Eighth International Conference on
Geometry, Integrability and Quantization
June 9–14, 2006, Varna, Bulgaria
Ivaïlo M. Mladenov and Manuel de León, Editors
SOFTEX, Sofia 2007, pp 322–333

CONSTRUCTION OF ELLIPTIC SOLUTIONS TO THE QUINTIC COMPLEX ONE-DIMENSIONAL GINZBURG–LANDAU EQUATION

SERGEY YU. VERNOV

Skobeltsyn Institute of Nuclear Physics, Moscow State University
Vorob'evy Gory, Moscow, 119992, Russia

Abstract. The Conte–Musette method has been modified for the search of only elliptic solutions to systems of differential equations. A key idea of this a priory restriction is to simplify calculations by means of the use of a few Laurent series solutions instead of one and the use of the residue theorem. The application of our approach to the quintic complex one-dimensional Ginzburg–Landau equation (CGLE5) allows us to find elliptic solutions in the wave form. We also find restrictions on coefficients, which are necessary conditions for the existence of elliptic solutions to the CGLE5.

1. Introduction

At present time the methods for construction of special solutions of nonintegrable systems in terms of elementary (more precisely, degenerated elliptic) and elliptic functions are still actively developed (see [12, 23] and references therein).

Elliptic and degenerate elliptic functions are single-valued functions, therefore, the necessary condition for the existence of such solutions of a nonintegrable system is the existence of the Laurent series solutions of it. Such local solutions can be constructed by means of the Ablowitz–Ramani–Segur algorithm of the Painlevé test [1] (see also [11, 12, 17]). Moreover for a wide class of dynamical systems using this method one can find all possible Laurent series expansions of solutions. In this way one obtains solutions only as formal series, that is sufficient, because really only a finite number of coefficients of these series are used. Examples of construction of such solutions are given in [13] and [18]. The Laurent series solutions give the information about the global behavior of differential systems and assist to look for exact solutions [14]. The Laurent series solutions can be used to prove the nonexistence of elliptic solutions [10, 22] as well.

In [14] a new method for construction of single-valued special solutions of nonintegrable differential equations has been proposed. A key idea of this method is the use of the Laurent series solutions to transform the initial differential equation into a nonlinear system of algebraic equations. Using this method one can in principle find all elliptic and degenerate elliptic solutions. Unfortunately, if the initial differential equation includes a large number of numerical parameters, then it is difficult to solve the obtained nonlinear system of algebraic equations.

The goal of this paper is to propose a modification of the Conte–Musette method, which allows to seek elliptic solutions only. We show that in this case it is possible to fix some parameters of the initial differential system and therefore simplify the resulting system of algebraic equations. To do this we use the Hone method, which has been proposed to prove the non-existence of elliptic solutions [10]. Note that using our approach one can find in principle all elliptic solutions.

In [14] the authors have transformed the initial system of two coupled ordinary differential equations into the equivalent single differential equation and only after that have constructed the Laurent series solutions. In our paper we demonstrate that the analysis of the system of differential equations may be more useful than the consideration of the equivalent differential equation. Moreover in this paper we show that if the system of differential equations includes a few functions it is possible to find the analytic form of a function, which satisfies this system, even without knowledge of other functions in the analytic form.

2. The Conte–Musette Method for the System of Differential Equations

In [14] Conte and Musette have proposed a method to search for elliptic and degenerate elliptic solutions to a polynomial autonomous differential equation. In this section we reformulate their method for a system of such equations

$$F_i(\tilde{\mathbf{y}}_{;t}^{(n)}, \tilde{\mathbf{y}}_{;t}^{(n-1)}, \ldots, \tilde{\mathbf{y}}_{;t}, \tilde{\mathbf{y}}) = 0, \qquad i = 1, \ldots, N \tag{1}$$

where $\tilde{\mathbf{y}} = \{y_1(t), y_2(t), \ldots, y_L(t)\}$ and $y_{j;t}^{(k)} = \mathrm{d}^k y_j/\mathrm{d}t^k$.

