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ON THE EQUIVALENCE BETWEEN MANEV AND KEPLER PROBLEMS

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Abstract. Here we demonstrate the existence of a *local* Darboux chart for the Manev model such that its dynamics becomes locally equivalent to the Kepler model. This explains lot of similarities between these two models and especially why they share common symmetry algebras. We also discuss the existence of group actions on the phase space for the algebras inherent in the Manev model.

1. Introduction

In the last decade Manev model had enjoyed an increased interest either as a very suitable approximation to Einstein's relativistic dynamics from astronomers' point of view or as a toy model for applying different techniques of the modern dynamics (see e.g. [4,7,8,17,18]). It was not invented as an approximation of relativity theory but as a consequence of Max Planck's (more general) action-reaction principle and is capable to describe both the perihelion advance of the inner planets and the Moon's perigee motion.

By **Manev model** [16] we mean here the dynamics given by the Hamiltonian

$$H = \frac{1}{2}(p_x^2 + p_y^2 + p_z^2) - \frac{A}{r} - \frac{B}{r^2} \tag{1}$$

where $r = \sqrt{x^2 + y^2 + z^2}$ while A and B are assumed to be arbitrary real constants whose positive values correspond to attractive forces. Due to the rotational invariance each component of the **angular momentum**

$$L_j = \varepsilon_{jkm} p_k x_m \quad \text{with} \quad (x_1, x_2, x_3) = (x, y, z)$$

is an obvious first integral, i.e., $\{H, L_j\} = 0$ and so, like the Kepler problem (and any central potential), the Manev model is integrable. Angular momentum components themselves are not in involution but span an $\mathfrak{so}(3)$ algebra with respect to the Poisson brackets

$$\{L_j\,, L_k\} = \varepsilon_{jkm} L_m \tag{2}$$

and if we approach the question of the integrability solely in L_j terms, we obtain the most simple example of non-commutative integrability [9, 19, 20].

The motion is confined on a plane which we assume to be XOY and correspondingly the angular momentum $L_z \equiv L$ is in the z-direction. From now on we shall concentrate on this dynamics on the phase space $\mathcal{M} = T^*\left(\mathbb{R}^2 \setminus \{0\}\right) = T^*\mathbb{R}^+ \times T^*\mathbb{S}^1$ which is separable in radial coordinates r and $\theta = \arctan(y/x)$ as it is governed by

$$H = \frac{1}{2}\left(p_r^2 + \frac{L^2 - 2B}{r^2}\right) - \frac{A}{r}, \qquad \omega = \mathrm{d}p_r \wedge \mathrm{d}r + \mathrm{d}L \wedge \mathrm{d}\theta. \tag{3}$$

Recently we reported [11] that Manev model has an additional independent globally defined constant of motion, albeit not for all initial data. Also, it has exactly the same additional symmetry algebras $\mathfrak{so}(3)$ (or $\mathfrak{so}(2,1)$ for positive energies) as the Kepler problem thus giving more arguments in favour of the view that Manev model is the natural generalization of Kepler's. We shall explain here why we have coinciding algebras for both models through an extension of the Newton's Revolving Orbits Theorem. Also we construct a (rather weak form of) canonical transformation connecting the Manev dynamics with the Kepler's for a set of initial data corresponding to large enough angular momentum.

2. The Kepler Problem Invariants

In the case of **Kepler problem** the Hamiltonian is

$$H_K = \frac{1}{2}\left(p_r^2 + \frac{L^2}{r^2}\right) - \frac{A}{r} \tag{4}$$

and we have more first integrals (for details and historical notes see e.g. [6, 12, 13, 21]) due to

$$\{H_K, \vec{J}\} = 0 \quad \text{with} \quad \vec{J} = \vec{p} \wedge \vec{L} - A\vec{r}/r$$

being the **Laplace-Runge-Lenz** vector whose components are not independent as $J^2 = 2H_K L^2 + A^2$. Together with the Hamiltonian and angular momentum they close on an algebra with respect to the Poisson brackets and after redefining $\vec{E} = \vec{J}/\sqrt{|-2h_K|}$ on each $H_K = h_K$ level set we get

$$\{L, E_x\} = E_y\,, \qquad \{L, E_y\} = -E_x\,, \qquad \{E_x, E_y\} = -\operatorname{sign}(h_K)\, L$$

