mu}u^{a} + \left({}^{w}\omega^{a}{}_{b\mu}\right)u^{b} \tag{182}$$

and similarly for the anti-Weyl spinors

$$\nabla_{[{}^{w}\omega]\mu}v_{\bar{a}} = \partial_{\mu}v_{\bar{a}} - v_{\bar{c}}\left({}^{w}\omega^{\bar{c}}{}_{\bar{a}\mu}\right). \tag{183}$$

The corresponding Dirac differentiation of bispinor fields has the unified form

$$\nabla_{[\omega]\mu}\Psi^r = \nabla_{[\omega]\mu}\begin{bmatrix} u^a \\ v_{\bar{b}} \end{bmatrix} = \begin{bmatrix} \nabla_{[{}^w\omega]\mu}u^a \\ \nabla_{[{}^w\omega]\mu}v_{\bar{b}} \end{bmatrix}. \tag{184}$$

Let us quote in addition some similar formulae for bispinors

$$\nabla_{[\omega]\mu}\widetilde{\Psi}_r = \partial_\mu\widetilde{\Psi}_r - \omega^s{}_{r\mu}\widetilde{\Psi}_s. \tag{185}$$

It is interesting that in the above formulae the scale standard, i.e., particular normalisation of η is non-essential and it does not occur at all. The two kinds of amorphous two-component Weyl spinors are completely sufficient here. It is not so in (152) where the indices of Dirac matrices are moved with the use of fixed-normalised metric η.

Starting from the Weyl spinors, it is instructive to discuss the important problem of metrical compatibility of our affine connections. First of all, an arbitrary $\mathrm{GL}(W)$-ruled connection form ${}^w\omega^a{}_{b\mu}$ gives rise to the connection Γ_μ, $\Gamma^A{}_{B\mu}$ (see (178) and (179)) dealing with the capital indices. Then, when some (co)tetrad field e is fixed, the true affine connection $\Gamma^\alpha{}_{\beta\mu}$ (135) may be fixed. For a quite arbitrary connection ${}^w\omega$, therefore, for a quite arbitrary affine connection $\Gamma^\alpha{}_{\beta\mu}$, and for the original definition of Dirac-Einstein metric (132) with somehow fixed normalisation of η, we have the following Einstein-Cartan-Weyl rule

$$\Gamma^\mu{}_{\nu\lambda} = \left\{ \begin{matrix} \mu \\ \nu\lambda \end{matrix} \right\} + \mathcal{K}^\mu{}_{\nu\lambda} \tag{186}$$

where $\left\{ \begin{matrix} \mu \\ \nu\lambda \end{matrix} \right\}$ is the Levi-Civita connection built of $g\,[e,\eta]$, the $\mathcal{K}$-tensor is given by

$$\mathcal{K}^\mu{}_{\nu\lambda} = S^\mu{}_{\nu\lambda} + S_{\nu\lambda}{}^\mu - S_\lambda{}^\mu{}_\nu + \frac{1}{2}\left(\delta^\mu{}_\nu Q_\lambda + \delta^\mu{}_\lambda Q_\nu - g_{\nu\lambda}Q^\mu\right) \tag{187}$$

in which $S^\mu{}_{\nu\lambda}$ is the torsion of $\Gamma^\mu{}_{\nu\lambda}$

$$S^\mu{}_{\nu\lambda} = \frac{1}{2}\left(\Gamma^\mu{}_{\nu\lambda} - \Gamma^\mu{}_{\lambda\nu}\right) \tag{188}$$

and Q_μ is the Weyl covector field, so that

$$\nabla_\lambda g_{\mu\nu} = -Q_\lambda g_{\mu\nu} \tag{189}$$

and the covariant derivative is meant in the total Γ-sense.

One can easily show that

$$Q_\mu = \frac{1}{2}\Gamma^A{}_{A\lambda} = 2\mathrm{Re}\left(\omega^a{}_{a\mu}\right). \tag{190}$$

However, the level of the Riemann-Cartan-Weyl space is not suited to the theory of dynamically interacting geometry and spinor matter. To include the internal Weyl invariance into theory one should introduce additional geometrodynamical quantities. One of explanations is just in (176). One would have to introduce into this formula a new version of tetrad $f^{\mu}{}_{A}$, independent on $e^{\mu}{}_{A}$, and postulate same dynamics for the system of geometrodynamical quantities.

So, let us go back to reporting the Einstein-Cartan-Dirac theory and then pass to our original ideas.

Let us start with the remark that in a sense, paradoxically, generally-relativistic theory of dynamical spinors, i.e., spinorial geometrodynamics, is less confusing than the seemingly simpler specially-relativistic theory. The point is again in artefacts and some hidden structures of Minkowski space. There is nothing like the smooth transition to Dirac theory and spinor theory in a curved space. The reason is that the covering group of $\mathrm{GL}(4,\mathbb{R})$ (any $\mathrm{GL}(n,\mathbb{R})$, $n > 3$) is not a linear group, so it does not admit a faithful realisation in terms of finite-dimensional matrices (like $\mathrm{SU}(2)$, $\mathrm{SL}(2,\mathbb{C})$, respectively for $\mathrm{SO}(3,\mathbb{R})$, $\mathrm{SO}^{\uparrow}(1,3)$). Affine spinors would have to be either nonlinear or infinite-dimensional objects. The only way out of this difficulty is to introduce the (co)tetrad field as an auxiliary object, gravitational potential, by definition orthonormal one. Then the group $\mathrm{SO}(1,3)$ ($\mathrm{SO}(\mathrm{H},\eta)$ geometrically speaking) "mixes" the (co)tetrad (co)legs in a way appropriately synchronised with the action of $\mathrm{SL}(2,\mathbb{C})$ ($\mathrm{SL}(W)$) on (bi)spinor fields. And the status of this (co)tetrad field is geometrically mysterious. It is neither the gauge field (at least in a standard sense; there were some non-standard attempts) nor Higgs field, nor matter field or anything of a well-established status. It is the field of reference frames that never occurs as a dynamical quantity in any physical theory. Just an additional motivation to do something with this problem. By the way, even in specially-relativistic spinor theory, the concept of tetrad is implicit present, although apparently hidden.

Matter Lagrangian for the Dirac field in the standard Einstein-Cartan theory is given by

$$\begin{aligned} L_{\mathrm{m}}(\Psi,e,\omega) &= \frac{\mathrm{i}}{2} g^{\mu\nu} e^{r}{}_{s\mu}\left(\widetilde{\Psi}_{r}\nabla_{\nu}\Psi^{s}-\nabla_{\nu}\widetilde{\Psi}_{r}\Psi^{s}\right)\sqrt{|g|}-m\widetilde{\Psi}_{r}\Psi^{r}\sqrt{|g|} \\ &= \frac{\mathrm{i}}{2} g^{\mu\nu} e_{\bar{r}s\mu}\left(\overline{\Psi}^{\bar{r}}\nabla_{\nu}\Psi^{s}-\nabla_{\nu}\overline{\Psi}^{\bar{r}}\Psi^{s}\right)\sqrt{|g|}-mG_{\bar{r}s}\overline{\Psi}^{\bar{r}}\Psi^{s}\sqrt{|g|} \end{aligned}$$

where, obviously, $e^{r}{}_{s\mu}$, $e_{\bar{r}s\mu}$ are given by (163), (164, and $\sqrt{|g|}$ is the W-density built canonically of $g_{\mu\nu}$

$$\sqrt{|g|}=\sqrt{|\det[g_{\mu\nu}]|}. \tag{191}$$

The same formula works in Minkowski space in curvilinear coordinates. No trick enables us to hide the metric tensor $g^{\mu\nu}$ by the formal use of

$$e^{\mu r}{}_{s} := e^{\mu}{}_{A}\gamma^{Ar}{}_{s} = g^{\mu\nu}e^{r}{}_{s\nu}$$

(and similarly for $e_{\bar{r}s\mu}$), because the scalar density of weight one $\sqrt{|g|}$ is necessary if L_m is to be a correctly defined Lagrangian density.

We remember that in the standard Dirac theory, $e^r{}_{s\mu}$, $e_{\bar{r}s\mu}$ represent vector-valued differential one-forms taking values in some proper subspaces V, $\widetilde{V}$ of $\mathrm{H}(\mathcal{D}, G)$ or $\mathrm{H}(\mathcal{D}^*)$. Unlike this, in Weyl theory for the field u one uses the most general differential one-form $e_{\bar{a}b\mu}$ without any restrictions on its values in $\mathrm{H}(W^*)$

$$L^{\mathrm{Weyl}} = \frac{\mathrm{i}}{2} g^{\mu\nu} e_{\bar{a}b\mu} \left(\overline{u}^{\,\bar{a}} \nabla_\nu u^b - \nabla_\nu \overline{u}^{\,\bar{a}} u^b \right) \sqrt{|g|}. \tag{192}$$

Let us remind that in all those formulae we are dealing with the Einstein-Cartan model, where the Weyl covector vanishes

$$Q_\mu = 0 \tag{193}$$

thus, the connection form ${}^w\omega^a{}_{b\mu}$ is trace-less, i.e., $\mathfrak{sl}(2, \mathbb{C})$-valued (and so are ${}^w\overline{\omega}^a{}_{b\mu}$, ${}^w\omega^r{}_{s\mu}$), and $\Gamma^K{}_{L\mu}$ is η-skew-symmetric

$$\Gamma^K{}_{L\mu} + \eta^{KM} \eta_{LN} \Gamma^N{}_{M\mu} = 0 \tag{194}$$

therefore, trace-less. Then automatically (194) implies that the affine connection $\Gamma^\alpha{}_{\beta\mu}$ built of the abstract connection $\Gamma^A{}_{B\mu}$ and the (co)frame $\left(e^A{}_\mu\right)$ $e^\mu{}_A$ is metrical, i.e., it is an Einstein-Cartan connection, thus, the metrical one

$$\nabla_\lambda g_{\mu\nu} = 0$$

although in general non-symmetric (the torsion S needs not vanish). This Levi-Civita assumption would be too strong and artificial.

When the gravitational degrees of freedom, i.e., $e_{\bar{a}b\mu}$, or $e^r{}_{s\mu}$, or $e_{\bar{r}s\mu}$ are fixed (and so is the Einstein-Dirac metric $g\left[e, \eta\right]_{\mu\nu}$) and not subject to the variational procedure, then we obtain the following generally-relativistic Dirac and Weyl equations formulated on the background of fixed geometry (gravitational field)

$$\mathrm{i} e^\mu{}_A \gamma^A \left(\nabla_\mu + S^\nu{}_{\nu\mu} I_4 \right) \Psi = m\Psi \tag{195}$$

(in natural units), i.e., more analytically

$$\mathrm{i} e^{\mu r}{}_s \left(\nabla_\mu{}^s{}_z + S^\nu{}_{\nu\mu} \delta^s{}_z \right) \Psi^z = m\Psi^r \tag{196}$$

for Dirac equations and

$$\mathrm{i} e^\mu{}_A \sigma^A \left(\nabla_\mu + S^\nu{}_{\nu\mu} I_2 \right) u = 0 \tag{197}$$

for generally-relativistic Weyl equation. Analytically

$$\mathrm{i} e^\mu{}_{\bar{a}c} \left(\nabla_\mu{}^c{}_b + S^\nu{}_{\nu\mu} \delta^c{}_b \right) u^b = 0. \tag{198}$$

Let us notice that the two-component Weyl equation is mass-less if it is to be self-adjoint and relativistically invariant. The terms involving the trace of torsion are remarkable. Only in the torsion-free Riemann space or in special Riemann-Cartan spaces with the vanishing trace of torsion, one obtains the expected equations. It is interesting that the self-adjoint structure of field equations, i.e., the existence of Lagrangian is in both cases based on the fixed metric normalisation η. What concerns the field equations themselves, the Dirac equation (195), (196) depends explicitly on η through its occurrence in γ^A-s. Unlike this, only the Weyl Lagrangian is based on some choice of η while the corresponding Weyl equation (197) or (198) is non-metrical-amorphous.

Some problems appear when we admit also anti-Weyl field v and wish to consider it simultaneously with the Weyl field; this union is just the Dirac field in a sense. Then we have to use anti-Weyl Lagrangian, analytically

$$\begin{aligned} L^{\mathrm{anti-Weyl}} &= \frac{\mathrm{i}}{2} e^{\nu}{}_{A} \widetilde{\sigma}^{Aa\bar{b}} \left(\overline{v}_a \nabla_\nu v_{\bar{b}} - \left(\nabla_\nu \overline{v}_a\right) v_{\bar{b}} \right) \sqrt{|g|} \\ &= \frac{\mathrm{i}}{2} \widetilde{\sigma}^{\nu a\bar{b}} \left(\overline{v}_a \nabla_\nu v_{\bar{b}} - \left(\nabla_\nu \overline{v}_a\right) v_{\bar{b}} \right) \sqrt{|g|} \end{aligned} \tag{199}$$

and even the resulting field equation will depend on the explicit fixation of η

$$\mathrm{i} e^{\mu}{}_{A} \widetilde{\sigma}^{A} \left(\nabla_\mu + S^{\nu}{}_{\nu\mu} I_2 \right) v = 0. \tag{200}$$

An alternative way of writing (199) would be

$$L^{\mathrm{anti-Weyl}} = \frac{\mathrm{i}}{2} g^{\mu\nu} f_{A\mu} \widetilde{\sigma}^{Aa\bar{b}} \left(\overline{v}_a \nabla_\nu v_{\bar{b}} - \left(\nabla_\nu \overline{v}_a\right) v_{\bar{b}} \right) \sqrt{|g|}. \tag{201}$$

In the geometrodynamical sector we would have to use then two independent tetrads $\left(e^{\mu}{}_{A}, f^{\mu A}\right)$ or cotetrads $\left(e^{A}{}_{\mu}, f_{A\mu}\right)$, with some independent dynamical interaction between them. One really does something like this in spinor theory invariant under the Weyl group. However, this would be a compromise and our proper idea goes further towards the full conformal group as the gauge group of geometrodynamics.

A natural procedure would be to consider e, f as logically independent, take the spinor connection fundamental one, construct the corresponding affine connections $\Gamma[\omega, e]$, $\Gamma[\omega, f]$, in H, and to include some dynamical term built of e, f, e.g., starting from the twice covariant tensor

$$t\,[e, f]_{\mu\nu} = e^{A}{}_{\mu} f_{A\nu}$$

or its symmetric part

$$\gamma\,[e, f]_{\mu\nu} = \frac{1}{2} \left(e^{A}{}_{\mu} f_{A\nu} + e^{A}{}_{\nu} f_{A\mu} \right).$$

Obviously, there is plenty of algebraic and differential concomitants of the objects $t\,[e, f]$, $\gamma\,[e, f]$ which might be a priori possible as interaction terms. And then, the independence (at least partial one) of e, f-tensors might be used for elimination the embarrassing standard of scale η_{AB} from the theory.