It is known (see details in [14]) that the elliptic functions (including any degenerate ones) are solutions of the following first order polynomial autonomous differential equations (the summation runs over nonnegative integers j that are less than or equal to $(p+1)(m-k)/p$)

$$\sum_{k=0}^{m} \sum_{j=0}^{(p+1)(m-k)/p} h_{j,k} y^j {y_t}^k = 0, \qquad h_{0,m} = 1 \tag{2}$$

where p is an order of poles of the elliptic solutions, m is a positive integer and $h_{j,k}$ are constants to be determined. It should be noted that the Conte–Musette algorithm is the following [14] (see also [5]):

1. Choose a positive integer m, define the form of equation (2) and calculate the number of unknown coefficients $h_{j,k}$.
2. Construct some solutions of the system (1) in the form of Laurent series. If such solutions do not exist or they correspond to known exact solutions, then no unknown single-valued solutions exist. Note that, since system (1) is autonomous, the coefficients of the Laurent series do not depend on the position of the singular point. They may depend on values of the numerical parameters of (1). In addition, some of these coefficients (the number of which is less than the order of system (1) may take arbitrary values and have to be considered as new numerical parameters. One should compute more coefficients of the Laurent series than the number of numerical parameters in the Laurent series plus the number of $h_{j,k}$.
3. Choose a Laurent series expansion for some function y_k and substitute the obtained Laurent series coefficients into equation (2). This substitution transforms (2) into a linear and overdetermined system in $h_{j,k}$ with coefficients depending on numerical parameters.
4. Eliminate the coefficients $h_{j,k}$ and get a nonlinear system in the parameters.
5. Solve the so obtained nonlinear system.

Conte and Musette note that a computer algebra package is highly recommended for using their method [5]. Some steps of this algorithm can be implemented in computer algebra systems separately.

For a given system it is easy to calculate the Laurent series solutions to any accuracy. These computations are based on the Painlevé test, which has been implemented in the most popular computer algebra systems. Note that when a sufficient number of the Laurent series coefficients has been computed one can forget about the system of differential equations and work only with coefficients of the obtained series. The first package of computer algebra procedures, which realize the third and the fourth steps of the algorithm, has been written in AMP by Conte. One can also use our `Maple` and `REDUCE` packages of procedures, which are accessible via the internet [19] and are described in [20] and [21]. So, one passes the first four steps of algorithm without any difficulties.

At the fifth (and last) step one should solve an overdetermined system of nonlinear algebraic equations. The complexity of algorithm, which allows to solve such systems, in particular, the required computer memory, depends, in the general case, exponentially on the number of the unknowns. Therefore, this number should be made as small as possible. The purpose of this paper is to show that we can essentially simplify the algebraic system of equations, which we have to solve in the last step of the Conte–Musette method, if we search the elliptic solutions only.

In [14] Conte and Musette have used their method to find wave solutions of the **complex cubic Ginzburg–Landau equation** (CGLE3). The nonexistence of elliptic travelling and standing wave solutions of the CGLE3 has been proved in [10] and [22] respectively. In Section 4 we seek the elliptic solutions of the **quintic complex Ginzburg–Landau equation** (CGLE5) using our modification of the Conte–Musette method. Note that both the CGLE3 and the CGLE5 have only one-parameter Laurent-series solutions in the wave form and there exist only a finite number of such solutions.

Our approach can be effectively used in investigation of any system (1) or a single differential equation, for which only a finite number of different Laurent-series solutions exist. Note that for a wide class of such differential equations it has been proven that all their meromorphic solutions are elliptic (maybe degenerated) functions [8].

3. Properties of the Elliptic Functions

Let us recall some definitions and theorems. The function $\varrho(z)$ of the complex variable z is a doubly-periodic function if there exist two numbers ω_1 and ω_2 with $\omega_1/\omega_2 \notin \mathbb{R}$, such that for all $z \in \mathbb{C}$

$$\varrho(z) = \varrho(z + \omega_1) = \varrho(z + \omega_2).$$

By definition a double-periodic meromorphic function is called an elliptic function [9]. These periods define parallelograms with vertices z_0, $z_0 + N_1\omega_1$, $z_0 + N_2\omega_2$ and $z_0 + N_1\omega_1 + N_2\omega_2$, where N_1 and N_2 are arbitrary natural numbers and z_0 is an arbitrary complex number. The fundamental parallelogram of periods does not include other parallelogram of periods and corresponds to $N_1 = N_2 = 1$.