with Casimir invariant

$$E_x^2 + E_y^2 + \operatorname{sign}(-H_K)\, L^2 = \frac{A^2}{|2H_K|} \tag{5}$$

which makes obvious the fact that we have an $\mathfrak{so}(3)$ algebra for negative energies and $\mathfrak{so}(2,1)$ for positive ones. In the case of the three-dimensional Kepler problem the components of the angular momentum give us another copy of $\mathfrak{so}(3)$, see equation eq2, so the full symmetry algebra is either $\mathfrak{so}(4)$ or $\mathfrak{so}(3,1)$ depending on the sign of h_K. Actually, the first use of these first integrals was made by J. Hermann (= J. Ermanno) in 1710 (in order to find *all* possible orbits under an inverse square law force) in the disguise of **Ermanno-Bernoulli constants**

$$J_\pm = J_x \pm \mathrm{i} J_y = \left(\frac{L^2}{r} - A \mp \mathrm{i}\, L p_r\right) \mathrm{e}^{\pm \mathrm{i}\theta}$$

satisfying

$$\{H_K, J_\pm\} = 0, \qquad \{L, J_\pm\} = \pm \mathrm{i} J_\pm, \qquad \{J_+, J_-\} = -4\mathrm{i} H_K L. \tag{6}$$

3. The Manev Problem Invariants and Symmetries

It has already been established the invariance of

$$\mathcal{J}_\pm = \nu L \left[p_r \pm \mathrm{i}\left(\nu p_\perp - \frac{A}{\nu L}\right)\right] \mathrm{e}^{\pm \mathrm{i}\nu\theta}\,, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{J}_\pm = 0 \tag{7}$$

where

$$p_\perp = \frac{L}{r} \quad \text{and} \quad \nu^2 = \frac{L^2 - 2B}{L^2}$$

and $\mathcal{J}_+$ and $\mathcal{J}_-$ are not independent as

$$\mathcal{J}_+ \mathcal{J}_- = 2\nu^2 H L^2 + A^2\,.$$

Obviously in the Kepler case ν equals *one* and (up to a multiplication by i) we recover the ‘Ermanno-Bernoulli’ constants.

3.1. Compact Motion Case

Trajectories always lie on the joint level sets of H and L which in the case when $0 \neq L^2 > 2B$ and $H < 0$ are two-dimensional tori. When ν is irrational trajectories fill densely these tori, and hence there are no new (continuous) constants of motion, and the fibration by two-dimensional tori is the finest fibration with invariant fibers.

In order to have *globally* defined constants of motion in this case we have to require that the real valued ν's be rational, i.e.,

$$\nu = \sqrt{L^2 - 2B} : L = m : k \tag{8}$$

with m and k mutually prime integers. Then $\mathcal{J}_\pm$ are conserved by the flow determined by equation (1) on any surface $L = \ell$ satisfying the rationality condition (8). Thus we have conditional constants of motion which exist only for disjoint but infinite set of values ℓ, otherwise $\mathcal{J}_\pm$ would not be well defined and we would have invariant submanifolds but not genuine constants of motion. Consequently, trajectories are periodic and we have a finer invariant fibration.

Let us remark that for a generic central potential we could have disjoint set of initial data corresponding to closed orbits but in our case *all* points on certain level sets of the angular momentum lie on closed orbits which are intersections with the level sets of the additional invariant.

Each trajectory in the configuration space looks like a "rosette" with m petals and this is connected to the fact that $\mathcal{J}_\pm$ (as well as H and L) are invariant under the action of the cyclic group generated by rotations by angle $\frac{2\pi k}{m}$

$$\theta \to \theta + 2\pi \frac{k}{m} n, \qquad n = 0, 1, \ldots, m-1 \,.$$

To visualize the intersection of the level sets we can fix the angular momentum and use x, y and p_r as coordinates on this level set. Both Hamiltonian and $\mathcal{J}_\pm$ level sets are represented as two-dimensional surfaces whose intersection gives the trajectories in the phase space (see Fig. 1).

While the integrable systems are characterized by a fibration by Liouville tori $\pi : \mathcal{M} \to \mathcal{B}$ mapping the phase space $\mathcal{M}$ to the base $\mathcal{B}$ which is the space of independent first integrals (with $2 \dim \mathcal{B} = \dim \mathcal{M}$), in the superintegrable case (i.e., when $2 \dim \mathcal{B} > \dim \mathcal{M}$) we have a more complicate structure. A Hamiltonian vector field is superintegrable if it is tangent to the fibers of a fibration $i : \mathcal{M} \to \mathcal{B}$ with connected and isotropic fibers such that there exists a second fibration with coisotropic fibers $c : \mathcal{M} \to \mathcal{A}$ such that the tangent spaces of the former and the latter are symplectically orthogonal. Such pair of fibrations denoted by $\mathcal{A} \xleftarrow{c} \mathcal{M} \xrightarrow{i} \mathcal{B}$ is called **bifibration** and is a particular case of a *dual pair* [22].