Obviously, in the specially-relativistic limit, when pseudo-Cartesian coordinates are chosen (and gravitational-geometric degrees of freedom are frozen), so that

$$e^{A}{}_{\mu} = \delta^{A}{}_{\mu}, \quad \Gamma^{A}{}_{B\mu} = 0, \quad {}^{w}\omega^{a}{}_{b\mu} = 0, \quad \omega^{r}{}_{s\mu} = 0, \quad g_{\mu\nu} = \eta_{\mu\nu} \tag{202}$$

then for the weak bispinor and spinor fields we obtain

$$L = \frac{\mathrm{i}}{2}\gamma^{\mu r}{}_{s}\left(\widetilde{\Psi}_{r}\partial_{\mu}\Psi^{s} - \partial_{\mu}\widetilde{\Psi}_{r}\Psi^{s}\right) - m\widetilde{\Psi}_{r}\Psi^{r} \tag{203}$$

$$L = \frac{\mathrm{i}}{2}\sigma^{\mu}{}_{\bar{a}b}\left(\overline{u}^{\bar{a}}\partial_{\mu}u^{b} - \partial_{\mu}\overline{u}^{\bar{a}}u^{b}\right) \tag{204}$$

respectively for the Dirac and Weyl fields and obviously

$$\mathrm{i}\gamma^{\mu r}{}_{s}\partial_{\mu}\Psi^{s} = m\Psi^{r}, \qquad \mathrm{i}\sigma^{\mu}{}_{\bar{a}b}\partial_{\mu}u^{b} = 0, \qquad \mathrm{i}\widetilde{\sigma}^{\mu a\bar{b}}\partial_{\mu}v_{\bar{b}} = 0 \tag{205}$$

respectively for the Dirac, Weyl and anti-Weyl fields in special relativity.

Usually one concentrates in physics on two extreme situations:

1. Matter wave equations on the basis of fixed geometry.
2. Dynamics of gravitational field under the influence of matter.

Obviously, the most important situations are those characterised by the mutual interactions. Especially when one is interested in evolution of the Early Universe or the dynamics of highly concentrated objects like, e.g., neutron stars.

Let us quote a few Lagrangian terms used in the Poincaré gauge models of gravitation, including the Einstein-Cartan model.

The most traditional Einstein-Cartan model is similar to the Palatini Lagrangian, however, without restricting assumptions about the symmetry of affine connection

$$L^{\mathrm{EC}}_{\mathrm{gr}}(e,\omega) = \frac{1}{k}g^{\mu\nu}R(\Gamma)^{\alpha}{}_{\mu\alpha\nu}\sqrt{|g|} \tag{206}$$

k denoting gravitational constants, up to units and normalisation, $R(\Gamma)^{\alpha}{}_{\mu\kappa\nu}$ is the Riemann tensor built of Γ. The existence of Lagrangian linear in curvature is a peculiarity of gravitation among all other gauge models of fundamental interactions. (206) is believed to describe macroscopic gravitation. The quantities (e,ω) or (e,Γ) are assumed to be two independent kinds of degrees of freedom. The label "gr" refers to "gravity" or "geometry".

The Yang-Mills term quadratic in curvature is constructed according to the standard prescription of gauge theories

$$L^{\mathrm{YM}}_{\mathrm{gr}}(e,\omega) = \frac{1}{\ell}R^{\alpha}{}_{\beta\mu\nu}R^{\beta}{}_{\alpha\kappa\lambda}g^{\mu\kappa}g^{\nu\lambda}\sqrt{|g|} \tag{207}$$

where the Riemann tensor just as previously is built of affine connection, without any direct use of g. The quantity ℓ is a "microgravitational" constant. It is assumed to be responsible for gravitation in the very microscopic scale, at very small distances.

A priori one can admit the "cosmological-like" term

$$L_{\mathrm{gr}}^{\mathrm{cosm}}(e,\omega) = \Lambda\sqrt{|g|} \tag{208}$$

in which Λ is a constant (unlike to certain popular views, no sign of Λ may be decided a priori, without comparison with experiments).

An important triple of terms is one quadratic in the torsion tensor. There are functionally independent ones, known as Weitzenböck invariants

$$\begin{aligned} L_{\mathrm{gr}}^{\mathrm{torsion}}(e,\omega) &= A g^{\mu\kappa} g^{\nu\lambda} S^{\alpha}{}_{\mu\nu} S^{\beta}{}_{\kappa\lambda}\sqrt{|g|} \\ &\quad + B g^{\mu\nu} S^{\alpha}{}_{\beta\mu} S^{\beta}{}_{\alpha\nu}\sqrt{|g|} + C g^{\mu\nu} S^{\alpha}{}_{\alpha\mu} S^{\beta}{}_{\beta\nu}\sqrt{|g|} \end{aligned} \tag{209}$$

with A, B, C being constants. It is also expected that these terms have to do with the microscale gravity. Let us observe the characteristic Killing-Cartan structure of the S-S expression in the second term.

There is an opinion, expressed by Obukhov, Sardanashvilli and Ivanenko [4, 5, 9] that combining approximately the above terms one obtains on the quantized level the renormalizable theory. This is interesting, because according to certain views the usual Einstein theory is so notoriously non-renormalizable like the old Fermi model of weak interactions. If the mentioned views are true, this would be a new argument in favour of gauge methodology in fundamental interactions.

The Einstein-Cartan gauge Poincaré model of the spinor-gravity (geometry) interaction is in fact a kind of gauge theory, although one must say a very peculiar one, with certain features fairly uncommon with other, in a sense "true" gauge theories of fundamental interactions.

Let us remind transformations like $U(A)$ (65), $u(a)$ (70), (71), A_{H} (80), $\alpha_{(\mathrm{H})} = \ell(\alpha)$ (88)–(90), $P(A)$, $p(a)$ (95)–(98) and apply them in the gauge theory context.

The gauge idea consists in that all transformations quoted above are local, i.e., x-dependent. More precisely, we are dealing with the infinite-dimensional group of pointwisely composed group-valued mappings

$$A : M \to \mathrm{GL}(W) \simeq \mathrm{GL}(2,\mathbb{C})$$

or in traditional Dirac-Einstein-Cartan theory

$$A : M \to \mathrm{SL}(W) \simeq \mathrm{SL}(2,\mathbb{C}).$$

On the level of spaces $\mathrm{H}(W)$, $\mathrm{H}(W^{*}) \simeq \mathrm{H}(W)^{*}$ we are dealing with the resulting mappings

$$A_{\mathrm{H}} = L(A) : M \to \mathbb{R}^{+}\mathrm{O}(1,3)$$

or in the traditional Dirac-Einstein-Cartan theory

$$A_{\mathrm{H}} = L(A) : M \to \mathrm{SO}^{\uparrow}(1,3)$$

when the range of A is restricted to $\mathrm{SL}(W) \simeq \mathrm{SL}(2,\mathbb{C})$. Similarly, we deal with mappings $U : M \to \mathrm{U}(\mathcal{D}, G)$, i.e., analytically, $U : M \to \mathrm{U}(2,2)$ and in Dirac-Einstein-Cartan theory U, as mentioned, takes the values in the subgroup of corresponding to the $D^{(1/2,0)} \otimes D^{(0,1/2)}$-representation of $\mathrm{SL}(2,\mathbb{C})$. Transformation properties of our objects are then as follows

$$[\Psi^r] \mapsto [U(A)^r{}_s \Psi^s] \tag{210}$$

$$\left[e^K{}_\mu\right] \mapsto \left[L(A)^K{}_M e^M{}_\mu\right] \tag{211}$$

$$\left[\Gamma^K{}_{M\mu}\right] \mapsto \left[L(A)^K{}_N \Gamma^N{}_{R\mu} L(A)^{-1R}{}_M - \frac{\partial L(A)^K{}_N}{\partial x^\mu} L(A)^{-1N}{}_M\right] \tag{212}$$

$$[\omega^r{}_{s\mu}] \mapsto \left[U(A)^r{}_z \omega^z{}_{t\mu} U(A)^{-1t}{}_s - \frac{\partial U(A)^r{}_z}{\partial x^\mu} U(A)^{-1z}{}_s\right]. \tag{213}$$

The first two rules are homogeneous-linear. The next two ones are non-homogeneous, i.e., affine. They contain the typical non-homogeneous-additive corrections for geometric objects of the connection type.

It is obvious that the total Einstein-Cartan-Dirac Lagrangians of the form

$$L = L_{\mathrm{m}}(\Psi, e, \omega) + L_{\mathrm{gr}}(e, \omega) \tag{214}$$

L_{m} given by (191), and L_{gr} obtained by summation of the terms (206), (207), (208) and (209) is invariant under the action of the above gauge group. In this sense Einstein-Cartan-Dirac theory is a kind of gauge theory with the internal (acting on bispinor and on the tetrad legs) $\mathrm{SL}(W) \simeq \mathrm{SL}(2,\mathbb{C})$ group of local transformations. There are, however, some quite non-typical features of this gauge theory in comparison with "true" gauge theories.

First of all, its main non-typical feature in comparison with commonly used gauge theories is that the gauge group $\mathrm{SL}(W) \simeq \mathrm{SL}(2,\mathbb{C})$, just like the dilatations-admitting $\mathrm{GL}(W) \simeq \mathrm{GL}(2,\mathbb{C})$, is non-compact. Much more strange is the dynamical use of the tetrad or equivalently cotetrad. The main meaning of this object is that of the reference frame. In no other "true" gauge theory such an object occurs as a dynamical quantity. In gauge gravitation it is an object which establishes a bundle monomorphism of an abstract principal $\mathrm{SO}^\uparrow(1,3)$-bundle over M into the bundle of linear frames FM; more in detail, onto some $\mathrm{SO}^\uparrow(1,3)$-reduction of FM. This reduction is dynamical, non-fixed, thus, it belongs to physical degrees of freedom.

Besides of this fact there are a few other doubts as to the gauge status and general structure of the theory.

1. The first one is that the differential vector-valued one-form

$$[e^r{}_{s\mu}]_x = \left[\gamma_A{}^r{}_s e^A{}_\mu\right]_x : T_x M \to V \subset \mathrm{iu}(\mathcal{D}, G) \simeq \mathrm{iu}(2,2)$$

takes values in the proper subspace V, not in the total space $\mathrm{i}\mathfrak{u}(\mathcal{D}, G) \simeq \mathrm{i}\mathfrak{u}(2,2)$. As mentioned, in this respect the comparison with generally-relativistic Weyl theory is rather instructive. When working with Weyl spinors, we use the total spaces of Hermitian tensors $\mathrm{H}(W)$, $\mathrm{H}(W^*) \simeq \mathrm{H}(W)^*$ as target spaces. Why not to follow this pattern when dealing with four-component spinors and basing on the conformal-motivated $\mathrm{U}(\mathcal{D}, G)$-symmetry?

2. The $(+,+,-,-)$ metric $G_{\bar{r}s}$ underlies $L_{\mathrm{m}}\left(\Psi, e, \omega\right)$. So a desirable idea is the symmetry just under conformally-motivated $\mathrm{U}(\mathcal{D}, G) \simeq \mathrm{U}(2,2)$, not only under $\mathrm{SL}(W) \simeq \mathrm{SL}(2, \mathbb{C})$ injected into $\mathrm{U}(2,2)$ by the bispinor representation $D^{(1/2,0)} \otimes D^{(0,1/2)}$. To achieve it in a structureless M, one must admit general vector-valued forms
$$\left[e^{r}{}_{s\mu}\right]_x : T_x M \to \mathrm{H}(\mathcal{D}, G) \simeq \mathrm{H}(2,2).$$
3. In special relativity, the x-independence of the representation subspace $V \subset \mathrm{H}(\mathcal{D}, G) \simeq \mathrm{H}(2,2)$ has to do with the Dirac-Clifford idea of taking the square root of the d'Alembert operator. In general relativity, this paradigm loses its conceptual coherence and convincing power. Classically, it is still true that
$$\left(\gamma^{\mu} p_{\mu}\right)^2 = g^{\mu\nu} p_{\mu} p_{\nu} I_4 \tag{215}$$
but on the operator level,
$$\left(\mathrm{i}\gamma^{\mu}\left(\nabla_{\mu} + S^{\nu}{}_{\nu\mu} I_4\right)\right)^2 \neq -g^{\mu\nu}\nabla_{\mu}\nabla_{\nu} I_4 \neq \left(\mathrm{i}\gamma^{\mu}\nabla_{\mu}\right)^2 \tag{216}$$
where
$$\gamma^{\mu} := e^{\mu}{}_{A}\gamma^{A}$$
are the "world Dirac matrices". There is no longer the Clifford square-rooting of the d'Alembert operator, and the artificial globality of V becomes a price paid for nothing.
Even in the simplest case of (artificial in this context) Levi-Civita connection, the right-hand side of (216) contains an additional term
$$-\frac{1}{4} R\, I_4 \tag{217}$$
which rather destroys the coherence of Clifford paradigm as a fundamental physical postulate (here R denotes the scalar curvature of the Riemann tensor).
Let us also stress another important point. Namely, Poincaré group is drastically non-semi-simple. Because of this the Lagrangian (214), especially in its geometric-gravitational part, is drastically non-unique and dependent on plenty of arbitrary parameterizing constants. The good thing is only

that according to views declared by people working in the topic, the resulting theory may be renormalizable on the quantum level and that it "macroscopic" part is compatible with standard general relativity, including such its well-established consequences as the Schwarzschild solution and post-Newtonian limit.

4. As mentioned, there is no dynamical use of frames in "genuine" gauge theories. Generally-relativistic spinors must use tetrads because of the mentioned geometric reasons (for $n \geq 3$, $\overline{\mathrm{GL}(n,\mathbb{R})}$ and $\overline{\mathrm{SL}(n,\mathbb{R})}$, the coverings of $\mathrm{GL}(n,\mathbb{R})$ and $\mathrm{SL}(n,\mathbb{R})$ are nonlinear groups). Would not it be better to reinterpret tetrads as additional gauge or Higgs fields? Or something else, but not reference frames. Let us remind, there is an idea by Hehl, Ne'emann and others [1–3] to interpret the total Poincaré group, not only the homogeneous Lorentz group, as the gauge group of relativity. But then translations are either "external" gauge transformations acting on arguments, not on values (like in "true" gauge theories), or they are internal transformations acting in the affine instead of linear tangent spaces which is also a rather difficult and exotic idea.
5. Dirac Lagrangian for the bispinor field is a rather surprising linear function of quantities

$$\mathcal{J}^{r}{}_{s\mu} := \left(\Psi^{r}\nabla_{\mu}\widetilde{\Psi}_{s} - \left(\nabla_{\mu}\Psi^{r}\right)\widetilde{\Psi}_{s}\right)\sqrt{|g|}. \tag{218}$$

This expression has the typical structure of the bosonic current. The natural question appears: does it mean anything deeper? Is this the true current following from some invariance principle via the Noether theorem? What would be the hypothetical primeval Lagrangian underlying such a system of currents? The tensor structure of internal variables of this current indicates directly on the pseudounitary group $\mathrm{U}(\mathcal{D},G) \simeq \mathrm{U}(2,2)$, the internal conformal geometry or a gauge scheme of gravitation and perhaps the second-order in derivatives fundamental $\mathrm{U}(\mathcal{D},G) \simeq \mathrm{U}(2,2)$-invariant Lagrangian. The point is only then how to reconciliate the second-order differential equation for Ψ with the physically well-established aspects of Dirac theory?

