Eighth International Conference on
Geometry, Integrability and Quantization
June 9–14, 2006, Varna, Bulgaria
Ivaïlo M. Mladenov and Manuel de León, Editors
SOFTEX, Sofia 2007, pp 292–301

ALGEBRAS WITH POLYNOMIAL IDENTITIES AND BERGMAN POLYNOMIALS

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Abstract. This paper is an introduction to the theory of algebras with polynomial identities. It stresses on matrix algebras and polynomial identities for them. The notion of Bergman polynomials is introduced. Such types of polynomials are investigated being identities for algebras with symplectic involution. In the Lie case more information is given for Bergman polynomials as Lie identities for the considered algebras.

1. Algebras with Polynomial Identities

We fix a countably infinite set $X = \{x_1, x_2, \dots\}$ and consider a field K of characteristic zero. We work in the algebra $K\langle X\rangle$ which has a basis the set of all words

$$x_{i_1} \dots x_{i_k}, \qquad x_{i_j} \in X$$

and multiplication defined by

$$(x_{i_1} \dots x_{i_m})(x_{j_1} \dots x_{j_n}) = x_{i_1} \dots x_{i_m} x_{j_1} \dots x_{j_n}.$$

Definition 1. i) *Let $f = f(x_1, \dots, x_n) \in K\langle X\rangle$ and let R be an associative algebra. We say that $f = 0$ is a **polynomial identity** for R if*

$$f(r_1, \dots, r_n) = 0, \qquad r_1, \dots, r_n \in R.$$

ii) *If the associative algebra R satisfies a non-trivial polynomial identity f (i.e., f is a nonzero element of $K\langle X\rangle$), we call it PI-algebra.*

It could be shown that $f \in K\langle X\rangle$ is a polynomial identity for R if and only if f is in the kernel of all homomorphisms $K\langle X\rangle \to R$. We give some examples:

Example 1. *The algebra R is **commutative** if and only if it satisfies the polynomial identity*

$$[x_1, x_2] = x_1x_2 - x_2x_1 = 0.$$

Example 2. *Let R be a finite dimensional associative algebra and let* $\dim R < n$. *Then R satisfies the standard identity of degree n*

$$s_n(x_1, \ldots, x_n) = \sum_{\sigma \in S_n} (\operatorname{sign} \sigma) x_{\sigma(1)} \ldots x_{\sigma(n)} = 0$$

*where S_n is the symmetric group of degree n. The algebra R also satisfies the **Capelli identity***

$$d_n(x_1, \ldots, x_n; y_1, \ldots, y_{n+1}) = \sum_{\sigma \in S_n} (\operatorname{sign} \sigma) y_1 x_{\sigma(1)} y_2 \ldots y_n x_{\sigma(n)} y_{n+1} = 0.$$

Example 3. *Let $M_2(K)$ be the 2×2 matrix algebra. It satisfies the following polynomial identities*:

1. *The standard identity $s_4(x_1, x_2, x_3, x_4) = 0$*
2. *The Hall identity $[[x_1, x_2]^2, x_3] = 0$.*

The algebra $M_2(K)$ does not satisfy the Capelli identity $d_4 = 0$ and the standard identity $s_3 = 0$.

Example 4. *The $n \times n$ matrix algebra $M_n(K)$ satisfies the identity of algebraicity*

$$\begin{aligned} &d_{n+1}(1, x, x^2, \ldots, x^n; 1, y_1, \ldots, y_n, 1) \\ &= \sum_{\sigma \in S_{n+1}} (\operatorname{sign} \sigma) x^{\sigma(0)} y_1 x^{\sigma(1)} y_2 \ldots y_n x^{\sigma(n)} = 0 \end{aligned}$$

where the symmetric group S_{n+1} acts on $\{0, 1, \ldots, n\}$, and the identity

$$s_n([x, y], [x^2, y], \ldots, [x^n, y]) = 0.$$

Example 5. *Let $U_n(K)$ be the algebra of $n \times n$ upper triangular matrices. It satisfies the identity*