From the classical theorems for elliptic functions [9] we know that:

- If an elliptic function has no poles then it is a constant
- The number of the poles of any elliptic function within any finite period parallelogram is finite
- The sum of residues within any finite period parallelogram is equal to zero (**the residue theorem**)
- If $\varrho(z)$ is an elliptic function then any rational function of $\varrho(z)$ and its derivatives is an elliptic function as well
- For each elliptic function $\varrho(z)$ there exist a natural number m ($m \geq 2$) and coefficients $h_{i,j}$ such that $\varrho(z)$ is a solution of equation (2).

Lemma 1. *An elliptic function can not have two poles with the same Laurent series expansions in its fundamental parallelogram of periods.*

Proof: Let some elliptic function $\varrho(\xi)$ has two poles ξ_0 and ξ_1, which belong to the fundamental parallelogram of periods. The corresponding Laurent series are the same and have the convergence radius R. In this case the function $\upsilon(\xi) = \varrho(\xi - \xi_0) - \varrho(\xi - \xi_1)$ is an elliptic function as a difference between two elliptic functions with the same periods. At the same time for all ξ such that $|\xi| < R$ we have $\upsilon(\xi) = 0$, therefore, $\upsilon(\xi) \equiv 0$ and $\varrho(\xi - \xi_0) \equiv \varrho(\xi - \xi_1)$ and $\xi_1 - \xi_0$ is a period of $\varrho(\xi)$. It contradicts however to our initial assumption that both points ξ_0 and ξ_1 belong to the fundamental parallelogram of periods. □

4. Construction of Elliptic Solutions

4.1. The Quintic Complex Ginzburg–Landau Equation

The one-dimensional CGLE5 is a generalization of the one-dimensional CGLE3, which is one of the most studied nonlinear equations (see [3, 4] and references therein). Moreover, the CGLE5 is a generic equation which describes many physical phenomena, for example, the behaviour of travelling patterns in binary fluid convection [16] and the large scale behavior of many nonequilibrium pattern-forming systems [6]. The CGLE5 is as follows

$$\mathrm{i}\mathcal{A}_t + p\mathcal{A}_{xx} + q|\mathcal{A}|^2\mathcal{A} + r|\mathcal{A}|^4\mathcal{A} - \mathrm{i}\gamma\mathcal{A} = 0 \tag{3}$$

where subscribes denote partial derivatives: $\mathcal{A}_t \equiv \partial\mathcal{A}/\partial t$, $\mathcal{A}_{xx} \equiv \partial^2\mathcal{A}/\partial x^2$, $p, q, r \in \mathbb{C}$ and $\gamma \in \mathbb{R}$.

One of the most important directions in the study of the CGLE5 is the consideration of its travelling wave reduction [2, 5, 7, 13, 15, 16]

$$\mathcal{A}(x,t) = \sqrt{M(\xi)}\mathrm{e}^{\mathrm{i}(\varphi(\xi)-\omega t)}, \quad \xi = x - ct, \qquad c \in \mathbb{R}, \quad \omega \in \mathbb{R}. \tag{4}$$

Substituting (4) in (3) and multiplying both sides of this equation by $4M^2/A$ we obtain

$$\begin{aligned} 2pM''M - pM'^2 + 4\mathrm{i}p\psi MM' + 2(2\omega - \mathrm{i}c - 2\mathrm{i}\gamma + 2c\psi - 2p\psi^2 + 2\mathrm{i}p\psi')M^2 \\ + 4qM^3 + 4rM^4 = 0 \end{aligned} \tag{5}$$

where $\psi \equiv \varphi' \equiv \mathrm{d}\varphi/\mathrm{d}\xi$, $M' \equiv \mathrm{d}M/\mathrm{d}\xi$. Equation (5) is a system of two equations: both real and imaginary parts of its left-hand side have to be equal to zero. Dividing (5) by p and separating real and imaginary terms, we obtain