Before going any further let us quote some heuristics concerning the last point. This will be a maximally-simplified heuristics concerning the interplay between second-order and first-order derivatives in wave equations. Let us forget for a moment about four-component spinors, two-component spinors, etc. The object we concentrate on for a while is the one-component specially-relativistic complex Klein-Gordon field $\Psi : M \to \mathbb{C}$.

The respective Lagrangian has the standard form

$$L_{\rm m}(\Psi) = \frac{1}{2} g^{\mu\nu} \partial_\mu \overline{\Psi} \partial_\nu \Psi \sqrt{|g|} - \frac{c}{2} \overline{\Psi} \Psi \sqrt{|g|} \tag{219}$$

with the obvious meaning of symbols and where c is constant (nothing to do with "velocity of light"). This Lagrangian is globally invariant under $\mathrm{U}(1)$ acting in the standard way

$$\Psi \mapsto \exp(\mathrm{i}\alpha)\Psi, \qquad \alpha \in \mathbb{R}. \tag{220}$$

Localization of this phase invariance, i.e., passing over to the x-dependent phase α results in a standard way into replacing

$$\partial_\mu \mapsto \nabla_\mu = \partial_\mu - \mathrm{i} q e_\mu \tag{221}$$

q denoting the coupling constant ("charge") and e_μ is the "gauge" field. Then the locally $\mathrm{U}(1)$-gauge-invariant Lagrangian for the field Ψ becomes

$$L_{\rm m}(\Psi, e) = \frac{1}{2} g^{\mu\nu} \overline{\nabla_\mu \Psi} \nabla_\nu \Psi \sqrt{|g|} - \frac{c}{2} \overline{\Psi} \Psi \sqrt{|g|}. \tag{222}$$

The usual Maxwell dynamics for e (it is difficult to be inventive here) is

$$L_{\rm g}(e) = -\frac{1}{4} f_{\mu\nu} f_{\kappa\lambda} g^{\mu\kappa} g^{\nu\lambda} \sqrt{|g|}, \qquad f_{\mu\nu} = \partial_\mu e_\nu - \partial_\nu e_\mu. \tag{223}$$

Therefore, the total dynamics is to be given by

$$L\left(\Psi, e\right) = L_{\rm m}(\Psi, e) + L_{\rm g}(e). \tag{224}$$

Violating the nice gauge aesthetics, but reducing everything to the brutal facts of ordering the differential operators, we can write

$$L_{\rm m}(\Psi, e) = L^\prime_{\rm m}(\Psi, e) + \frac{1}{2} g^{\mu\nu} \overline{\partial_\mu \Psi} \partial_\nu \Psi \sqrt{|g|} \tag{225}$$

where

$$\begin{aligned} L^\prime_{\rm m}(\Psi, e) &= q g^{\mu\nu} e_\mu \frac{\mathrm{i}}{2} \left(\overline{\Psi} \partial_\nu \Psi - \left(\partial_\nu \overline{\Psi} \right) \Psi \right) \sqrt{|g|} \\ &\quad - \left(\frac{c}{2} - \frac{q^2}{2} g^{\mu\nu} e_\mu e_\nu \right) \overline{\Psi} \Psi \sqrt{|g|}. \end{aligned} \tag{226}$$

In this language the resulting field equations may be written as

$$q \mathrm{i} e^\mu \partial_\mu \Psi - \left(\frac{c}{2} - \frac{q^2}{2} e^\mu e_\mu - \frac{\mathrm{i} q}{2} e^\mu{}_{;\mu} \right) \Psi - \frac{1}{2} g^{\mu\nu} \partial_\mu \partial_\nu \Psi = 0 \tag{227}$$

$$\partial_\nu f^{\mu\nu} = q^2 e^\mu \overline{\Psi} \Psi + \frac{\mathrm{i} q}{2} g^{\mu\nu} \left(\overline{\Psi} \partial_\nu \Psi - \overline{\partial_\nu \Psi} \Psi \right). \tag{228}$$

So, we have the coupled system of "Dirac" equation for Ψ, "parasitically" disturbed by the second-order derivative term, and "Maxwell equation" for the "potential" e_μ with the strange "current" composed of the term algebraic in Ψ (the "Dirac" term) and the term involving derivatives (the "Klein-Gordon" term). There is a natural question: do exist situations ("slowly-varying" fields), when the last terms on the right-hand sides of (227), (228) are negligible? If so, there is a range of applications for the truncated "Dirac" Lagrangian and truncated "Dirac" dynamics

$$\begin{aligned} L'_{\mathrm{m}}(\Psi, e) = \frac{\mathrm{i}}{2} q e^\mu \left(\overline{\Psi} \partial_\mu \Psi - \left(\partial_\mu \overline{\Psi} \right) \Psi \right) \sqrt{|g|} \\ - \left(\frac{c}{2} - \frac{q^2}{2} e^\mu e_\mu \right) \overline{\Psi} \Psi \sqrt{|g|} \end{aligned} \tag{229}$$

$$\mathrm{i} e^\mu \partial_\mu \Psi = \left(\frac{c}{2q} - \frac{q}{2} e^\mu e_\mu - \frac{\mathrm{i}}{2} e^\mu{}_{;\mu} \right) \Psi \tag{230}$$

$$f^{\mu\nu}{}_{;\nu} = \partial_\nu f^{\mu\nu} = q^2 e^\mu \overline{\Psi} \Psi. \tag{231}$$

This "truncated model" is rather artificial, but it brings about the following *question*:
Is not the "genuine" Dirac theory also a truncated part of some more fundamental Klein-Gordon theory with the gauge group $\mathrm{U}(2,2)$?
Let us proceed.

3. $\mathrm{U}(2,2)$ as an Expected Fundamental Symmetry in Spinor Geometrodynamics

3.1. Some Objections Against Dirac Theory

Generally-relativistic Dirac theory deals with a triple of mutually interacting objects: the bispinor matter wave Ψ and two geometrodynamical quantities, namely, the tetrad field e and the $\mathrm{SL}(2,\mathbb{C})$-ruled bispinor connection ω, which gives rise to the covariant differentiation of bispinors

$$\nabla_\mu \Psi^r = \partial_\mu \Psi^r + \omega^r{}_{s\mu} \Psi^s. \tag{232}$$

The target spaces of e and Ψ, i.e., $\mathbb{R}^4$ and $\mathbb{C}^4$, are endowed with certain geometric structures. Namely, $\mathbb{R}^4$ is Minkowskian space with the scalar product η, whereas in $\mathbb{C}^4$ a neutral-signature hermitian form G is fixed. Analytically

$$[\eta_{AB}] = \mathrm{diag}\,(1,-1,-1,-1)\,, \qquad [G_{\bar{r}s}] = \mathrm{diag}\,(1,1,-1,-1)\,.$$

G gives rise to the Dirac conjugation of bispinors

$$\widetilde{\Psi}_r := \overline{\Psi}^{\bar{s}} G_{\bar{s}r}. \tag{233}$$

Within the matrix algebra $\mathrm{L}(4, \mathbb{C})$ one fixes a quadruplet of $G-$hermitian Dirac matrices γ^A satisfying Clifford anticommutation rules

$$\gamma^A \gamma^B + \gamma^B \gamma^A = 2\eta^{AB} I_4. \tag{234}$$

The tetrad field e and the internal metric η give rise to the metric tensor g on the space-time manifold M

$$g_{\mu\nu} := \eta_{AB} e^A{}_\mu e^B{}_\nu. \tag{235}$$

Similarly, the pair (e, ω) induces the Einstein-Cartan affine connection

$$\Gamma^\lambda{}_{\mu\nu} := e^\lambda{}_A \Gamma^A{}_{B\nu} e^B{}_\mu + e^\lambda{}_A e^A{}_{\mu,\nu}, \qquad \Gamma^A{}_{B\mu} := \frac{1}{2} \operatorname{Tr}\left(\gamma^A \omega_\mu \gamma_B\right). \tag{236}$$

where the shift of capital indices is meant in the η-sense.

Matter Lagrangian is given by

$$L_{\mathrm{m}}(\Psi; e, \omega) = \frac{\mathrm{i}}{2} e^\mu{}_A \gamma^{Ar}{}_s \left(\widetilde{\Psi}_r \nabla_\mu \Psi^s - \nabla_\mu \widetilde{\Psi}_r \Psi^s\right) \sqrt{|g|} - m \widetilde{\Psi}_r \Psi^r \sqrt{|g|}. \tag{237}$$

A few choices of geometrodynamical Lagrangians are logically consistent and compatible with experimental data. The simplest of them, used in Einstein-Cartan theory, is proportional to the curvature scalar $R(\Gamma, g)$ built of Γ and g. There are also more sophisticated models, admitting the Yang-Mills terms quadratic in curvature, and algebraic terms quadratic in torsion [2–5, 9].

This scheme is a kind of gauge theory in which $\mathrm{SL}(2, \mathbb{C})$ is its structural group, the cotetrad field $\left[e^A{}_\mu\right]$, or rather its $2:1$ spinorial covering object, is a reference frame (cross-section of the corresponding principal bundle), $[\omega^r{}_{s\mu}]$ is a connection form on the principal bundle, and the matter field $[\Psi^r]$ represents a cross-section of an associate bundle with the standard fibre $\mathbb{C}^4$. Although this theory works perfectly in usual applications, some principal objections may be raised against it. Let us quote them.

1. The tetrad field e enters the Lagrangian L_{m} through the differential one-form
$$[e^r{}_{s\mu}] = \left[\gamma_A e^A{}_\mu\right] \tag{238}$$
with values in the $\mathbb{R}$-linear span of Dirac matrices $V := \bigoplus_{A=0}^{3} \mathbb{R}\gamma^A$. This linear subspace of the space of all G-hermitian matrices is fixed once for all and used as the value-space of (238) at all space-time points. This is a global, rigidly fixed structure that drastically violates the local paradigm of gauge theories. In a sense, it is an action-at-distance concept. It would be much more compatible with the local philosophy of gauge theories if we admitted the linear mappings $[e^r{}_{s\mu}]_x$ to be general injections of $T_x M$ into the space of G-hermitian operators in $\mathbb{C}^4$.

2. In the genuine gauge theories of elementary particle physics, the reference frame never occurs explicitly as a dynamical quantity. Field equations are imposed on associate bundle objects (matter) and connections in principal bundles (interaction). Unlike this, in spinor theory, the tetrad field is an important dynamical variable from the gravitational sector. One cannot avoid $\left[e^{A}{}_{\mu}\right]$ when constructing Lagrangians. But if so, there is a temptation to modify the theory in such a way as to turn the cotetrad into a gauge field of some kind.
3. The internal metric G is explicitly used in the construction of Lagrangian. This suggests that it is rather the total pseudounitary group $\mathrm{U}(4, G) \simeq \mathrm{U}(2,2)$ than its injected subgroup $\mathrm{SL}(2, \mathbb{C})$ that should be used as a proper group of physical symmetries. $\mathrm{SU}(2,2)$ is in fact used in twistor geometry and conformal field theory [6, 8], because it is the covering group of the conformal group $\mathrm{CO}(1,3)$. However, without serious and complicated modifications, this approach is applicable only to massless particles in Minkowskian space-time. Moreover, although in this treatment field equations are invariant under $\mathrm{SU}(2,2)$ combined with the conformal action on the wave function argument, the Lagrangian itself is not invariant. Thus, the resulting symmetries are non-Noetherian, and do not lead to conservation laws.
4. An intriguing structural feature of the Lagrangian (237) is that it is built of quantities
$$\mathcal{J}^{r}{}_{s\mu} := (\nabla_{\mu}\widetilde{\Psi}_{s}\Psi^{r} - \widetilde{\Psi}_{s}\nabla_{\mu}\Psi^{r})\sqrt{|g|} \tag{239}$$
with a characteristic structure of bosonic Noether currents. What does it mean? What is the hypothetical primeval Lagrangian leading to these currents? What group is to be used? The very algebraic structure of $\mathcal{J}^{r}{}_{s\mu}$ suggests the group $\mathrm{U}(4, G)$ and the Klein-Gordon Lagrangian for Ψ with G as an internal metric.

3.2. Second-Order Derivatives Model with the Internal $\mathrm{U}(2,2)$-Symmetry

The four-dimensional space-time manifold M of our model is not endowed with any absolute geometry apart, of course, the very differential structure. There are two basic target spaces, namely, the complex linear algebra $\mathrm{L}(4, \mathbb{C})$ and its natural domain $\mathbb{C}^4$. The algebra $\mathrm{L}(4, \mathbb{C})$ appears as the faithful irreducible realisation of the complexified Clifford algebra for the standard Minkowskian space $(\mathbb{R}^4, \eta)$, where $[\eta_{AB}] = \mathrm{diag}(1, -1, -1, -1)$. Therefore, the amplitude space $\mathbb{C}^4$ will be endowed with the neutral signature $(+,+,-,-)$ pseudo-unitary geometry. The corresponding hermitian form will be denoted by G, analytically, $G_{\bar{r}s}$. This form appears in physics in the mass term of Dirac equation. It is also an intertwining operator interrelating two mutually hermitian-conjugate representations of γ-matrices. The G-shift of indices enables one to construct the Dirac conjugation $\widetilde{\Psi}_r := \overline{\Psi}^{\bar{s}} G_{\bar{s}r}$, it is an antilinear isomorphism of $\mathbb{C}^4$ onto its dual $\mathbb{C}^{4*} \simeq \mathbb{C}^4$. The scalar product G gives rise to the pseudo-unitary group $\mathrm{U}(4,G) \subset \mathrm{GL}(4,\mathbb{C})$, if we put $[G_{\bar{r}s}] = \mathrm{diag}(1,1,-1,-1)$, then, of course, $\mathrm{U}(4,G) = \mathrm{U}(2,2)$. The corresponding Lie algebra $\mathfrak{u}(4,G) \subset \mathrm{L}(4,\mathbb{C})$, isomorphic with $\mathfrak{u}(2,2)$, consists of matrices A which are G-antihermitian, i.e., satisfy $G(Au,v) = -G(u,Av)$ for any $u,v \in \mathbb{C}^4$ (where, of course, $G(u,v) = G_{\bar{r}s}\overline{u}^{\bar{r}} v^s$). The imaginary unit multiple $\mathrm{i}\mathfrak{u}(4,G)$ of $\mathfrak{u}(4,G)$ consists of G-hermitian matrices; in particular, Dirac matrices belong to this class.