$$[x_1, x_2] \ldots [x_{2n-1}, x_{2n}] = 0.$$

Some important properties of the associative algebras are expressed in the language of polynomial identities. We have seen this for the commutativity. Other examples come from the nonunitary algebras. The algebra R is nilpotent of bounded index if there exists $n \in N$ such that $x^n = 0$ is an identity for R. The algebra R is nilpotent of class $\leq n$ if $x_1 \ldots x_n = 0$ for R. More details could be found in [4]. It turns out that the class of all PI-algebras has a structure and combinatorial properties similar to those of the commutative and the finite dimensional algebras.

2. Bergman Polynomials in Associative PI-Algebras

We define a class of homogeneous associative polynomials, called Bergman polynomials [1]. These are homogeneous and multilinear in $y_1, \dots, y_n$ polynomials $f(x, y_1, \dots, y_n)$ from the free associative algebra $K\langle x, y_1, \dots, y_n\rangle$ which can be written as

$$f(x, y_1, \dots, y_n) = \sum_{i=(i_1,\dots,i_n)\in S_n} v(g_i)(x, y_{i_1}, \dots, y_{i_n}) \tag{1}$$

where $g_i \in K[t_1, \dots, t_{n+1}]$ are homogeneous polynomials in commuting variables

$$g_i(t_1, \dots, t_{n+1}) = \sum \alpha_p t_1^{p_1} \dots t_{n+1}^{p_{n+1}} \tag{2}$$

and

$$v(g_i) = v(g_i)(x, y_{i_1}, \dots, y_{i_n}) = \sum \alpha_p x^{p_1} y_{i_1} \dots x^{p_n} y_{i_n} x^{p_{n+1}}. \tag{3}$$

The following theorem of Bergman shows how one could investigate PI-algebras using commutative theory approach.

Proposition 1 ([1, Section 6, (27)]). i) *The polynomial $v(g_i)$ from* (3) *is an identity for $M_n(K)$ if and only if*

$$\prod_{1\le p<q\le n+1} (t_p - t_q)$$

divides $g_i(t_1, \dots, t_{n+1})$ for all $i = (i_1, \dots, i_n)$.
ii) *The polynomial $f(x, y_1, \dots, y_n)$ from* (1) *is an identity for $M_n(K)$ if and only if every summand $v(g_i)$ is also an identity for $M_n(K)$.*

If the algebra has some additional properties the analog of the Bergman theorem could be formulated. These properties are concerned with the existence of involutions (i.e., antiautomorphisms of second order) in the considered algebras.

We recall that in the matrix algebra $M_{2n}(K, *)$ the symplectic involution $*$ is defined by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^* = \begin{pmatrix} D^t & -B^t \\ -C^t & A^t \end{pmatrix}$$

where A, B, C, D are $n \times n$ matrices and t is the usual transpose.

For an algebra R with involution $*$ we have the splitting $(R, *) = R^+ \oplus R^-$, where $R^\pm = \{r \in R, r^* = \pm r\}$.

We call $f(x_1, \dots, x_n) \in K\langle X\rangle$ a $*$-polynomial identity for $(R, *)$ in symmetric variables if $f(r_1^+, \dots, r_n^+) = 0$ for all $r_1^+, \dots, r_n^+ \in R^+$. Analogously $f(x_1, \dots, x_s) \in K\langle X\rangle$ is a $*$-polynomial identity for $(R, *)$ in skew-symmetric variables if $f(r_1^-, \dots, r_s^-) = 0$ for all $r_1^-, \dots, r_s^- \in R^-$.

The algebra R^+ is a Jordan algebra with respect to the multiplication $r_1^+ \circ r_2^+ = r_1^+ r_2^+ + r_2^+ r_1^+$ where $r_1^+, r_2^+ \in R^+$ and the identities in symmetric variables are weak polynomial identities for the pair (R, R^+).