$$\begin{aligned} &2MM'' - M'^2 - 4M^2\tilde{\psi}^2 - 2\tilde{c}MM' + 4g_iM^2 + 4d_rM^3 + 4u_rM^4 = 0 \\ &M\tilde{\psi}' + \tilde{\psi}(M' - \tilde{c}M) - g_rM + d_iM^2 + u_iM^3 = 0 \end{aligned} \tag{6}$$

where the new real variables are defined as follows

$$u_r + \mathrm{i}u_i = \frac{r}{p}, \qquad d_r + \mathrm{i}d_i = \frac{q}{p}, \qquad s_r - \mathrm{i}s_i = \frac{1}{p}, \qquad \tilde{c} \equiv cs_i$$

$$g_r + \mathrm{i}g_i = (\gamma + \mathrm{i}\omega)(s_r - \mathrm{i}s_i) + \frac{1}{2}c^2 s_i s_r + \frac{\mathrm{i}}{4}c^2 s_r^2, \qquad \tilde{\psi} \equiv \psi - \frac{cs_r}{2}.$$

Our new system (6) includes seven numerical parameters: g_r, g_i, d_r, d_i, u_r, u_i and $\tilde{c}$. Note that in order to obtain (6) from (5) we have assumed that the functions $M(\xi)$ and $\psi(\xi)$ are real.

The standard way to construct exact solutions to the system (6) is to transform it into the equivalent third order differential equation for M. We rewrite the first equation of (6) as

$$\tilde{\psi}^2 = \frac{G}{M^2} \tag{7}$$

where

$$G \equiv \frac{1}{2}MM'' - \frac{1}{4}M'^2 - \frac{\tilde{c}}{2}MM' + g_i M^2 + d_r M^3 + u_r M^4.$$

Using (7) we express $\tilde{\psi}$ in terms of M and its derivatives

$$\tilde{\psi} = \frac{G' - 2\tilde{c}G}{2M^2(g_r - d_i M - u_i M^2)} \tag{8}$$

and obtain the third order equation for M

$$(G' - 2\tilde{c}G)^2 + 4GM^2(g_r - d_i M - u_i M^2)^2 = 0. \tag{9}$$

Below we consider the case $p/r \notin \mathbb{R}$, which corresponds to the condition $u_i \neq 0$. In this case the equation (9) is not integrable [13] and its general solution (which should depend on three arbitrary integration constants) is not known. Using the Painlevé analysis [13] it has been shown that single-valued solutions of (6) can depend on only one arbitrary parameter. System (6) is autonomous, so this parameter is ξ_0 and if $M = f(\xi)$ is a solution, then $M = f(\xi - \xi_0)$, where $\xi_0 \in \mathbb{C}$ has to be also a solution. Special solutions in terms of elementary functions have been found in [2, 5, 13, 15]. All known exact solutions of (6) are elementary (rational or hyperbolic) functions. The full list of these solutions is presented in [5]. The purpose of this section is to find at least one elliptic solution of (6).

Let us note that (6) is invariant under the transformation

$$\tilde{\psi} \to -\tilde{\psi}, \quad g_r \to -g_r, \quad d_i \to -d_i, \quad u_i \to -u_i$$

and therefore we can assume without loss of generality that $u_i > 0$. Moreover, using the scale transformations

$$M \to \lambda M, \quad d_r \to \frac{d_r}{\lambda}, \quad d_i \to \frac{d_i}{\lambda}, \quad u_r \to \frac{u_r}{\lambda^2}, \quad u_i \to \frac{u_i}{\lambda^2}$$

we can always put $u_i = 1$.

4.2. The Laurent Series Solutions

Let us construct Laurent series solutions to the system (6). We assume that in a sufficiently small neighborhood of the singularity point ξ_0 the functions $\tilde{\psi}$ and M tend to infinity as some powers of $\xi - \xi_0$, i.e.,

$$\tilde{\psi} = A(\xi - \xi_0)^{\alpha} \qquad \text{and} \qquad M = B(\xi - \xi_0)^{\beta} \tag{10}$$

where α and β are negative integer numbers and, of course, $A \neq 0$ and $B \neq 0$. Substituting (10) into (6) we obtain that two or more terms in the equations of system (6) balance if and only if $\alpha = \beta = -1$. In other words these terms have equal powers and the other terms can be ignored as $\xi \longrightarrow \xi_0$. We obtain the values of A and B from the following algebraic system