Our model involves three kinds of independent dynamical variables:

i) the matter field, i.e., wave amplitude $\Psi : M \to \mathbb{C}^4$
ii) the normal-hyperbolic metric tensor g
iii) the $\mathrm{U}(4,G)$-ruled connection on M, locally represented as a $\mathfrak{u}(4,G)$-valued differential one-form

$$M \ni x \to A_x \in \mathrm{L}\left(T_x M, \mathfrak{u}(4,G)\right).$$

The corresponding analytical symbols are Ψ^r, $g_{\mu\nu}$, $A^r{}_{s\mu}$, where $r = 1,2,3,4$, $\mu,\nu = 0,1,2,3$.

Geometrodynamical sector is described by two field quantities (g,A). There is no dynamical use of tetrad, affine connection, or $\mathrm{SL}(2,\mathbb{C})$-ruled spinor connection. Instead, all these quantities will appear as byproducts of A, after the $\mathrm{SL}(2,\mathbb{C})$-reduction.

Local transformations $U : M \to \mathrm{U}(4,G)$ act on our field quantities according to the standard rule

$$\begin{aligned} \left(U\Psi\right)(x) &= U(x)\Psi(x), \qquad Ug = g \\ \left(UA\right)_x &= U(x)A_x U(x)^{-1} - \mathrm{d}U_x U(x)^{-1}. \end{aligned} \tag{240}$$

Covariant differentiation of wave amplitudes is defined as

$$\begin{aligned} \nabla_\mu \Psi &= \partial_\mu \Psi + g\left(A_\mu - \frac{1}{4}\operatorname{Tr} A_\mu I\right)\Psi + \frac{q}{4}\operatorname{Tr} A_\mu \Psi \\ &= \partial_\mu \Psi + gA_\mu \Psi + \frac{q-g}{4}\operatorname{Tr} A_\mu \Psi \end{aligned} \tag{241}$$

where the coupling constants g and q correspond, respectively, to the subgroups $\mathrm{SU}(4,G)$, $\mathrm{e}^{\mathrm{i}\mathbb{R}}I$. The curvature form F depends only on the "semisimple" coupling constant g,

$$F_{\mu\nu} = DA_{\mu\nu} = \mathrm{d}A_{\mu\nu} + g\left[A_\mu, A_\nu\right] = \partial_\mu A_\nu - \partial_\nu A_\mu + g\left[A_\mu, A_\nu\right]. \tag{242}$$

Matter Lagrangian will be assumed in the Klein-Gordon form

$$\begin{aligned} L_{\mathrm{m}}(\Psi; A, g) &= \frac{b}{2} g^{\mu\nu} \nabla_\mu \widetilde{\Psi} \nabla_\nu \Psi \sqrt{|g|} - \frac{c}{2} \widetilde{\Psi} \Psi \sqrt{|g|} \\ &= \frac{b}{2} g^{\mu\nu} \nabla_\mu \overline{\Psi}^{\bar{r}} \nabla_\nu \Psi^s G_{\bar{r}s} \sqrt{|g|} - \frac{c}{2} G_{\bar{r}s} \overline{\Psi}^{\bar{r}} \Psi^s \sqrt{|g|} \end{aligned} \tag{243}$$

b, c denoting constants. This is the only reasonable model locally invariant under $\mathrm{U}(4, G)$. Dirac-like models based on first-order differential equations are incompatible with our choice of degrees of freedom, because we have no tetrad or any other vector-valued differential one-form transforming under (240) according to a homogeneous-linear rule.

The gauge-invariant Noether current corresponding to the $\mathrm{U}(4, G) \simeq \mathrm{U}(2, 2)$-symmetry is given by

$$\mathcal{J}(\Psi; A, g)^r{}_{s\mu} := \frac{b}{2} \left(\Psi^r \nabla_\mu \widetilde{\Psi}_s - \nabla_\mu \Psi^r \widetilde{\Psi}_s \right) \sqrt{|g|}. \tag{244}$$

Just as in electrodynamics, it is algebraically equivalent to derivatives of L_m with respect to the gauge potential

$$\frac{\partial L_{\mathrm{m}}(\Psi; A, g)}{\partial A^r{}_{s\mu}} = g \mathcal{J}^s{}_r{}^\mu + \frac{q-g}{4} \mathcal{J}^z{}_z{}^\mu \delta^s{}_r. \tag{245}$$

The only reasonable dynamical model for A is that based on the Yang-Mills Lagrangian

$$L_{\mathrm{YM}}(A, g) = \frac{a}{4} \mathrm{Tr}\left(F_{\mu\nu} F_{\kappa\lambda}\right) g^{\mu\kappa} g^{\nu\lambda} \sqrt{|g|} + \frac{a'}{4} \mathrm{Tr}\left(F_{\mu\nu}\right) \mathrm{Tr}\left(F_{\kappa\lambda}\right) g^{\mu\kappa} g^{\nu\lambda} \sqrt{|g|} \tag{246}$$

where a, a' are constants depending on the choice of units; they refer, respectively, to the subgroups $\mathrm{SU}(4, G)$ and $\mathrm{e}^{\mathrm{i}\mathbb{R}} I$ of $\mathrm{U}(4, G)$.

There is less aprioric evidence as to the choice of the dynamical term for g. Let us quote three natural possibilities:

i) Palatini-like model. In this scheme there is no separate Lagrangian for g. The total Lagrangian reduces to $L_{\mathrm{m}}(\Psi; A, g) + L_{\mathrm{YM}}(A, g)$, and the metric tensor enters it in a purely algebraic way. Nevertheless, just as in the usual Palatini model, g is a dynamical variable, subject to the variational procedure in $L_{\mathrm{m}} + L_{\mathrm{YM}}$. The usual gravitational constant of Einstein theory will be proportional to the inverse of a.

ii) Hilbert-Einstein model

$$L_{\mathrm{HE}}(g) = -dR(g)\sqrt{|g|} + l\sqrt{|g|} \tag{247}$$

in which d, l are constants, and $R(g)$ denotes the scalar curvature of g. The correspondence of our model with the standard gravitation theory enables one to identify some linear combination of d, a with the inverse of the gravitational constant. Formally, the parameter l has the cosmological constant status, there are, however, no aprioric restrictions on its sign. Obviously, putting $d = 0$, $l = 0$, we obtain Palatini-like model.

iii) g might be a byproduct of something else, like, e.g., some vector-valued differential one-form E on M, transforming under (240) according to a homogeneous rule (generalized cotetrad). It is reasonable to assume Lagrangian quadratic in the A-covariant differential of E.

The analogy with the usual Palatini principle enables one to suppose that the model i) will be more suitable and reasonable than ii) with non-vanishing d, l. However, at this stage, we refrain from any choice and assume ii) with the possibility of putting $d = 0$, $l = 0$.

The gauge field momentum ${H^r}_s{}^{\mu\nu}$ is defined as usually

$$
{H^r}_s{}^{\mu\nu} := \frac{\partial L_{\mathrm{YM}}}{\partial {A^s}_{r\mu,\nu}} \tag{248}
$$

and thus

$$
H^{\mu\nu} = -aF^{\mu\nu}\sqrt{|g|} - a' I_4 \operatorname{Tr} F^{\mu\nu}\sqrt{|g|} \tag{249}
$$

where the shift of the spatio-temporal indices is meant in the $g-$sense.

The metrical energy-momentum tensor of matter and gauge fields is given by

$$
T^{\mu\nu} = {T_{\mathrm{m}}}^{\mu\nu} + {T_{\mathrm{YM}}}^{\mu\nu} = -\frac{2}{\sqrt{|g|}}\left(\frac{\partial(L_{\mathrm{m}} + L_{\mathrm{YM}})}{\partial g_{\mu\nu}} - \left(\frac{\partial(L_{\mathrm{m}} + L_{\mathrm{YM}})}{\partial g_{\mu\nu,\alpha}}\right)_{,\alpha}\right)
$$

after short calculations, one obtains

$$
T_{\mathrm{m}\mu\nu} = b\nabla_{(\mu}\widetilde{\Psi}\nabla_{\nu)}\Psi - \frac{b}{2}\nabla_\alpha\widetilde{\Psi}\nabla_\beta\Psi g^{\alpha\beta}g_{\mu\nu} + \frac{c}{2}\widetilde{\Psi}\Psi g_{\mu\nu} \tag{250}
$$

$$
\begin{aligned}
T_{\mathrm{YM}\mu\nu} &= a\operatorname{Tr}\left(F_{\mu\kappa}{F_\nu}^\kappa\right) - \frac{a}{4}\operatorname{Tr}\left(F_{\alpha\beta}F^{\alpha\beta}\right) g_{\mu\nu} \\
&\quad + a'\operatorname{Tr}\left(F_{\mu\kappa}\right)\operatorname{Tr}\left({F_\nu}^\kappa\right) - \frac{a'}{4}\operatorname{Tr}\left(F_{\alpha\beta}\right)\operatorname{Tr}\left(F^{\alpha\beta}\right) g_{\mu\nu}.
\end{aligned} \tag{251}
$$

Obviously, the total Lagrangian underlying our variational principle is given by

$$
L = L_{\mathrm{m}} + L_{\mathrm{YM}} + L_{\mathrm{HE}}
$$

and then the resulting Euler-Lagrange equations may be written in the following form

$$g^{\mu\nu}\nabla_{[g]\mu}\nabla_{[g]\nu} + \frac{c}{b}\Psi = 0 \tag{252}$$

$$\nabla_{[g]\nu}H^{\mu\nu} = g\mathcal{J}^{\mu} + \frac{q-g}{4}\operatorname{Tr}\mathcal{J}^{\mu}I_4 \tag{253}$$

$$d\left(R(g)^{\mu\nu} - \frac{1}{2}R(g)g^{\mu\nu}\right) = -\frac{l}{2}g^{\mu\nu} + \frac{1}{2}T^{\mu\nu} \tag{254}$$

where $\nabla_{[g]\mu}$ denotes the total covariant differentiation corresponding to the simultaneous use of the Yang-Mills connection $A^{r}{}_{s\mu}$ (internal indices) and the Levi-Civita connection $\left\{\begin{array}{c}\alpha\\ \beta\mu\end{array}\right\}$ (spatio-temporal indices), $R(g)^{\mu\nu}$ denotes the Ricci tensor of g.

Let us observe that the left-hand side of (253) may be rewritten as

$$\nabla_{[g]\nu}H^{\mu\nu} = \partial_{\nu}H^{\mu\nu} + g\left[A_{\nu}, H^{\mu\nu}\right] = H^{\mu\nu}{}_{;\nu} + g\left[A_{\nu}, H^{\mu\nu}\right] \tag{255}$$

where the semicolon denotes the Levi-Civita covariant differentiation. Although $H^{\mu\nu}{}_{;\nu} = \partial_{\nu}H^{\mu\nu}$ (because H is a skew-symmetric contravariant tensor density of weight one), it will be convenient in our later calculations to use the form with $H^{\mu\nu}{}_{;\nu}$.

If we use the Palatini-like pattern, $d = 0$, $l = 0$, then (254) becomes $T^{\mu\nu} = 0$, just as in Poincaré-gauge models of gravitation. If $d = 0$, but the "cosmological" term is admitted, them $T^{\mu\nu} = lg^{\mu\nu}$.

Apparently, the above $\mathrm{U}(2,2)$-invariant Klein-Gordon-Yang-Mills system has nothing to do with the physically well-established Dirac equation and Einstein-Cartan geometrodynamics. The wave equation (252) is a second-order differential equation, and it is difficult to expect any reasonable correspondence with the first-order Dirac equation and its Clifford background. However, a more detailed analysis reveals not only the correspondence but also certain promising features of the model.

3.3. Expressing Everything in Terms of the Internal Symmetry $\mathrm{SL}(2,\mathbb{C})$

The correspondence with generally-covariant Dirac theory and with an Einstein-Cartan type geometrodynamics becomes readable when one expands all internal quantities with respect to a basis adapted to an appropriate monomorphism of $\mathrm{SL}(2,\mathbb{C})$ into $\mathrm{U}(4,G) \simeq \mathrm{U}(2,2)$. This monomorphism corresponds to the standard injection of the proper Lorentz group $\mathrm{SO}^{\uparrow}(1,3)$ into the conformal group $\mathrm{CO}(1,3)$. Let γ^{A}, $A = 0,1,2,3$, be any quadruplet of Dirac matrices adapted to the hermitian form G, thus, $\mathrm{i}\gamma^{A} \in \mathfrak{u}(4,G)$, and

$$\gamma^{A}\gamma^{B} + \gamma^{B}\gamma^{A} = 2\eta^{AB}I_4. \tag{256}$$

It is well-known that complexified Clifford algebra $\mathrm{L}(4,\mathbb{C})$ is generated by Dirac matrices. Besides of γ^A's themselves the most natural γ-adapted basis contains the following standard matrices

$$\gamma^5 = -\gamma_5 = -\gamma^0\gamma^1\gamma^2\gamma^3, \qquad {}^A\gamma = \mathrm{i}\gamma^A\gamma^5 = -\mathrm{i}\gamma^5\gamma^A \tag{257}$$

$$\Sigma^{AB} = \frac{1}{4}\left(\gamma^A\gamma^B - \gamma^B\gamma^A\right) = -\Sigma^{BA}. \tag{258}$$

The quadruplet of ${}^A\gamma$'s obeys the Clifford rules with the reversed signature, i.e., $(-,+,+,+)$

$${}^A\gamma{}^B\gamma + {}^B\gamma{}^A\gamma = -2\eta^{AB}I_4. \tag{259}$$

The Lie algebra $\mathfrak{u}(4,G)$ is an $\mathbb{R}$-linear shell of matrices

$$\mathrm{i}\gamma^A, \quad \mathrm{i}{}^A\gamma, \quad \Sigma^{AB}, \quad \mathrm{i}\gamma^5, \quad \mathrm{i}I_4. \tag{260}$$

Geometrically and physically relevant subalgebras of $\mathfrak{u}(4,G)$ are $\mathbb{R}$-linear shells of the following subsystems:

- $\mathfrak{su}(4,G)$: $\mathrm{i}\gamma^A$, $\mathrm{i}{}^A\gamma$, Σ^{AB}, $\mathrm{i}\gamma^5$.
- Lorentz algebra, i.e., injected $\mathfrak{sl}(2,\mathbb{C})$: Σ^{AB}.
- Weyl algebra, i.e., injected $\mathfrak{sl}(2,\mathbb{C}) \oplus \mathbb{R}I_2$: Σ^{AB}, $\mathrm{i}\gamma^5$.
- injected $\mathfrak{gl}(2,\mathbb{C})$: Σ^{AB}, $\mathrm{i}\gamma^5$, $\mathrm{i}I$ (Weyl algebra and $\mathrm{U}(1)$-gauges).