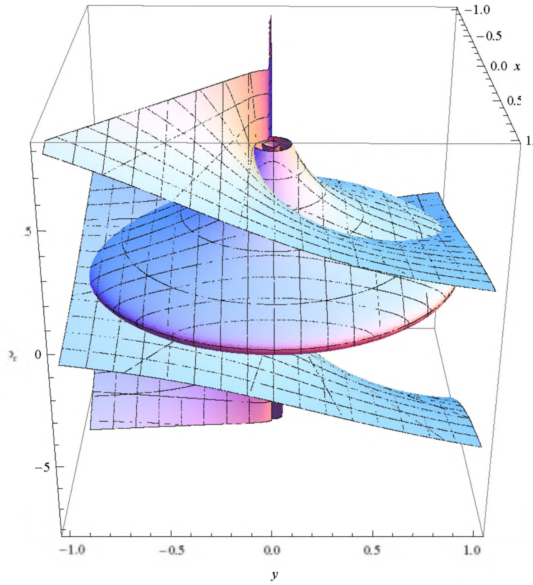

Figure 1. The level sets of H (the central "bubble") and of $\mathcal{J}_+$ (the helicoid-like surface) for $L = 1$, $A = 2$, $\nu = 1/3$, $H = -1.95$ and $\arg(\mathcal{J}_+) = -0.9$.

In our case $\mathcal{B}$ is the three-dimensional space of *first integrals* $\{H,\ L,\ \mathcal{J}\}$, which carries a natural Poisson structure and $\mathcal{A}$ is the space of *actions*, i.e., Casimirs of this structure, which consists of the Hamiltonian (1) only. The fibers of i are the periodic orbits, and the fibres of c are the level sets of H, and each level set is fibred by the periodic orbits, i.e., we have a third fibration $s : \mathcal{B} \to \mathcal{A}$ with $c = s \circ i$.

We shall now introduce another Darboux chart for our symplectic form equation (3) in the case when $L^2 - 2B > 0$ by defining new local coordinates $\vartheta = \nu(L)\theta$, and $\mathsf{L} = \nu(L)L$ which are canonically conjugate as

$$\mathrm{d}\mathsf{L} \wedge \mathrm{d}\vartheta = \mathrm{d}L \wedge \mathrm{d}\theta \quad \text{and hence} \quad \omega = \mathrm{d}p_r \wedge \mathrm{d}r + \mathrm{d}\mathsf{L} \wedge \mathrm{d}\vartheta\,. \tag{9}$$

It should be noted that ϑ is not even an angular type of coordinate (i.e., a coordinate ϕ which does not exist globally but $\mathrm{d}\phi$ is still well defined closed one-form). In our case even $\mathrm{d}\vartheta = \nu(L)\mathrm{d}\theta + \frac{\theta}{\nu(L)}\mathrm{d}L$ is not well defined globally due to the second term but $\mathrm{d}\mathsf{L} \wedge \mathrm{d}\vartheta$ still makes sense. When written in the new coordinates Manev's Hamiltonian takes the form of Kepler's

$$H = \frac{1}{2}\left(p_r^2 + \frac{\mathsf{L}^2}{r^2}\right) - \frac{A}{r} \tag{10}$$

and this link between the two models gives us a direct method to demonstrate that Manev's model possess exactly the same symmetry algebra as Kepler's. The fact that symplectic structure is the same in both charts means that for every pair of phase space functions we will have

$$\{F(r,p_r,\theta,L),G(r,p_r,\theta,L)\} = \{F(r,p_r,\vartheta,\mathsf{L}),G(r,p_r,\vartheta,\mathsf{L})\} \tag{11}$$

and hence from any (Poisson brackets) algebra of the Kepler model we can immediately produce identical algebra of the Manev problem by just taking the same functions depending now on the local variables ϑ and L. This observation may be viewed as a trivial extension of the Newton's Revolving Orbits Theorem[1]

Theorem 1. *Let $r(\theta)$ be an orbit generated by any central force $F(r)$. Then the revolving orbit $r(\widetilde{\theta}) = r(\alpha\theta)$ is generated by a central force $\widetilde{F}(r)$ that differs from $F(r)$ by an inverse-cube force, and conversely. In particular, if L and $\widetilde{L}$ are the angular momenta corresponding to $r(\theta)$ and $r(\widetilde{\theta})$, respectively, then*

$$\widetilde{F}(r) = F(r) + \frac{L^2-\widetilde{L}^2}{r^3} \quad \text{and} \quad \alpha = \frac{\widetilde{L}}{L}\cdot$$

In our notations $\alpha = \nu$ and $2B = L^2 - \widetilde{L}^2$, and we may state that all Poisson brackets (and hence algebras) for the *tilde*-system are identical to the ones of the original system provided we replace the arguments of the phase space functions L and θ with $\mathsf{L} = \widetilde{L}$ and $\vartheta = \alpha\theta$.