$$B^2\left(3 - 4A^2 + 4u_r B^2\right) = 0, \qquad 2A - B^2 = 0. \tag{11}$$

The last system has four nonzero solutions

$$A_1 = u_r + \frac{1}{2}\sqrt{4u_r^2 + 3}, \qquad B_1 = \sqrt{2u_r + \sqrt{4u_r^2 + 3}} \tag{12}$$

$$A_2 = u_r + \frac{1}{2}\sqrt{4u_r^2 + 3}, \qquad B_2 = -\sqrt{2u_r + \sqrt{4u_r^2 + 3}} \tag{13}$$

$$A_3 = u_r - \frac{1}{2}\sqrt{4u_r^2 + 3}, \qquad B_3 = \sqrt{2u_r - \sqrt{4u_r^2 + 3}} \tag{14}$$

and

$$A_4 = u_r - \frac{1}{2}\sqrt{4u_r^2 + 3}, \qquad B_4 = -\sqrt{2u_r - \sqrt{4u_r^2 + 3}}. \tag{15}$$

Correspondingly the system (6) has four types of Laurent series solutions. Let us denote them as follows

$$\tilde{\psi}_k = \frac{A_k}{\xi} + a_{k,0} + a_{k,1}\xi + \ldots, \quad M_k = \frac{B_k}{\xi} + b_{k,0} + b_{k,1}\xi + \ldots, \quad k = 1, \ldots, 4. \tag{16}$$

4.3. Elliptic Solutions

Let $M(\xi)$ is a nontrivial elliptic function. Note that if $\tilde{\psi}$ is a constant, then from the second equation of system (6) it follows that M can not be a nontrivial elliptic function. Therefore, using (8), we conclude that $\tilde{\psi}(\xi)$ has to be a nontrivial elliptic function as well.

Let us consider the fundamental parallelogram of periods for the function $M(\xi)$ and define a number of its poles in this domain. Let M has a pole of type M_1, hence, according to the residue theorem, it should have a pole of type M_2 (it can not have a pole of type M_4 because u_r is a real parameter). So $\tilde{\psi}$ has poles with the Laurent series $\tilde{\psi}_1$ and $\tilde{\psi}_2$. As an elliptic function it should have a pole of type $\tilde{\psi}_3$ or $\tilde{\psi}_4$ as well. This means that the function $M(\xi)$ should have a pole of type M_3 and, hence, a pole of type M_4. So $M(\xi)$ should have at least four different poles in its fundamental parallelogram of periods. Using Lemma 1, we obtain that the function $M(\xi)$ cannot have the same poles in the fundamental parallelogram of periods. Therefore, $M(\xi)$ has exactly four poles in its fundamental parallelogram of periods. In this case by means of the residue theorem for $\tilde{\psi}$ we obtain

$$u_r = 0. \tag{17}$$

Actually we have obtained that the CGLE5 with $u_r \neq 0$ has no elliptic solution in the wave form. In the case $u_r = 0$ all possible elliptic solutions should have four simple poles in the fundamental parallelogram of periods, and, therefore, have the following form [9]

$$M(\xi - \xi_0) = C + \sum_{k=1}^{4} B_k \zeta(\xi - \xi_k) \tag{18}$$

where the function $\zeta(\xi)$ is an integral of the Weierstrass elliptic function multiplied by -1

$$\zeta'(\xi) = -\wp(\xi)$$

and where C and ξ_k are constants to be defined. We should also define periods of the Weierstrass elliptic function.

To obtain restrictions on other parameters, we use the Hone method [10] and apply the residue theorem to the functions $\tilde{\psi}^2$, $\tilde{\psi}^3$, and so on. The residue theorem for the function $\tilde{\psi}^2$ gives the equation

$$\sum_{k=1}^{4} A_k a_{k,0} = 0. \tag{19}$$

Substituting A_k and $a_{k,0}$ in (19), we obtain

$$\sum_{k=1}^{4} A_k a_{k,0} = \frac{3}{4}\tilde{c} = 0 \quad \Rightarrow \quad \tilde{c} = 0.$$