It is convenient to use the following mixtures of two kinds of γ's

$$\tau_A := \frac{1}{2}\left(\gamma_A + {}_A\gamma\right) = \frac{1}{2}\eta_{AB}\left(\gamma^B + {}^B\gamma\right) \tag{261}$$

$$\chi^A := \frac{1}{2}\left(\gamma^A - {}^A\gamma\right). \tag{262}$$

They generate Abelian Lie algebras

$$[\tau_A, \tau_B] = 0, \qquad \left[\chi^A, \chi^B\right] = 0.$$

Within the twistor formalism, the group generated by τ_A's is identified with spatio-temporal translations and that generated by χ^A's with proper conformal transformations of the four-dimensional Minkowskian space. Σ^{AB}'s generate Lorentz transformations and $\mathrm{i}\gamma^5$ – dilatations. In our, generally-covariant, approach, the group $\mathrm{U}(4,G) \simeq \mathrm{U}(2,2)$, describes purely internal symmetries without any direct relationship to subgroups of $\mathrm{Diff}M$.

As γ-matrices do not generate Lie subgroups, it is convenient to replace the system (260) by the following one, better suited to the group structure of $\mathrm{U}(4,G)$.

$$\mathrm{i}\tau_A, \qquad \mathrm{i}\chi^A, \qquad \Sigma^{AB}, \qquad \mathrm{i}\gamma^5, \qquad \mathrm{i}I_4. \tag{263}$$

The Yang-Mills field A will be expanded as follows

$$A_\mu = \frac{1}{2}\check{\Omega}^{AB}{}_\mu\Sigma_{AB} + B_\mu\frac{1}{\mathrm{i}}\gamma_5 + A'_\mu\mathrm{i}I + e^A{}_\mu\mathrm{i}\tau_A + f_{A\mu}\mathrm{i}\chi^A \tag{264}$$

where $\check{\Omega}^{AB}{}_{\mu} = -\check{\Omega}^{BA}{}_{\mu}$, and all indicated coefficient fields are real. Let us define:

$$\Omega^{A}{}_{B\mu} := \check{\Omega}^{A}{}_{B\mu} + 2B_{\mu}\delta^{A}{}_{B} \tag{265}$$

and

$$\check{\Omega}^{A}{}_{B\mu} = \Omega^{A}{}_{B\mu} - \frac{1}{4}\Omega^{C}{}_{C\mu}\delta^{A}{}_{B}, \qquad B_{\mu} = \frac{1}{8}\Omega^{A}{}_{A\mu}. \tag{266}$$

Then, (264) may be rewritten as

$$A_{\mu} = \frac{1}{2}\Omega^{AB}{}_{\mu}\left(\Sigma_{AB} + \frac{1}{4}\eta_{AB}\frac{1}{\mathrm{i}}\gamma^{5}\right) + e^{A}{}_{\mu}\mathrm{i}\tau_{A} + f_{A\mu}\mathrm{i}\chi^{A} + A'_{\mu}\mathrm{i}I. \tag{267}$$

The correspondence analysis will be based on rescaled quantities:

$$\Gamma^{A}{}_{B\mu} := g\Omega^{A}{}_{B\mu}, \qquad \check{\Gamma}^{A}{}_{B\mu} = \Gamma^{A}{}_{B\mu} - \frac{1}{4}\Gamma^{C}{}_{C\mu}\delta^{A}{}_{B} \tag{268}$$

$$Q_{\mu} := 4gB_{\mu} = \frac{g}{2}\Omega^{A}{}_{A\mu} = \frac{1}{2}\Gamma^{A}{}_{A\mu} \tag{269}$$

$$\varepsilon^{A}{}_{\mu} := ge^{A}{}_{\mu}, \qquad \varphi_{A\mu} := gf_{A\mu}. \tag{270}$$

Expressing the Yang-Mills field in terms of Γ, we obtain

$$A_{\mu} = \frac{1}{2g}\Gamma^{AB}{}_{\mu}\left(\Sigma_{AB} + \frac{1}{4}\eta_{AB}\frac{1}{\mathrm{i}}\gamma_{5}\right) + e^{A}{}_{\mu}\mathrm{i}\tau_{A} + f_{A\mu}\mathrm{i}\chi^{A} + A'_{\mu}\mathrm{i}I \tag{271}$$

$$\nabla_{\mu}\Psi = D_{\mu}\Psi + ge^{A}{}_{\mu}\mathrm{i}\tau_{A}\Psi + gf_{A\mu}\mathrm{i}\chi^{A}\Psi \tag{272}$$

where

$$D_{\mu}\Psi = \partial_{\mu}\Psi + \frac{1}{2}\Gamma^{AB}{}_{\mu}\left(\Sigma_{AB} + \frac{1}{4}\eta_{AB}\frac{1}{\mathrm{i}}\gamma_{5}\right)\Psi + qA'_{\mu}\mathrm{i}\Psi. \tag{273}$$

Let us now describe how the spinorial group $\mathrm{SL}(2,\mathbb{C})$ acts of these objects. Any choice of γ-matrices gives rise to same monomorphism

$$U : \mathrm{SL}(2,\mathbb{C}) \to \mathrm{U}(4,G) \subset \mathrm{GL}(4,\mathbb{C})$$

it is given by the formulae

$$U(A) = \exp\left(\frac{1}{2}t_{AB}\Sigma^{AB}\right) \tag{274}$$

$$A = \exp\left(\frac{1}{8}t_{AB}\left(\tilde{\sigma}^{A}\sigma^{B} - \tilde{\sigma}^{B}\sigma^{A}\right)\right), \qquad \tilde{\sigma}^{A} = \eta^{AB}\sigma_{B} \tag{275}$$

where σ^{A}, $A = 0,1,2,3$, denotes the relativistic quadruplet of Pauli matrices. In the last formula it is put, exceptionally, $\sigma^{A} = \sigma_{A}$, whereas the η-shift of indices is indicated by the tilde symbol. For certain geometric reasons, there is no escaping this inconsistency without introducing an obscuring crowd of additional symbols

[11, 12]. The Lorentz transformation $L(A) \in \mathrm{SO}^{\uparrow}(1,3)$ assigned to A satisfies, obviously, the following conditions

$$U(A)\gamma_K U(A)^{-1} = \gamma_M L(A)^M{}_K \tag{276}$$

$$L(A)^K{}_M = \frac{1}{4}\operatorname{Tr}\left(\gamma^K U(A)\gamma_M U(A)^{-1}\right) = \frac{1}{2}\operatorname{Tr}\left(\sigma^K A\sigma_M A^+\right). \tag{277}$$

Obviously, local transformations $A{:}M \to \mathrm{SL}(2,\mathbb{C})$ act on wave amplitudes through $U(A)$, like in (240)

$$\left(U(A)\Psi\right)(x) = U\left(A(x)\right)\Psi(x). \tag{278}$$

The gauge field components $e^K{}_\mu$, $f_{K\mu}$, $\Gamma^K{}_{L\mu}$, A'_μ suffer in virtue of (278), the following transformations:

$$e^K{}_\mu{}' = L(A)^K{}_M e^M{}_\mu \tag{279}$$

$$f_{K\mu}{}' = f_{M\mu} L(A)^{-1M}{}_K \tag{280}$$

$$\Gamma^K{}_{N\mu}{}' = L(A)^K{}_M \Gamma^M{}_{H\mu} L(A)^{-1H}{}_N - \frac{\partial L(A)^K{}_M}{\partial x^\mu} L(A)^{-1M}{}_N \tag{281}$$

$$(A'_\mu)' = A'_\mu. \tag{282}$$

It is important that if the local $\mathrm{U}(4,G) \simeq \mathrm{U}(2,2)$-symmetry is restricted to the subgroup $\mathrm{U}(\mathrm{SL}(2,\mathbb{C}))$, the transformation rule for e becomes homogeneous and algebraic in $L(A)$. The field f transforms contragradiently to e, thus, roughly speaking. e, f are, respectively, contravariant and covariant vectors in Minkowskian space $(\mathbb{R}^4,\eta)$. The Γ-coefficients transform under $\mathrm{SL}(2,\mathbb{C})$ exactly as non-holonomic coefficients of some $\mathrm{SO}^{\uparrow}(1,3)$-ruled spatio-temporal connection. If we extend $\mathrm{SL}(2,\mathbb{C})$ to $\mathrm{GL}(2,\mathbb{C})$, faithfully generated in bispinor space by Σ^{AB}, $\mathrm{i}\gamma^5$, $\mathrm{i}I_4$, the Lorentz group $\mathrm{SO}^{\uparrow}(1,3)$ is replaced by the Weyl group $\mathrm{e}^{\mathbb{R}}\mathrm{SO}^{\uparrow}(1,3)$, and the above-quoted transformation properties of e, f, Γ, A', remain true. Thus, if we assume in addition that $\det\left[e^A{}_\mu\right] \neq 0$, then, from the point of view of the reduced symmetry $\mathrm{GL}(2,\mathbb{C})$, e becomes the cotetrad, and coefficients $\Gamma^K{}_{M\mu}$ are, related to this cotetrad, non-holonomic coefficients of some Einstein-Cartan-Weyl affine connection [2]. The trace-less part $\check{\Gamma}^K{}_{M\mu}$ is an Einstein-Cartan connection, and $Q_\mu = (1/2)\Gamma^A{}_{A\mu}$ becomes the Weyl covector. The fields f_A form an additional cotetrad; we have used it in our $\mathrm{GL}(2,\mathbb{C})$-invariant approach to spinors, to compensate the effect of dilatations. Within our model the cotetrad field e is interpreted dynamically, as a part of the $\mathrm{U}(4,G)$-ruled Yang-Mills field, not as a reference frame for some $\mathrm{SL}(2,\mathbb{C})$-ruled bispinor connection. Admitting also Q_μ and A'_μ we obtain the $\mathrm{GL}(2,\mathbb{C})$-connection, which compensates also local dilatations and local electromagnetic gauges. The part $D_\mu\Psi$ (273) of the covariant derivative $\nabla_\mu\Psi$ is just the corresponding $\mathrm{GL}(2,\mathbb{C})$-invariant covariant differentiation of bispinors. It becomes the usual $\mathrm{SL}(2,\mathbb{C})$-derivative if we put $Q_\mu = 0$, $A'_\mu = 0$.

If e, f are co-tetrads (in general, they need not be so), i.e., if $\det\left[e^{A}{}_{\mu}\right] \neq 0$, $\det[f_{A\mu}] \neq 0$, then Yang-Mills field A gives rise to certain spatio-temporal objects, assigned to A in a locally $\mathrm{GL}(2,\mathbb{C})$-invariant way. For example, we can define following algebraic objects

$$t\,(e,f)_{\mu\nu} := e^{A}{}_{\mu} f_{A\nu}, \qquad G\,(e,f)_{\mu\nu} = t\,(e,f)_{(\mu\nu)}\,. \tag{283}$$

The latter is a metric-like quantity. We can also introduce $\mathrm{SL}(2,\mathbb{C})$-invariant Dirac-Einstein matrices

$$h\,(e,\eta)_{\mu\nu} = \eta_{AB} e^{A}{}_{\mu} e^{B}{}_{\nu}, \qquad h\,(f,\eta)_{\mu\nu} = \eta^{AB} f_{A\mu} f_{B\nu}. \tag{284}$$

We can also construct the following spatio-temporal affine connections $\Gamma(e)^{\lambda}{}_{\mu\nu}$, $\Gamma(f)^{\lambda}{}_{\mu\nu}$, assigned to A in a locally $\mathrm{GL}(2,\mathbb{C})$-invariant manner

$$\Gamma(e)^{\lambda}{}_{\mu\nu} = e^{\lambda}{}_{A}\Gamma^{A}{}_{B\nu}e^{B}{}_{\mu} + e^{\lambda}{}_{\mu}e^{A}{}_{\mu,\nu} \tag{285}$$

$$\Gamma(f)^{\lambda}{}_{\mu\nu} = -f_{A\mu}\Gamma^{A}{}_{B\nu}f^{\lambda B} + f^{\lambda A}f_{A\mu,\nu}. \tag{286}$$

They are Einstein-Cartan-Weyl connections in the sense of metrics $h\,(e,\eta)$ and $h\,(f,\eta)$, respectively

$$\nabla_{[\Gamma(e)]\lambda} h\,(e,\eta)_{\mu\nu} = -Q_{\lambda} h\,(e,\eta)_{\mu\nu}\,, \qquad \nabla_{[\Gamma(f)]\lambda} h\,(f,\eta)_{\mu\nu} = -Q_{\lambda} h\,(f,\eta)_{\mu\nu}\,.$$

It must be stressed, however, that, in view of the independent dynamical status of $g_{\mu\nu}$, they need not be metrical with respect to $g_{\mu\nu}$, thus, in general, the tensors

$$K(e)^{\lambda}{}_{\mu\nu} := \Gamma(e)^{\lambda}{}_{\mu\nu} - \left\{ \begin{array}{c} \lambda \\ \mu\nu \end{array} \right\}, \qquad K(f)^{\lambda}{}_{\mu\nu} := \Gamma(f)^{\lambda}{}_{\mu\nu} - \left\{ \begin{array}{c} \lambda \\ \mu\nu \end{array} \right\} \tag{287}$$

will not have any special algebraic properties.