Applying this to Kepler's invariants and defining

$$K_1 = \frac{\mathcal{J}_+ + \mathcal{J}_-}{2\sqrt{|2H|}}, \qquad K_2 = \frac{\mathrm{i}(\mathcal{J}_+ - \mathcal{J}_-)}{2\sqrt{|2H|}}, \qquad K_3 = \mathsf{L}$$

we obtain $\mathfrak{so}(3)$ or $\mathfrak{so}(2,1)$ algebra

$$\{K_1,K_2\} = \mathrm{sign}(-H)K_3\,, \qquad \{K_2,K_3\} = K_1\,, \qquad \{K_3,K_1\} = K_2$$

with Casimir invariant

$$C \equiv K_1^2 + K_2^2 + \mathrm{sign}(-H)\,K_3^2 = \frac{A^2}{|2H|}$$

and so, the space of invariants (i.e., the fibres of s, or equivalently, the space of first integrals for fixed value of the Hamiltonian) is a sphere or a hyperboloid (which degenerate to a point or cone if $A = 0$), i.e., exactly the same as in the Kepler model. In this way dynamical vector fields of Manev and Kepler problems provide different fibrations over isomorphic base spaces $\mathcal{B}$.

[1]Comments on the original Newton's wording may be found in [15] and a standard exposition of this matter in [23].

The specifics of the superintegrable systems are reflected in the Poisson structures on the base, which for the three-dimensional case are, fortunately, completely classified [10] by just two smooth functions u and φ. The most general Poisson bracket for a basis $\{y_n\}$ in $\mathbb{R}^3$ has the form

$$\{y_i, y_j\} = \varepsilon_{ijk} u \frac{\partial \varphi}{\partial y_k} \tag{12}$$

with φ being a Casimir of the Poisson structure. Here $\varphi = C$ and $u = 1$ (if we identify K_i with y_i) and we have as a bonus a Lie algebraic structure whose existence is not guaranteed in generic superintegrable systems.

3.2. Noncompact Motion Cases

- When $0 \neq \ell^2 > 2B$ and $H \geq 0$ the additional invariants are *always* globally defined and have the form and symmetry algebras just described.
- When $\ell^2 = 2B$ we have the first integral

$$j = Lp_r + A\theta$$

  satisfying $\{H, j\} = 0$, $\{L, j\} = A$.
- In the case when $0 \neq \ell^2 < 2B$ we may denote $\upsilon = \mathrm{i}\nu$ with υ real and

$$\mathcal{J}_\pm = \upsilon L \left[p_r \pm \left(\upsilon p_\perp + \frac{A}{\upsilon L} \right) \right] \mathrm{e}^{\pm \upsilon \theta}$$

  will be first integrals for *any* ℓ.

In the last case (which has no direct analogue in the Kepler mechanics) we can again introduce new Darboux chart denoting $\mathsf{L} = \upsilon(L)L$ and $\vartheta = \upsilon(L)\theta$. When written in the new coordinates Manev's Hamiltonian takes the form

$$H = \frac{1}{2}\left(p_r^2 - \frac{\mathsf{L}^2}{r^2} \right) - \frac{A}{r} \tag{13}$$

and we can define

$$K_1 = \frac{\mathcal{J}_+ + \mathcal{J}_-}{2\sqrt{|2H|}}, \qquad K_2 = \frac{\mathcal{J}_+ - \mathcal{J}_-}{2\sqrt{|2H|}}, \qquad K_3 = \mathsf{L}$$

to obtain the $\mathfrak{so}(2,1)$ algebra

$$\{K_1, K_2\} = \operatorname{sign}(-H) K_3\,, \qquad \{K_2, K_3\} = K_1\,, \qquad \{K_3, K_1\} = -K_2$$

for both choices of the sign of H. Its Casimir invariant is

$$C \equiv K_1^2 - K_2^2 + \operatorname{sign}(-H)\, K_3^2 = -\frac{A^2}{|2H|}$$

and thus the space of invariants is one- or two-sheet hyperboloid, and this again corresponds to $\varphi = C$ and $u = 1$ case of the general Poisson bracket equation (12).