For the function $\tilde{\psi}^3$ the residue theorem gives

$$\sum_{k=1}^{4} A_k \left(A_k a_{k,1} + a_{k,0}^2 \right) = 0. \tag{20}$$

Equation (20) is equivalent to

$${d_i}^2 + 27{d_r}^2 = 0 \quad \Rightarrow \quad d_i = \pm \mathrm{i}\sqrt{27} d_r.$$

The parameters d_r and d_i should be real, therefore, $d_r = 0$ and $d_i = 0$. So, consideration of $\tilde{\psi}^2$ and $\tilde{\psi}^3$ gives three restrictions

$$\tilde{c} = 0, \qquad d_r = 0 \qquad \text{and} \qquad d_i = 0. \tag{21}$$

The residue theorem for $\tilde{\psi}^4$ gives the restriction

$$g_i g_r = 0. \tag{22}$$

Taking into account (17) and (21) we obtain that the system (6) has the following form

$$\begin{aligned} &2MM'' - M'^2 - 4M^2\tilde{\psi}^2 + 4g_i M^2 = 0 \\ &\tilde{\psi}'M + \tilde{\psi}M' - g_r M + M^3 = 0. \end{aligned} \tag{23}$$

In order to find the corresponding elliptic solutions to the system (23) we use the Conte–Musette method. The function $M(\xi)$ in the parallelogram of periods has four different Laurent series expansions, so we should choose the parameter m such that solutions of equation (2) have four poles in its fundamental parallelogram of periods. The minimal possible value of m is equal to four. The general form of (2) for $m = 4$ and $p = 1$ is the following

$$\begin{aligned} &M'^4 + (h_{2,3}M^2 + h_{1,3}M + h_{0,3})M'^3 \\ &+ (h_{4,2}M^4 + h_{3,2}M^3 + h_{2,2}M^2 + h_{1,2}M + h_{0,2})M'^2 \\ &+ (h_{6,1}M^6 + h_{5,1}M^5 + h_{4,1}M^4 + h_{3,1}M^3 + h_{2,1}M^2 + h_{1,1}M + h_{0,1})M' \\ &+ h_{8,0}M^8 + h_{7,0}M^7 + h_{6,0}M^6 + h_{5,0}M^5 \\ &+ h_{4,0}M^4 + h_{3,0}M^3 + h_{2,0}M^2 + h_{1,0}M + h_{0,0} = 0. \end{aligned} \tag{24}$$

Substituting the Laurent series M_k from (16), we transform the left hand side of (24) into the Laurent series, which has to be equal to zero. Therefore, we obtain the algebraic system in $h_{i,j}$ and g_r. The first algebraic equation, which corresponds to $1/\xi^8$ is

$$B_k^4(h_{8,0}B_k^4 - h_{6,1}B_k^3 + h_{4,2}B_k^2 - h_{2,3}B_k + 1) = 0 \tag{25}$$

where B_k are defined by (12)–(15). Using the standard Conte–Musette method, which allows to find all elliptic and degenerate elliptic solutions, we can use only one of the Laurent series solutions (only one B_k) and can express, for example, $h_{8,0}$ via $h_{6,1}$, $h_{4,2}$ and $h_{2,3}$. As we seek only elliptic solutions, all B_k have to satisfy the system (25) which can be rewritten in the following form

$$\begin{aligned} h_{8,0}B_1^4 - h_{6,1}B_1^3 + h_{4,2}B_1^2 - h_{2,3}B_1 + 1 = 0 \\ h_{8,0}B_2^4 - h_{6,1}B_2^3 + h_{4,2}B_2^2 - h_{2,3}B_2 + 1 = 0 \\ h_{8,0}B_3^4 - h_{6,1}B_3^3 + h_{4,2}B_3^2 - h_{2,3}B_3 + 1 = 0 \\ h_{8,0}B_4^4 - h_{6,1}B_4^3 + h_{4,2}B_4^2 - h_{2,3}B_4 + 1 = 0. \end{aligned} \tag{26}$$