Torsion and curvature tensors of $\Gamma(e)$, $\Gamma(f)$ will be denoted by $S(e)$, $S(f)$, $R(e)$, $R(f)$, thus

$$S(e)^{\lambda}{}_{\mu\nu} = \Gamma(e)^{\lambda}{}_{[\mu\nu]}, \qquad S(f)^{\lambda}{}_{\mu\nu} = \Gamma(f)^{\lambda}{}_{[\mu\nu]} \tag{288}$$

$$R(e)^{\lambda}{}_{\kappa\mu\nu} = 2\Gamma(e)^{\lambda}{}_{\kappa[\nu,\mu]} + 2\Gamma(e)^{\lambda}{}_{\rho[\mu}\Gamma(e)^{\rho}{}_{|\kappa|\nu]} \tag{289}$$

$$R(f)^{\lambda}{}_{\kappa\mu\nu} = 2\Gamma(f)^{\lambda}{}_{\kappa[\nu,\mu]} + 2\Gamma(f)^{\lambda}{}_{\rho[\mu}\Gamma(f)^{\rho}{}_{|\kappa|\nu]}. \tag{290}$$

Let us now express the curvature form of A in terms of the $\mathrm{SL}(2,\mathbb{C})$-reduction. After some calculations one obtains

$$F = DA = T(e)^{A}\mathrm{i}\tau_{A} + T(f)_{A}\mathrm{i}\chi^{A} + \frac{1}{2}\widetilde{R}^{AB}\Sigma_{AB} + \widetilde{G}\frac{1}{\mathrm{i}}\gamma^{5} + F'\mathrm{i}I \tag{291}$$

where

$$\begin{aligned}
T(e)^A &= \mathrm{d}e^A + g\Omega^A{}_B \wedge e^B = \mathrm{d}e^A + \Gamma^A{}_B \wedge e^B \\
\widetilde{R}^A{}_B &= R(\Omega)^A{}_B - \frac{1}{4}R(\Omega)^C{}_C\delta^A{}_B - 2ge^A \wedge f_B + 2g\eta^{AC}\eta_{BD}e^D \wedge f_C \\
&= \frac{1}{g}R(\Gamma)^A{}_B - \frac{1}{4g}R(\Gamma)^C{}_C\delta^A{}_B - 2ge^A \wedge f_B + 2g\eta^{AC}\eta_{BD}e^D \wedge f_C \\
\widetilde{G} &= \frac{1}{4g}\mathrm{d}Q - ge^A \wedge f_A = \frac{1}{g}\left[\frac{1}{8}R(\Gamma)^A{}_A - g^2e^A \wedge f_A\right] \\
F' &= \mathrm{d}A'
\end{aligned}$$

and $R(\Gamma)$ denotes the curvature form of the $\mathrm{e}^{\mathbb{R}}\mathrm{SO}(1,3)^{\uparrow}$-ruled connection Γ

$$R(\Gamma)^A{}_B = \mathrm{d}\Gamma^A{}_B + \Gamma^A{}_C \wedge \Gamma^C{}_B, \qquad R(\Omega)^A{}_B = \mathrm{d}\Omega^A{}_B + g\Omega^A{}_C \wedge \Omega^C{}_B.$$

At this stage Γ is an abstract $\mathrm{e}^{\mathbb{R}}\mathrm{SO}^{\uparrow}(1,3)$-ruled connection, and $R(\Gamma)$ – its abstract curvature. An affine connection in M and its curvature may be assigned to them only after some choice of tetrad had been done, and it is not yet the case. In general e, f are free to be singular, or even vanishing.

The torsion-like structure of quantities $T(e)$, $T(f)$ is easily recognised. There is nothing surprising in it as for example, we know that the torsion of a linear connection may be reinterpreted as a part of curvature of the corresponding affine connection. Here one is dealing with the conformal connection A. Some of its part may interpreted as translational torsion, "proper conformal" torsion, and Lorentz-rotational curvature.

If e, f are co-tetrads, i.e., $\det\left[e^A{}_\mu\right] \neq 0$, $\det\left[f_{A\mu}\right] \neq 0$, then the following relationships hold

$$S(e)^\lambda{}_{\mu\nu} = -\frac{1}{2}e^\lambda{}_A T(e)^A{}_{\mu\nu}, \qquad S(f)^\lambda{}_{\mu\nu} = -\frac{1}{2}f^{\lambda A}T(f)_{A\mu\nu} \tag{292}$$

$$R(e)^\lambda{}_{\kappa\mu\nu} = e^\lambda{}_A e^B{}_\kappa R^A{}_{B\mu\nu}, \qquad R(f)^\lambda{}_{\kappa\mu\nu} = -f_{A\kappa}f^{\lambda B}R^A{}_{B\mu\nu} \tag{293}$$

where, of course

$$e^A{}_\lambda e^\lambda{}_B = \delta^A{}_B, \qquad f_{A\lambda}f^{\lambda B} = \delta_A{}^B.$$

3.4. Reducing Everything to the Subgroups $\mathrm{SL}(2,\mathbb{C})$ and $\mathrm{GL}(2,\mathbb{C})$

The above reduction concerned only the kinematical concepts. Let us now express the dynamics of our model in terms of $\mathrm{SL}(2,\mathbb{C})$ and $\mathrm{GL}(2,\mathbb{C})$ objects. To some extent, the correspondence with standard theory is readable already on the level of Lagrangians, without manipulations on field equations.

Matter Lagrangian may be expressed in the following form

$$\begin{aligned} L_{\mathrm{m}}\left(\Psi; A, g\right) = {} & bg\frac{\mathrm{i}}{2}g^{\mu\nu}e^{K}{}_{\mu}\left(D_{\nu}\widetilde{\Psi}\tau_{K}\Psi - \widetilde{\Psi}\tau_{K}D_{\nu}\Psi\right)\sqrt{|g|} \\ & + bg\frac{\mathrm{i}}{2}g^{\mu\nu}f_{K\mu}\left(D_{\nu}\widetilde{\Psi}\chi^{K}\Psi - \widetilde{\Psi}\chi^{K}D_{\nu}\Psi\right)\sqrt{|g|} \\ & + bg\widetilde{\Psi}W\Psi\sqrt{|g|} + \frac{b}{2}g^{\mu\nu}D_{\mu}\widetilde{\Psi}D_{\nu}\Psi\sqrt{|g|} \end{aligned} \tag{294}$$

where the matrix W is given by

$$W = \frac{g}{2}g^{\mu\nu}e^{K}{}_{\mu}f_{K\nu}I_{4} - \frac{c}{2bg}I_{4} - \frac{g}{2}\mathrm{i}g^{\mu\nu}e^{K}{}_{\mu}f_{L\nu}\varepsilon_{K}{}^{L}{}_{AB}\Sigma^{AB}. \tag{295}$$

Let us quote another, perhaps more intuitive, expression

$$\begin{aligned} L_{\mathrm{m}}\left(\Psi; A, g\right) = {} & bg\frac{\mathrm{i}}{2}g^{\mu\nu}E^{K}{}_{\mu}\left(D_{\nu}\widetilde{\Psi}\gamma_{K}\Psi - \widetilde{\Psi}\gamma_{K}D_{\nu}\Psi\right)\sqrt{|g|} \\ & + bg\frac{\mathrm{i}}{2}g^{\mu\nu}F^{K}{}_{\mu}\left(D_{\nu}\widetilde{\Psi}{}_{K}\gamma\Psi - \widetilde{\Psi}{}_{K}\gamma D_{\nu}\Psi\right)\sqrt{|g|} \\ & + bg\widetilde{\Psi}W\Psi\sqrt{|g|} + \frac{b}{2}g^{\mu\nu}D_{\nu}\widetilde{\Psi}D_{\mu}\Psi\sqrt{|g|} \end{aligned} \tag{296}$$

where

$$E^{A}{}_{\mu} := \frac{1}{2}\left(e^{A}{}_{\mu} + \eta^{AB}f_{B\mu}\right), \qquad F^{A}{}_{\mu} := \frac{1}{2}\left(e^{A}{}_{\mu} - \eta^{AB}f_{B\mu}\right). \tag{297}$$

The separate terms are invariant under the local $\mathrm{SL}(2,\mathbb{C})$ group. We easily recognise the usual Dirac structure in the first term of (296). The second term corresponds to the Dirac model with the reversed signature. There are also algebraic mass terms; they may bee non-vanishing even if $c = 0$. The only feature of (296) discouraging from the point of view of Dirac theory is the d'Alembert term quadratic is derivatives $D_{\mu}\Psi$ as it leads to second derivatives in field equations. It will be shown, however, that, surprisingly enough, this term is rather harmless.

The Yang-Mills Lagrangian has the following $\mathrm{SL}(2,\mathbb{C})$-structure

$$\begin{aligned} L_{\mathrm{YM}}(A,g) = {} & \frac{a}{8}\widetilde{R}^{A}{}_{B\mu\nu}\widetilde{R}^{B}{}_{A\kappa\lambda}g^{\mu\kappa}g^{\nu\lambda}\sqrt{|g|} - aT(e)^{A}{}_{\mu\nu}T(f)_{A\kappa\lambda}g^{\mu\kappa}g^{\nu\lambda}\sqrt{|g|} \\ & + aG_{\mu\nu}G_{\kappa\lambda}g^{\mu\kappa}g^{\nu\lambda}\sqrt{|g|} - \left(a + 4a'\right)F_{\mu\nu}F_{\kappa\lambda}g^{\mu\kappa}g^{\nu\lambda}\sqrt{|g|}. \end{aligned} \tag{298}$$

The correspondence with Poincaré-gauge theories of gravitation (including Einstein-Cartan scheme) becomes more readable when we assume that e, f are cotetrads and express (298) in terms of purely spatio-temporal quantities. We obtain that

$$
\begin{aligned}
L_{\mathrm{YM}}(A,g) \;=\; & \frac{a}{8g^2} R(e)^{\kappa}{}_{\lambda\mu\nu} R(e)^{\lambda}{}_{\kappa}{}^{\mu\nu}\sqrt{|g|} - \frac{a}{64g^2} R(e)^{\kappa}{}_{\kappa\mu\nu} R(e)^{\lambda}{}_{\lambda}{}^{\mu\nu}\sqrt{|g|} \\
& + 2aR(e)^{\kappa}{}_{\lambda\mu}{}^{\lambda} t_{\kappa}{}^{\mu}\sqrt{|g|} - 4at_{\lambda\kappa} S(e)^{\lambda}{}_{\mu\nu} S(f)^{\kappa\mu\nu}\sqrt{|g|} \\
& + 4ag^2 t_{\mu\nu} t^{\mu\nu}\sqrt{|g|} - 2ag^2 t_{\mu\nu} t^{\nu\mu}\sqrt{|g|} - 2ag^2 t^{\mu}{}_{\mu} t^{\nu}{}_{\nu}\sqrt{|g|} \\
& + 2ag^2 h(e)^{\mu}{}_{\nu} h(f)^{\nu}{}_{\mu}\sqrt{|g|} - 2ag^2 h(e)^{\mu}{}_{\mu} h(f)^{\nu}{}_{\nu}\sqrt{|g|} \\
& - (a+4a')\, F_{\mu\nu} F^{\mu\nu}\sqrt{|g|}
\end{aligned} \tag{299}
$$

where the shift of indices is meant in the g-sense. In particular, distinction must be made between $\widetilde{t}^{\mu\nu} = e^{\mu}{}_{A} f^{\nu A}$ $\left(\widetilde{t}^{\mu\kappa} t_{\kappa\nu} = \delta^{\mu}{}_{\nu}\right)$ and $t^{\mu\nu} = g^{\mu\kappa} g^{\nu\lambda} t_{\kappa\lambda}$.

We easily recognize in (299) the Yang-Mills term quadratic in curvature $R^{\alpha}{}_{\beta\mu\nu}$ and the Einstein-Cartan-Palatini term linear in $R^{\alpha}{}_{\beta\mu\nu}$. Both occur in Poincaré-gauge theories of gravitation [2–5, 9]. The main difference is that instead of terms quadratic in torsion, we have now the term bilinear in translational torsion $S(e)$ and proper-conformal torsion $S(f)$. There is however a correspondence range, where $S(e)$ and $S(f)$ become equal, at least approximately. Namely, as we shall see, our model admits solutions satisfying Einstein-Cartan constrains

$$
f_K = \eta_{KM} e^{M}, \qquad g = h(e,\eta) = G(e,f) = t(e,f). \tag{300}
$$

So, restricting our Lagrangians (294), (299) to Einstein-Cartan constraints (300) (compatible with equations (252), (253), (254) although not implied by them), one obtains

$$
\begin{aligned}
L_{\mathrm{m}}\left(\Psi; A, g\right)\big|_{\mathrm{EC}} = {} & bg\frac{\mathrm{i}}{2} g^{\mu\nu} e^{K}{}_{\mu}\left(D_{\nu}\widetilde{\Psi}\gamma_K\Psi - \widetilde{\Psi}\gamma_K D_{\nu}\Psi\right)\sqrt{|g|} \\
& + \left(2bg^2 - \frac{c}{2}\right)\widetilde{\Psi}\Psi\sqrt{|g|} + \frac{b}{2} g^{\mu\nu} D_{\mu}\widetilde{\Psi} D_{\nu}\Psi\sqrt{|g|}
\end{aligned} \tag{301}
$$

$$
\begin{aligned}
L_{\mathrm{YM}}(A,g)\big|_{\mathrm{EC}} = {} & \frac{a}{8g^2} R^{\kappa}{}_{\lambda\mu\nu} R^{\lambda}{}_{\kappa}{}^{\mu\nu}\sqrt{|g|} + 2aR^{\mu}{}_{\nu\mu}{}^{\nu}\sqrt{|g|} \\
& - 4aS^{\kappa}{}_{\mu\nu} S_{\kappa}{}^{\mu\nu}\sqrt{|g|} - 48ag^2\sqrt{|g|}.
\end{aligned} \tag{302}
$$

$L_{\mathrm{m}}|_{\mathrm{EC}}$ is a superposition of the usual Dirac and Klein-Gordon Lagrangians. The algebraic (mass) term exists even if we put $c = 0$ in the primary Lagrangian (243). Expression for $L_{\mathrm{YM}}|_{\mathrm{EC}}$ predicts some correspondence with metric-affine theories of gravitation. Lagrangians used there are superpositions of terms appearing in (302) (curvature-quadratic, curvature-linear, torsion-quadratic and "cosmological"). The ratios of constant coefficients in (302) follow from the assumed $\mathrm{U}(4, G)$-symmetry. In gauge-affine theories their values are not a priori prescribed and occur as certain control parameters. There are some indication that the first term should be relevant for the microscopic gravitation, whereas the linear term is necessary for the correct macroscopic limit [4,5,9].

It is, obviously, not the same to substitute constraints (300) to Lagrangians or to field equations (problem of Lagrange multipliers). The correct correspondence analysis should, and just will be, carried over on the level of field equations. Nevertheless, some heuristics, and guidance is also provided, in a concise form, by Lagrangians themselves.

To express concisely field equations in terms of the $\mathrm{SL}(2,\mathbb{C})$-ruled geometry, we must introduce auxiliary dynamical quantities. First of all, we express the conformal current of matter in terms of the complete system (263)

$$\begin{aligned} I_\mu = {}^A\theta_\mu \sqrt{|g|}\mathrm{i}\tau_A + \theta_{A\mu}\sqrt{|g|}\mathrm{i}\chi^A + \frac{1}{2}\theta^{AB}{}_\mu\sqrt{|g|}\Sigma_{AB} \\ + \theta_\mu\sqrt{|g|}\frac{1}{\mathrm{i}}\gamma^5 + \theta'_\mu\sqrt{|g|}\mathrm{i}I. \end{aligned} \tag{303}$$

Instead of θ-multiplets we shall also use the following world tensors

$$\theta^\mu{}_\nu := e^\mu{}_A{}^A\theta_\nu, \qquad {}^\mu{}_\nu\theta := f^{\mu A}\theta_{A\nu}, \qquad \theta^\kappa{}_{\mu\nu} := e^\kappa{}_A e^B{}_\mu \theta^A{}_{B\nu}. \tag{304}$$

These θ-quantities are quadratic-sesquilinear forms of the field Ψ and for details see [13], where also the $\mathrm{SL}(2,\mathbb{C})$-expansion of energy-momentum tensors (250), (251) is presented.