Similarly, the algebra R^- is a Lie algebra with respect to the new multiplication $[r_1^-, r_2^-] = r_1^- r_2^- - r_2^- r_1^-$ where $r_1^-, r_2^- \in R^-$ and the identities in skew-symmetric variables for $(R, *)$ are weak polynomial identities for the pair (R, R^-).

In order to state the next result we introduce the following notation, namely

$$g_{2n,0} = \prod_{\substack{1 \le p < q \le n+1 \\ (p,q) \ne (1,n+1)}} (t_p^2 - t_q^2)(t_1 - t_{n+1}).$$

Proposition 2 ([9, Theorem 3]). *Considered in $M_{2n}(K, *)$ the polynomial f introduced by* (1) *satisfies $f(a, r_1, \ldots, r_n) = 0$ for any skew-symmetric matrix a and all matrices $r_1, \ldots, r_n$ if and only if $(t_1 + t_{n+1})g_{2n,0}$ divides the polynomials $g_i(t_1, \ldots, t_{n+1})$ for all $i = (i_1, \ldots, i_n)$.*

The sufficient condition of this proposition could be improved.

Proposition 3 ([6, Theorem 1]). *Let the polynomial* (1) *be a $*$-identity in skew-symmetric variables for the algebra $M_{2n}(K, *)$. Then the polynomial $g_{2n,0}$ divides the polynomials g_i in equation* (2) *for all $i = (i_1, \ldots, i_n)$.*

Some other results are the following:

Proposition 4 ([8, Proposition 3]). *The linearization in y of the standard polynomial $S_3([x^3, y], [x^2, y], [x, y])$ is an identity in symmetric variables for $M_6(K, *)$ of minimal degree.*

Proposition 5 ([5, Theorem 3]). *A polynomial f is a Bergman type identity in skew-symmetric variables for $M_4(K, *)$ if and only if it has the form*

$$\begin{aligned} f = \alpha(v(g_1)(x, y_1, y_2) + v(g_2)(x, y_2, y_1)) \\ + \beta v(g_3)(x, y_1, y_2) + \gamma v(g_4)(x, y_2, y_1) \end{aligned}$$

where

1. *$g_1 = g_{4,0} \prod_i (a_i t_1 + b_i t_2 + c_i t_3)$, $g_2 = g_{4,0} \prod_i (-c_i t_1 - b_i t_2 - a_i t_3)$ and $t_1 + t_3$ is not a factor of the polynomials g_1 and g_2*
2. *The polynomial $(t_1 + t_3)g_{4,0}$ divides g_3 and g_4*
3. *The identity $v(g_1)(x, y_1, y_2) + v(g_2)(x, y_2, y_1) = 0$ follows from the identity $f_0(x, y_1, y_2) = \sum_{\sigma \in S_2} v(g_{4,0})(x, y_{\sigma(1)}, y_{\sigma(2)}) = 0$.*

We are able to formulate and prove a result for $M_{2n}(K, *)$ generalizing "only if" part of Proposition 5 for $n = 2$.

Theorem 1 ([8, Theorem 3]). *For $n \equiv 2,3 \pmod 4$ every Bergman type polynomial of degree k of the form*

$$f = \alpha \sum_i v(g_i)(x, y_{i_1}, \dots, y_{i_n}) + \beta \sum_j v(g_j)(x, y_{j_1}, \dots, y_{j_n})$$

is a $*$-identity in skew-symmetric variables for $M_{2n}(K,*)$, where

1. $g_i \;=\; g_{2n,0} \prod_{l=1}^{k-n^2-2n+1} \sum_{m=1}^{n} a_{i,m}^{(l)} t_m$

   $g_{i+\frac{n!}{2}} \;=\; g_{2n,0} \prod_{l=1}^{k-n^2-2n+1} (-\sum_{m=1}^{n} a_{i,n+1-m}^{(l)} t_m)$

   $i = 1, \dots, \frac{n!}{2}$ *and $t_1 + t_{n+1}$ is not a factor of these polynomials*
2. *The polynomial $(t_1 + t_{n+1}) g_{2n,0}$ divides g_j*
3. *The identity $\sum v(g_i)(x, y_{i_1}, \dots, , y_{i_n}) = 0$ follows from the identity*

   $$v(g_{2n,0})(x, y_{i_1}, y_{i_2}, \dots, y_{i_n}) + v(g_{2n,0})(x, y_{i_n}, y_{i_{n-1}}, \dots, y_{i_1})$$

   for $(i_1, i_2, \dots, i_n) \in S_n$.