From (26) and other equations of the algebraic system we obtain $g_i = 0$ and that the equation for M has the form

$$M'^4 = \frac{1}{81} M^2 \left(3M^2 - 4g_r\right)^3. \tag{27}$$

If $g_r \neq 0$ then the solution of (27) is an elliptic function. On the other side, the equation (9) with $u_i = 1$, $u_r = 0$, $\tilde{c} = 0$, $d_r = 0$, $d_i = 0$ and $g_i = 0$ has the form

$$\frac{1}{4} \left(M'''\right)^2 - \left(2MM'' - M'^2\right) \left(M^2 - g_r\right)^2 = 0. \tag{28}$$

We multiply the equation (28) by M'^2 and use equation (27) to express all derivatives of $M(\xi)$ in terms of the function $M(\xi)$. A straightforward calculation shows that any solution of (27) satisfies (28). So, we obtain elliptic wave solutions of the CGLE5. If $s_i \neq 0$ these solutions are standing wave solutions, in the opposite case ($s_i = 0$) the solutions can have an arbitrary speed c.

Note that we have obtained (9) from (5) using the condition that $M(\xi)$ is a real function. For $g_r < 0$ and any initial value of M we obtain real solutions. In the case $g_r > 0$ there exists a minimal possible initial value of M for which real solutions exist and only particular solutions of (27) are suitable elliptic solutions to the CGLE5. The function $M(\xi)$ has the form (18) and the values of constants can be determined from (27).

Summing up we can conclude that our modification of the Conte–Musette method allows us to get two results: we have obtained new elliptic wave solutions of the CGLE5, and we have proved that these solutions are the unique elliptic solutions for the CGLE5 with $g_r \neq 0$.

From (22) it follows that elliptic solutions can exist if either g_r or g_i is equal to zero. For all nonzero values of g_r and zero g_i we have found elliptic solutions. In the case $g_r = g_i = 0$ there is no elliptic solution. In the case of zero g_r and nonzero g_i we substitute the so obtained Laurent series solutions M_k into equation (2) with $m = 1, \ldots, 4$ and obtain neither elliptic functions nor degenerate elliptic solutions. We hope that the more detailed analysis of this case will allows us to find all elliptic solutions for the CGLE5.

5. Conclusion

In this paper we have proposed a new approach for the search of elliptic solutions to some systems of differential equations. The proposed algorithm is a modification of the Conte–Musette method [14]. We restrict ourselves to the search of elliptic solutions only. A key idea of this restriction is to simplify the calculations by means of the use of a few Laurent series solutions instead of one and the use of the residue theorem.

The application of our approach to the quintic complex one-dimensional Ginzburg–Landau equation allows to find elliptic solutions in the wave form. Note that these solutions are the first elliptic solutions for the CGLE5. Using the investigation of CGLE5 as an example, we demonstrate that to find elliptic solutions the analysis of a system of differential equations is more preferable than the analysis of the equivalent single differential equation.

We also find restrictions on the coefficients, which are necessary conditions for the existence of elliptic solutions of CGLE5. To do this we improve the Hone method [10]. We show that this method is useful not only to prove the nonexistence of elliptic solutions, but also to find new elliptic solutions. Note that the Hone method and, therefore, our approach, are so effective in the case of the CGLE5, because the coefficients of the Laurent series solutions depend only on the parameters of the equations, i.e., they do not include additional arbitrary parameters (have no resonances). It is an important problem to generalize the Hone method for the Laurent series solutions with resonances.

Acknowledgements

This work has been supported in part by Russian Federation President's Grant NSh–8122.2006.2.

References

[1] Ablowitz M., Ramani A. and Segur H., *A Connection Between Nonlinear Evolution Equations and Ordinary Differential Equations of P-type I & II*, J. Math. Phys. **21** (1980) 715–721, 1006–1015.

[2] Akhmediev N., Afanasjev V. and Soto-Crespo J., *Singularities and Spesial Soliton Solutions of the Cubic-Quintic Complex Ginzburg–Landau Equation*, Phys. Rev. E **53** (1996) 1190–1201.

[3] Aranson I. and Kramer L., *The World of the Complex Ginzburg–Landau Equation*, Rev. Mod. Phys. **74** (2002) 99–143; cond-mat/0106115.

[4] Conte R. and Musette M., *Linearity Inside Nonlinearity: Exact Solutions to the Complex Ginzburg–Landau Equation*, Physica D **69** (1993) 1–17.