To write down in a concise form the wave equation (252), we have introduced the unified covariant differentiation $\nabla_{[g]}$ (more precisely, we should denote it by $\nabla_{[g,A]}$). When acting on mixed geometric quantities with spatio-temporal and internal ($\mathrm{U}(4,G)$-ruled) indices, this operator combines, in the Leibniz-rule sense, the Levi-Civita and the internal A-based covariant differentiations. Restriction of $\mathrm{U}(4,G)$ to its injected subgroups $\mathrm{GL}(2,\mathbb{C})$, $\mathrm{SL}(2,\mathbb{C})$ gives rise to the internal covariant differentiation D defined in (273). Combining it with the Levi-Civita differentiation, we obtain the unified operator $D_{[g,\Gamma]}$. But now, an additional floor of mixed objects appears, namely, quantities with spatio-temporal and capital indices; the latter ones refer to the linear subspace of $L(4,\mathbb{C})$ spanned on γ-matrices. The operator $D_{[g,\Gamma]}$ extends in a natural way onto the realm of such objects. When e is a co-frame, then the pair, e, Γ gives rise to the affine connection $\Gamma(e)$ defined in (285), (286). The mixed tensor objects with indices of the type μ, A may be identified, by means of e, with purely spatio-temporal tensors endowed only with the μ-type indices. Under this identification, the $D_{[g,\Gamma]}$-differentiation of (μ, A)-objects becomes an operator $D_{[g,\Gamma(e)]}$ acting on the spatio-temporal tensor fields. These fields may be also differentiated covariantly in the sense of $\Gamma(e)$ or $\{g\}$. The corresponding operators $D_{[\Gamma(e)]}$, $D_{[\{g\}]}$ differ from $D_{[g,\Gamma(e)]}$ by certain terms involving the tensor field $K(e)$.

Let us summarise: $\mathrm{GL}(2,\mathbb{C})$-invariant covariant differentiation $D_{[g,\Gamma]}$ is defined as a Leibniz-rule extension of the following formulae

$$D_{[g,\Gamma]\mu}X_\nu := \partial_\mu X_\nu - X_\lambda \left\{ \begin{array}{c} \lambda \\ \nu\mu \end{array} \right\} = D_{[\{g\}]\mu}X_\nu \tag{305}$$

$$D_{[g,\Gamma]\mu}\Psi^r := \partial_\mu \Psi^r + \omega^r{}_{s\mu}\Psi^s \tag{306}$$

$$D_{[g,\Gamma]\mu}Y^A := \partial_\mu Y^A + \Gamma^A{}_{B\mu}Y^B \tag{307}$$

where

$$\omega^r{}_{s\mu} := \frac{1}{2}\Gamma^{AB}{}_\mu \left(\Sigma_{AB}{}^r{}_s + \frac{1}{2}\eta_{AB}\frac{1}{\mathrm{i}}\gamma^{5r}{}_s \right) + qA_\mu \mathrm{i}\delta^r{}_s. \tag{308}$$

Thus, e.g.,

$$D_{[g,\Gamma]\mu}Z^r{}_{\nu A} = \partial_\mu Z^r{}_{\nu A} + \omega^r{}_{s\mu}Z^s{}_{\nu A} - \Gamma^B{}_{A\mu}Z^r{}_{\nu B} - \left\{ \begin{array}{c} \lambda \\ \nu\mu \end{array} \right\} Z^r{}_{\lambda A}. \tag{309}$$

When e is a co-frame, and $g = h(e,\eta)$, then (308) becomes the usual relationship between the bispinor connection ω and the non-holonomic representation $\Gamma^A{}_{B\mu}$ of the Einstein-Cartan-Weyl affine connection $\Gamma(e)$.

The above way of introducing $\nabla_{[g]}$, $D_{[g,\Gamma]}$ as operators combining the $\{g\}$-differentiation of world tensors and ∇- or D-differentiation of internal objects, was rather technical. A rigorous treatment should have used the Cartesian product of two principle fibre bundles over M: the soldered bundle FM of linear frames and an abstract, yet non-specified $\mathrm{U}(4,G)$-ruled bundle over M, or its subbundles ruled by $\mathrm{GL}(2,\mathbb{C})$ and $\mathrm{SL}(2,\mathbb{C})$. Within this framework, the corresponding connections and covariant derivatives should be analysed. However, the explicit description of all appearing objects would enormously extend the treatment without any essential profit for our subject. Thus, at this stage, we decide to avoid this superfluous purism.

It is convenient to use the quantities $E^A{}_\mu$, $F^A{}_\mu$ defined in (297), because they are directly coupled to Dirac matrices

$$e^A{}_\mu \tau_A + f_{A\mu}\chi^A = E^A{}_\mu \gamma_A + F^A{}_\mu{}^A\gamma.$$

We shall also use the corresponding contravariant objects

$$E^\mu{}_A := g^{\mu\nu}\eta_{AB}E^B{}_\nu, \qquad F^\mu{}_A := g^{\mu\nu}\eta_{AB}F^B{}_\nu. \tag{310}$$

The W-operator (295) may be expressed as follows

$$\begin{aligned} W = {} & \frac{g}{2}\eta_{AB}E^A{}_\mu E^B{}_\nu g^{\mu\nu}I - \frac{g}{2}\eta_{AB}F^A{}_\mu F^B{}_\nu g^{\mu\nu}I \\ & - \frac{c}{2gb}I + gig^{\mu\nu}E^A{}_\mu F^B{}_\nu \varepsilon_{ABCD}\Sigma^{CD}. \end{aligned} \tag{311}$$

After some calculation, which in view of their large volume and rather purely technical character, do not deserve reporting here, we obtain the $\mathrm{GL}(2,\mathbb{C})$-reduced form of field equations (252), (253), (254).

When expressed in terms of $\mathrm{GL}(2,\mathbb{C})$-ruled objects, the wave equation (252) becomes

$$\begin{aligned} & \mathrm{i}\gamma^A \left(E^\mu{}_A D_\mu \Psi + \frac{1}{2} \left(D_{[g,\Gamma]\mu} E^\mu{}_A \right) \Psi \right) \\ & + \mathrm{i}^A\gamma \left(F^\mu{}_A D_\mu \Psi + \frac{1}{2} \left(D_{[g,\Gamma]\mu} F^\mu{}_A \right) \Psi \right) \\ & - W\Psi + \frac{1}{2g} g^{\mu\nu} D_{[g,\Gamma]\mu} D_{[g,\Gamma]\nu} \Psi = 0. \end{aligned} \tag{312}$$

On the left-hand side we recognise a superposition of two Dirac operators, corresponding to mutually opposite normal-hyperbolic signatures $(+,-,-,-)$ and $(-,+,+,+)$. There is also an algebraic "mass" term $W\Psi$ and the second-order "d'Alembert operator". The divergence-type corrections have to do with Lie derivatives of the pseudo-Riemannian volume element $\sqrt{|g|}$ and are necessary for the self-adjoint character of the wave equation. By abuse of language, we could rewrite (312) in the following, suggestive form

$$\mathrm{i}\gamma^A \mathcal{L}_{E_A}\Psi + \mathrm{i}^A\gamma \mathcal{L}_{F_A}\Psi - W\Psi + \frac{1}{2g} g^{\mu\nu} D_{[g,\Gamma]\mu} D_{[g,\Gamma]\nu}\Psi = 0 \tag{313}$$

where the "covariant Lie derivative" operators act on Ψ as on complex density of weight $1/2$, and contain compensating terms responsible for the local $\mathrm{GL}(2,\mathbb{C})$-invariance. There is also a second-order covariant d'Alembert operator, locally invariant under $\mathrm{GL}(2,\mathbb{C})$. It is seen that some Dirac-like structure emerges from these equations, but there are two terms perturbing it – an additional Dirac operator and the second-order terms. As we shall see, they are not so embarrassing as they could seem.

Expressing the Yang-Mills equations (253) in terms of the $\mathrm{GL}(2,\mathbb{C})$-splitting (291) we obtain

$$D_{[g,\Gamma]\beta} T(e)^{A\alpha\beta} + g e^B{}_\beta \widetilde{R}_B{}^{A\alpha\beta} - 2g e^A{}_\beta \widetilde{G}_B{}^{\alpha\beta} = -\frac{g}{a}{}^A\theta^\alpha \tag{314}$$

$$D_{[g,\Gamma]\beta} T(f)_A{}^{\alpha\beta} + g f_{B\beta} \widetilde{R}^B{}_A{}^{\alpha\beta} + 2g f_{A\beta} \widetilde{G}^{\alpha\beta} = -\frac{g}{a}\theta_A{}^\alpha \tag{315}$$

$$\begin{aligned} D_{[g,\Gamma]\beta} \widetilde{R}^A{}_B{}^{\alpha\beta} + 2g e_{B\beta} T(f)^{A\alpha\beta} - 2g e^A{}_\beta T(f)_B{}^{\alpha\beta} & \\ -2g f^A{}_\beta T(e)_B{}^{\alpha\beta} + 2g f_{B\beta} T(e)^{A\alpha\beta} &= -\frac{g}{a}\theta^A{}_B{}^\alpha \end{aligned} \tag{316}$$

$$D_{[g,\Gamma]\beta} \widetilde{G}^{\alpha\beta} - g e^A{}_\beta T(f)_A{}^{\alpha\beta} + g f_{A\beta} T(e)^{A\alpha\beta} = -\frac{g}{a}\theta^\alpha \tag{317}$$

$$\left(1 + \frac{4a'}{a}\right) D_{[g,\Gamma]\beta} F'^{\alpha\beta} = -\frac{g}{a}\theta'^\alpha. \tag{318}$$

3.5. Some Special Solutions and Correspondence with Standard Theory

As yet we did not assume non-singularity of e, f, moreover they we free to vanish. If they happen to be frames, i.e., $\det\left[e^{A}{}_{\mu}\right] \neq 0$, $\det\left[f_{A_{\mu}}\right] \neq 0$, then the above equation, involving mixed quantities, may be expressed in terms of world tensors, free of any internal indices. This form is more convenient when the correspondence with Einstein theory and gauge theories of gravitation is studied. The resulting equations have a rather complicated form and occupy much place, thus, we do not quote them here but for certain details cf. [13]. They establish relationships between curvature and torsion tensors $R(e)$, $R(f)$, $S(e)$, $S(f)$ (288), (289), (290) and their g-covariant derivatives. There are terms characteristic for Einstein, Einstein-Cartan, and Poincaré-gauge theories of gravitation. To discuss the correspondence in more detail, we begin with the purely geometrodynamical sector, putting $\Psi = 0$. Obviously, the wave equation (312) is then trivially satisfied and it turns out, there are also solutions to (253), (314)–(318). Namely, let us substitute to equations (314)-(318) the following Einstein Ansatz

$$f_{A\mu} = \eta_{AB} e^{B}{}_{\mu}, \qquad t_{\mu\nu} = g_{\mu\nu} = h(e)_{\mu\nu} = h(f)_{\mu\nu} \tag{319}$$

$$\Gamma(e)^{\lambda}{}_{\mu\nu} = \Gamma(f)^{\lambda}{}_{\mu\nu} = \left\{ \begin{matrix} \lambda \\ \nu\mu \end{matrix} \right\}, \qquad S(e)^{\lambda}{}_{\mu\nu} = S(f)^{\lambda}{}_{\mu\nu} = 0 \tag{320}$$

$$Q_{\mu} = 0, \qquad A'_{\mu} = 0. \tag{321}$$

Let us stress, these constructions are substituted now to the field equations (252)–(254), not to the variational principle based on (243), (246), (247). Thus, we do not modify the dynamics, but search for particular solutions.

Equations (317), (318) become identities under substitution (319)–(321), whereas (314) and (315) both reduce to the same form

$$R^{\mu\nu} - 12g^{2}g^{\mu\nu} = 0 \tag{322}$$

where $R_{\mu\nu}$ denotes the Ricci tensor of the metric $g_{\mu\nu} = h(e)_{\mu\nu}$. If matter is admitted, then (314) leads to

$$R^{\mu\nu} - 12g^{2}g^{\mu\nu} = \frac{g}{a}\theta^{\mu\nu} \tag{323}$$

the corresponding equation for (315) has $\theta^{\mu\nu}$ on the right-hand side. Using the Einstein tensor, we can rewrite these equations as follows

$$R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu} = -12g^{2}g^{\mu\nu} \tag{324}$$

$$R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu} = -12g^{2}g^{\mu\nu} + \frac{g}{a}\left(\theta^{\mu\nu} - \frac{1}{2}\theta^{\alpha}{}_{\alpha}\, g^{\mu\nu}\right). \tag{325}$$

Finally, equation (316) reduces to

$$R^{\alpha}{}_{\beta\mu}{}^{\nu}{}_{;\nu} = -\frac{g^2}{a}\theta^{\alpha}{}_{\beta\mu} \tag{326}$$

or without matter to

$$R^{\alpha}{}_{\beta\mu}{}^{\nu}{}_{;\nu} = 0. \tag{327}$$

Let us observe that, the last equation is redundant, because it follows from (322) in virtue of Bianchi identities.

Yang-Mills equations (314)–(318) are compatible with the Einstein Ansatz (319)–(321) and are reduced by it to (324), (325), i.e., to Einstein equations with the negative “cosmological constant” $-12g^2$, determined by the coupling constant g of the $\mathrm{U}(2,2)$-gauge field. Gravitational constant is given by g/a. If we admitted in (319)–(321) a non-vanishing torsion $S(e)$, then the resulting equations would correspond with those of Poincaré-gauge theories of gravitation.

The Ansatz (319)–(321) may be weakened without any violation of the equations (314)–(318), namely, by admitting arbitrary constant factors at f, g

$$\begin{gathered} f_{A\mu} = k\eta_{AB}e^{B}{}_{\mu}, \qquad t_{\mu\nu} = kh(e)_{\mu\nu}, \qquad h(f)_{\mu\nu} = k^2 h(e)_{\mu\nu} \\ \Gamma(e)^{\lambda}{}_{\mu\nu} = \Gamma(f)^{\lambda}{}_{\mu\nu} = \left\{ \begin{array}{c} \lambda \\ \nu\mu \end{array} \right\}, \quad g_{\mu\nu} = ph(e)_{\mu\nu}, \quad Q_{\mu} = 0, \quad A'_{\mu} = 0. \end{gathered} \tag{328}$$

Substituting these assumptions to equations (4.16), we obtain

$$R\left(h(e)\right)_{\mu\nu} - \frac{1}{2}\tilde{h}(e)^{\alpha\beta}R(h(e))_{\alpha\beta}h(e)_{\mu\nu} = -12g^2 h(e)_{\mu\nu}. \tag{329}$$

The factors k, p become essential when we consider the last subsystem (254), obtained from the variation of the action functional with respect to the metric $g_{\mu\nu}$.