3.3. From Symmetry Algebras to Group Actions

The mere existence of an algebra of well defined first integrals does not presuppose suitable group action on the phase space. Even as simple systems as the commensurate two-dimensional oscillator present obstructions to group actions on the phase space [2]. Also, the symmetry algebra of the Kepler model does not lead to global group action unless the problem is "regularized" [14]. Here we have a more immediate obstacle for the existence of group actions in the compact motion case as $\mathcal{J}_{\pm}$ (or K_1, K_2) do not commute with L and hence do not preserve any $L = \ell$ surface and destroy rationality condition (8). Even if we take the stance that one could analyze also first integrals without global meaning, so not bothering about the rationality condition, there always exist "candidates for group orbits" which reach $L^2 = 2B$ level set where the symmetry algebra itself changes and hence could not be prolonged.

If we would like to find an algebra having any chance to yield a group action in the phase space it should be an algebra of rotationally invariant functions (i.e., commuting with L). Similar to the bifibration of the previous sections with fibers formed by the Hamiltonian vector field and level sets of H, we could construct a bifibration with angular momentum L taking the role of H. Among the many possible such choices (starting e.g. with the first guess $\{r^2, p^2, \vec{p}, \vec{r}\}$) we would prefer an algebra more closely connected to the dynamics of the problem.

Such an $\mathfrak{so}(2,1)$ algebra had actually been obtained at the end of 60's as a tool for determining the energy levels in the quantum Manev model (but without calling it so) [1]. It is worth noting that this algebra somehow distinguishes the Manev model as it was demonstrated soon after in [5] that this is the most general model (under some set of physically sensible assumptions) with discrete and continuous spectrum having this algebra. A more recent survey [3] reported that the only explicit potentials realizing $\mathfrak{su}(1,1)$ (isomorphic to $\mathfrak{so}(2,1)$) algebra with discrete spectrum are Manev, Morse and "spiked oscillator" (i.e., $V = ar^2 + b/r^2$) ones. For the classical Manev model the algebra's basis is defined by

$$T_1 = \frac{1}{2}\left(rp^2 - \frac{2B}{r} - r\right), \qquad T_2 = \vec{p}\,.\,\vec{r}, \qquad T_3 = \frac{1}{2}\left(rp^2 - \frac{2B}{r} + r\right)$$

such that

$$\{T_1, T_2\} = T_3\,, \qquad \{T_2, T_3\} = -T_1\,, \qquad \{T_3, T_1\} = -T_2. \tag{14}$$

We shall make use also of

$$\begin{gathered} T_+ = T_3 + T_1 = rp^2 - \frac{2B}{r}\,, \qquad T_- = T_3 - T_1 = r \\ \text{with } \{T_+, T_-\} = 2T_2 \quad \text{and} \quad \{T_2, T_\pm\} = \mp T_\pm. \end{gathered} \tag{15}$$

Let us note that this is not a symmetry algebra as its elements do not commute with the Hamiltonian but with L. The Hamiltonian does not depend linearly on T_i-s but the combination $rH = T_+/2 - A$ does and hence through an appropriate reparametrization the radial motion could be represented as a linear dynamics on $\{T_1, T_2, T_3\}$.

The algebra has the Casimir invariant

$$T_3^2 - T_1^2 - T_2^2 = T_+T_- - T_2^2 = L^2 - 2B$$

and thus the space of invariants of the Hamiltonian vector field of L is one- or two-sheet hyperboloid for negative/positive $\ell^2 - 2B$, or a cone if $\ell^2 = 2B$.

If we want to keep the correspondence with the phase space we have to restrict ourselves to the region where $r \geq 0$, i.e., $T_3 \geq T_1$. This selects the upper hyperboloid if $\ell^2 - 2B > 0$, or half of the single hyperboloid if $\ell^2 - 2B < 0$. As a result there will be obstructions to a possible group action on the phase space, coming from the fact that the vector fields are not complete vector fields. While we will have well defined action on the upper hyperboloid for $\ell^2 - 2B > 0$. This will be no longer true in the case when $\ell^2 - 2B < 0$ as any point on the single hyperboloid could be made to hit $r = 0$ locus by a suitable rotation around T_3-axis.

4. Conclusions

We have demonstrated the existence of a *local* Darboux chart for the Manev model such that its dynamics becomes locally equivalent to the Kepler model. This explains why we observe so many similarities between these two models and especially why they have common symmetry algebras. We also discuss the problem of existence of group actions on the phase space for the algebras inherent in the Manev model.

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