[5] Conte R. and Musette M., *Solitary Waves of Nonlinear Nonintegrable Equations*, A chapter in the book *Dissipative Solitons*, Lect. Notes Phys. vol. 661, N. Akhmediev and A. Ankiewicz (Eds), Springer, Berlin, 2005; nlin.PS/0407026.

[6] Cross M. and Hohenberg P., *Pattern Formation Outside of Equilibrium*, Rev. Mod. Phys. **65** (1993) 851–1112.

[7] Doelman A., *Breaking the Hidden Symmetry in the Ginzburg–Landau Equation*, Physica D **97** (1996) 398–428.

[8] Eremenko A., *Meromorphic Traveling Wave Solutions of the Kuramoto–Sivashinsky Equation*; nlin.SI/0504053.

[9] Hurwitz A., *Allgemeine Funktionentheorie und elliptische Funktionen*, Courant R., *Geometrische Funktionentheorie*, Springer, Berlin, 1964.

[10] Hone A., *Non-existence of Elliptic Travelling Wave Solutions of the Complex Ginzburg–Landau Equation*, Physica D **205** (2005) 292–306.

[11] Hone A., *Painlevé Test, Singularity Structure, and Integrability*; nlin.SI/0502017.

[12] Kudryashov N., *Analytic Theory of Nonlinear Differential Equations*, RCD, Moscow-Izhevsk, 2004.

[13] Marcq Ph., Chaté H. and Conte R., *Exact Solutions of the One-Dimensional Quintic Complex Ginzburg–Landau Equation*, Physica D **73** (1994) 305–317; patt-sol/9310004.

[14] Musette M. and Conte R., *Analytic Solitary Waves of Nonintegrable Equations*, Physica D **181** (2003) 70–79; nlin.PS/0302051.

[15] van Saarloos W. and Hohenberg P., *Fronts and Pulses in the Complex Ginzburg–Landau Equation near a Subcritical Bifurcations*, Phys. Rev. Lett. **64** (1990) 749–752.

[16] van Saarloos W. and Hohenberg P., *Fronts, Pulses, Sources and Sinks in Generalized Complex Ginzburg–Landau Equations*, Physica D **56** (1992) 303–367, Erratum **69** (1993) 209.

[17] Tabor M., *Chaos and Integrability in Nonlinear Dynamics*, Wiley, New York, 1989.

[18] Vernov S., *Constructing Solutions for the Generalized HÉnon–Heiles System through the Painlevé Test*, Theor. Math. Phys. (Teor. Mat. Fiz.) **135** (2003) 409–419 (in Russian), 792–801 (in English); math-ph/0209063.

[19] Vernov S., *Packages of Procedures for* `Maple 9` (`'ellipso'`) *and* `REDUCE 3.7`, 2005, `http://theory.sinp.msu.ru/~svernov/programs/`.

[20] Vernov S., *Construction of Single-Valued Solutions for Nonintegrable Systems with the Help of the Painlevé Test*, In: Computer Algebra in Scientific Computing, V. Ganzha, E. Mayr and E. Vorozhtsov (Eds), Technische Universitat, Munchen, 2004, pp. 457–465; nlin.SI/0407062

[21] Vernov S., *Interdependence Between the Laurent-Series and Elliptic Solutions of Nonintegrable Systems*, In: Lecture Notes of Computer Science, vol. 3718 (Proceedings of the International Conference "Computer Algebra in Scientific Computing" (CASC 2005, Kalamata, Greece)), V. Ganzha, E. Mayr and E. Vorozhtsov (Eds), Springer, Berlin, 2005, pp. 457–468.

[22] Vernov S., *Proof of the Absence of Elliptic Solutions of the Cubic Complex Ginzburg–Landau Equation*, Theor. Math. Phys. (Teor. Mat. Fiz.) **146** (2006) 131–139 (in English), 161–171 (in Russian); nlin.PS/0503009.

[23] Vernov S., *Painlevé Analysis and Exact Solutions of Nonintegrable Systems*, In: Geometry, Integrability and Quantization VII, I. Mladenov and M. de León (Eds), SOFTEX, Sofia, 2006, pp. 280–291.