On the right-hand side of Einstein equations (254) we must substitute the total energy-momentum tensor

$$T^{\mu\nu} = T_{\mathrm{YM}}{}^{\mu\nu} + T_{\mathrm{m}}{}^{\mu\nu}.$$

If there is no matter, $T^{\mu\nu}$ reduces to $T_{\mathrm{YM}}{}^{\mu\nu}$, and substituting to (254) the weakened Ansatz (328), we obtain

$$\left(lp - 24g^2dk\right) h(e)_{\mu\nu} = T'_{\mathrm{YM}\mu\nu} \tag{330}$$

where T'_{YM} is given by

$$\begin{aligned} T'_{\mathrm{YM}\mu\nu} = & \frac{a}{8g^2p}R(h(e))^{\alpha}{}_{\beta\mu\lambda}R(h(e))^{\beta}{}_{\alpha\nu\sigma}\tilde{h}(e)^{\lambda\sigma} \\ & - \frac{a}{8g^2p}R(h(e))^{\alpha}{}_{\beta\kappa\lambda}R(h(e))^{\beta}{}_{\alpha\rho\sigma}\tilde{h}(e)^{\kappa\rho}\tilde{h}(e)^{\lambda\sigma}h(e)_{\mu\nu} \\ & + \frac{8ak}{p}\left(R(h(e))_{\mu\nu} - \frac{1}{4}\tilde{h}(e)^{\alpha\beta}R(h(e))_{\alpha\beta}h(e)_{\mu\nu}\right). \end{aligned} \tag{331}$$

Equation (330) impose rather strong algebraic conditions on the curvature tensor of the metric $h(e)_{\mu\nu}$. A priori it is not clear whether they are compatible with (323). One can show that, fortunately, there is no contradiction between (323) and (330), because there exist constant-curvature vacuum solutions. These solutions may be derived from flat $\mathrm{U}(4, G)$-connections A.

Let us assume that A is flat, i.e., $F = DA = 0$. Using the expansion (291), we express the condition $F = 0$ as follows

$$\mathrm{d}e^{A} + \Gamma^{A}{}_{B} \wedge e^{B} = 0 \tag{332}$$

$$\mathrm{d}f_{A} + f_{B} \wedge \Gamma^{B}{}_{A} = 0 \tag{333}$$

$$\begin{aligned} R(\Gamma)^{A}{}_{B} - \frac{1}{4}R(\Gamma)^{C}{}_{C}\delta^{A}{}_{B} - 2g^{2}e^{A} \wedge f_{B} & \\ +2g^{2}\eta^{AC}\eta_{BD}e^{D} \wedge f_{C} &= 0 \end{aligned} \tag{334}$$

$$\frac{1}{8}R(\Gamma)^{A}{}_{A} - g^{2}e^{A} \wedge f_{A} = 0 \tag{335}$$

$$F' = \mathrm{d}A' = 0. \tag{336}$$

Under the substitution $F = 0$, $\Psi = 0$, the Yang-Mills and wave equations (252), (253) become identities. Thus, the problem reduces to the system consisting of (332)–(336) and Einstein equations (254). There are geometrically distinguished solutions which provide a natural basis for the correspondence analysis of (252)–(254). They are based on the weakened Einstein Ansatz (328). Substituting it into (332)–(336), we observe that (332), (333), (335), (336) become identities. More precisely, it is sufficient to assume that

$$f_{A\mu} = k\eta_{AB}e^{B}{}_{\mu}, \qquad g_{\mu\nu} = ph(e)_{\mu\nu}, \qquad Q_{\mu} = 0, \qquad A'_{\mu} = 0. \tag{337}$$

Then (332) and (333) just imply that $\Gamma(e) = \Gamma(f) = \{g\} = \{h(e)\}$. And finally, equation (334) simply states that (M, g) is a constant-curvature space.

The $\mathrm{U}(2,2)$-flatness condition (332)–(336) reduces under the Ansatz (337) to the following equation

$$R(g)_{\alpha\beta\mu\nu} = \frac{4g^{2}k}{p}\left(g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\nu}g_{\beta\mu}\right). \tag{338}$$

This means that (M, g) is a constant-curvature space. It is conformally-flat, and this link between conformal flatness of g and the $\mathrm{U}(2,2)$-flatness of A is rather natural, if we remember that $\mathrm{SU}(2,2)$ is the universal covering group of the conformal group $\mathrm{CO}(1,3)$.

The equation (338) implies that

$$R(g)_{\mu\nu} - \frac{1}{2}R(g)g_{\mu\nu} = -\frac{12g^{2}k}{p}g_{\mu\nu}. \tag{339}$$

This is consistent with (330) and (332)–(336) if and only if

$$lp = 24g^2 dk. \tag{340}$$

It is a dissatisfying redundancy of the model, that the Einstein equations emerge from (252)–(254) in two different forms, with independent constants. One can avoid this disadvantage by following the Palatini scheme, i.e., putting $d = 0, l = 0$. The metric tensor g preserves then its status of independent dynamical variable, however, it is no longer represented by a separate term in Lagrangian. Instead, it enters algebraically the matter and Yang-Mills Lagrangian. The above integration constants, p, k are then completely arbitrary.

The presented solutions of geometrodynamical equations provide a convenient framework for investigating the Dirac limit of our wave equation. The most convenient choice of Ansatz constants is $k = 1$, corresponding to the balanced bitetrad (e, f). Let us mention, incidentally, there are also solutions corresponding to $k = 0$. They describe a geometric background for the strange world admitting only one kind of Weyl spinors. The trivial geometrodynamical vacuum $A = 0$ is incompatible with any concept of spinors.

Let us now go back to the wave equation (313). Due to the small value of the gravitational constant, geometrodynamical sector is weakly sensitive to the material one. Thus, in a small spatio-temporal scale of elementary particle physics, it is a satisfactory approximation to consider the wave dynamics (313) as played on the fixed geometric arena provided by the above solutions (337), (338) without any feedback through geometry.

Let us substitute to (313) the Ansatz

$$f_{A\mu} = \eta_{AB} e^{B}{}_{\mu}, \qquad g_{\mu\nu} = h(e)_{\mu\nu} = \eta_{AB} e^{A}{}_{\mu} e^{B}{}_{\nu}. \tag{341}$$

Therefore

$$E^{A}{}_{\mu} = e^{A}{}_{\mu}, \qquad F_{A\nu} = 0.$$

The Ansatz (341) reduces the wave equation to

$$\mathrm{i}\gamma^{A}\left(e^{\mu}{}_{A} D_{\mu}\Psi + \frac{1}{2}\left(D_{[g,\Gamma]\mu} e^{\mu}{}_{A}\right)\Psi\right) - \frac{4bg^2 - c}{2bg}\Psi + \frac{1}{2g} g^{\mu\nu} D_{[g,\Gamma]\mu} D_{[g,\Gamma]\nu}\Psi = 0.$$

The divergence correction term has the same status as in (312). Following (313), we can also use the symbolic representation

$$\mathrm{i}\gamma^{A}\mathcal{L}_{e_A}\Psi - \frac{4bg^2 - c}{2bg}\Psi + \frac{1}{2g} g_{\mu\nu} D_{[g,\Gamma]\mu} D_{[g,\Gamma]\nu}\Psi = 0. \tag{342}$$

It is easy to show that

$$D_{[g,\Gamma]\mu} e^{\mu}{}_{A} = -K(e)^{\alpha}{}_{\mu\alpha} e^{\mu}{}_{A} \tag{343}$$

thus,

$$e^{\mu}{}_{A}\mathrm{i}\gamma^{A}\left(D_{\mu}-\frac{1}{2}K(e)^{\alpha}{}_{\mu\alpha}\right)\Psi-\frac{4bg^{2}-c}{2bg}\Psi+\frac{1}{2g}g^{\mu\nu}D_{[g,\Gamma]\mu}D_{[g,\Gamma]\nu}\Psi=0.$$

If we assume in addition that $\Gamma(e)$ is an Einstein-Cartan connection, then

$$K(e)^{\alpha}{}_{\mu\alpha}=-2S^{\alpha}{}_{\mu\alpha}. \tag{344}$$

Finally, if Γ is a Levi-Civita connection, or, at least, if S is trace-less,

$$e^{\mu}{}_{A}\mathrm{i}\gamma^{A}D_{\mu}\Psi-\frac{4bg^{2}-c}{2bg}\Psi+\frac{1}{2g}g^{\mu\nu}D_{[g,\Gamma]\mu}D_{[g,\Gamma]\nu}\Psi=0. \tag{345}$$

In the specially-relativistic limit, when $e^{\mu}{}_{A}=\delta^{\mu}{}_{A}$, $\Gamma=0$, $g_{\mu\nu}=\eta_{\mu\nu}$, this equation becomes as follows

$$\mathrm{i}\gamma^{\mu}\partial_{\mu}\Psi-\frac{4bg^{2}-c}{2bg}\Psi+\frac{1}{2g}\partial^{\mu}\partial_{\mu}\Psi=0. \tag{346}$$

If our model is to be viable, this equation must somehow correspond with the Dirac equation, in spite of the term with second derivatives.

Let us consider, more generally, the following specially-relativistic Klein-Gordon-Dirac equation

$$V\mathrm{i}\gamma^{\mu}\partial_{\mu}\Psi-W\Psi-U\partial^{\mu}\partial_{\mu}\Psi=0 \tag{347}$$

which is derivable from the Lagrangian

$$L=V\frac{\mathrm{i}}{2}\left(\widetilde{\Psi}\gamma^{\mu}\partial_{\mu}\Psi-\partial_{\mu}\widetilde{\Psi}\gamma^{\mu}\Psi\right)-W\widetilde{\Psi}\Psi+U\partial^{\mu}\widetilde{\Psi}\partial_{\mu}\Psi \tag{348}$$

where V, W, U are real constants, and Minkowskian coordinates are used. Obviously, equation (347) does not correspond to any irreducible representation of the Poincaré group, and in this sense it is not admitted by the Wigner-Bargmann classification as a relativistic wave equation for elementary particles. Nevertheless, there are no principal obstacles against considering a continuous dynamical system ruled by (347). A more detailed analysis, together with quantization attempts, is presented in [14]. Here we restrict ourselves to an elementary analysis of the physical viability of (347).

Due to linearity of (347), one can expect solutions in the form of continuous superpositions of harmonic plane waves. Equation (347) yields, under substitution of

$$\Psi(x)=\varphi\exp\left(-\mathrm{i}p_{\mu}x^{\mu}\right)$$

that

$$\gamma^{\mu}p_{\mu}\varphi=m\varphi \tag{349}$$

where

$$m^{2}=p^{2}=\frac{1}{2U^{2}}\left(2UW+V^{2}\pm\sqrt{V^{4}+4UWV^{2}}\right). \tag{350}$$

Thus, the general solution of (347) is a superposition of two Dirac waves with masses m_-, m_+ given by (350). In general, when no restrictions on coefficients U, W, V are imposed, tachyonic situations $p^2 < 0$ are possible. To avoid this and warrant real non-negative solutions for p^2, we must assume

$$V^2 + 4UW \geq 0. \tag{351}$$

The appearance of two mass shells in the general solution of (347) need not be so embarrassing as it could seem, and namely, for the following reasons:

i) If the splitting of masses $m_+ - m_-$ is large, than, in usual conditions, it may be difficult to excite the m_+-states, because the frequency spectrum of external perturbations will have to contain frequencies of the order $(m_+ - m_-)c^2/h$. For example, if $U \to 0$, then $m_- \to |W|/|V|$, $m_+ \to \infty$ (compare this with the idea of the Pauli-Villars-Rayski regularization).
ii) It is not excluded that superposition of states with two masses might be just desirable, for example, one could try to explain in this way a mysterious kinship between heavy leptons and their neutrions, or the corresponding pairing between quarks. If there is no algebraic term, $W = 0$, then $m_- = 0$, $m_+ = |V/U|$, thus, in spite of the purely differential character of (347), massive states appear, and are paired with massless ones.
iii) For special values of U, V, W namely, when $V^2 + 4UW = 0$, the mass gap vanishes, $m_- = m_+ = |W/U|$, and (347) exactly reduces to usual Dirac equation.

Comparing equations (346), (347) we obtain that

$$V = 1, \qquad W = \frac{4bg^2 - c}{2bg}, \qquad U = -\frac{1}{2g} \tag{352}$$

and the Klein-Gordon-Dirac equation (347) is controlled by two parameters g and c/b. Therefore, one of masses in the doublet will vanish when $c = 4bg^2$ and the non-vanishing partner equals $m_+ = 2|g|$.

The mass splitting vanishes when $c = 3bg^2$. The Klein-Gordon-Dirac equation (347) reduces then to usual Dirac equation, and $m_- = m_+ = |g|$. The general formula for the mass doublet reads

$$m_{\mp}^2 = \frac{c}{b} - 2g^2 \left(1 \pm \sqrt{\frac{c}{bg^2} - 3}\right). \tag{353}$$

It is seen that below the threshold $c/b = 3g^2$ there is no correspondence with Dirac equation unless we accept complex masses, tachyons and decays. At $c/b = 3g^2$ there is a kind of "phase transition", and, due to interaction between fields Ψ and A, the effective Dirac particle of mass $|g|$ emerges. When c/b increases above the threshold, the mass splitting appears and the wave field Ψ becomes a superposition of Dirac waves with masses $m_{\pm}$ given by (353). The quadrat-of-mass gap

$$\Delta(m^2) = 4|g|\sqrt{\frac{c}{b} - 3g^2} \qquad (354)$$

becomes negligible for small coupling constants. Some primary "mass" c/b is necessary if the effective Dirac behaviour is to emerge from the original Klein-Gordon model due to the spontaneous breaking of the $\mathrm{U}(2,2)$-symmetry to $\mathrm{SL}(2,\mathbb{C})$. It is not excluded that extending the internal symmetry from $\mathrm{U}(2,2)$ to $\mathrm{GL}(4,\mathbb{C})$ we could obtain a model where the effective Dirac equation with mass would appear even in the absence of the algebraic term $(c/2)\widetilde{\Psi}\Psi\sqrt{|g|}$ in the Klein-Gordon Lagrangian for Ψ. The theory invariant under $\mathrm{GL}(4,\mathbb{C})$ involves more degrees of freedom, because $G_{\bar{r}s}$ becomes a dynamical quantity.

Final Remarks

We were motivated by some physical ideas concerning the status of nonlinearity in fundamental wave equations appearing in quantum theory of strongly condensed field-particles systems. In fundamental problems one must take relativistic phenomena into account. Even in situations where they are not quantitatively very strong, they strongly influence certain qualitative aspects. The nonlinearity we mean is usually implied by strong and non-commutative symmetry groups. When we deal with relativistic theory of phenomena in which particles with half-integer spin appear, it is just necessary to introduce additional geometric objects, well-known in mechanics of structured continua, like the non-holonomic fields of frames (cotetrad fields). And then this way the strong nonlinearity appears. This is because then by the very geometric nature of used fields, groups of conformal symmetry and general covariance appear. In spite of our dealing with the very fundamental physical fields, the mathematical methodology of our treatment is in a sense common with characteristic nonlinearities appearing in elasticity and theory of shells and membranes.

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