3. Lie Algebras

Starting with the free Lie algebra $L(X)$ we can define the notion of a polynomial identity for a Lie algebra G. We give some examples:

Example 6. *Let G be the two-dimensional Lie algebra with basis $\{a, b\}$ as a vector space and multiplication $[a, b] = a$. G satisfies the polynomial identity (namely the metabelian identity)*

$$[[x_1, x_2], [x_3, x_4]] = 0.$$

Example 7. *If G is a finite dimensional Lie algebra and* $\dim G < n$, *then G satisfies the Lie standard identity of degree $n+1$ (but in n skew-symmetric variables)*

$$x_0 s_n(\operatorname{ad} x_1, \dots, \operatorname{ad} x_n) = \sum_{\sigma \in S_n} (\operatorname{sign} \sigma)[x_0, x_{\sigma(1)}, \dots, x_{\sigma(n)}] = 0.$$

Example 8. *The Lie algebra $(U_n(K))^{(-)}$ of all upper triangular $n \times n$ matrices satisfies the identity*

$$[[x_1, x_2], \dots, [x_{2n-1}, x_{2n}]] = 0.$$

Example 9. *Let W_n be the set of all derivations of the polynomial algebra in n variables. (The linear operator δ of the vector space $K[x_1, \dots, x_n]$ is a derivation if $\delta(uv) = \delta(u)v + u\delta(v)$ where $u, v \in K[x_1, \dots, x_n]$.) W_n is a Lie algebra with respect to the operation $[\delta_1, \delta_2]$.*

1. *The algebra W_1 satisfies the standard Lie identity*

$$x_0 s_4(\operatorname{ad} x_1, \operatorname{ad} x_2, \operatorname{ad} x_3, \operatorname{ad} x_4) = 0.$$

2. *The algebra W_n satisfies some standard Lie identity.*

As in the case of associative algebras, some classical properties and results for Lie algebras can be stated in the language of polynomial identities. A Lie algebra G is abelian if it satisfies the identity $[x_1, x_2] = 0$ meaning that G has a trivial multiplication. The algebra G is nilpotent of class $\leq n$ if it satisfies $[x_1, \dots, x_n] = 0$. The algebra G is solvable of class $\leq n$ if it satisfies the identity $f_n = 0$, where $f_n = f_n(x_1, \dots, x_{2^n})$, is defined inductively by

$$\begin{aligned} f_1(x_1, x_2) &= [x_1, x_2] \\ f_n &= [f_{n-1}(x_1, \dots, x_{2^{n-1}}), f_{n-1}(x_{2^{n-1}+1}, \dots, x_{2^n})], \qquad n > 1. \end{aligned}$$

The solvable Lie algebras of class two are called metabelian. Any solvable finite dimensional Lie algebra satisfies the identity

$$[[x_1, x_2], \dots, [x_{2n-1}, x_{2n}]] = 0$$

for some positive n.

4. Bergman Polynomials in Lie Algebras

It is a natural question to consider Bergman polynomials in Lie algebras as well. Working in the Lie algebra $\mathfrak{so}(4, K, *)$ of the skew-symmetric with respect to the symplectic involution $*$ variables of the matrix algebra of fourth order $M_4(K, *)$ we are interested in finding the minimal degree of these polynomials. In [7] using the Hall basis of the free algebra $L(X)$ for $X = \{x, y_1, y_2\}$ we have considered the following elements of a given degree $k + 2$: $[y_{i_1}, \underbrace{x, \dots, x}_{k}, y_{i_2}]$ and $[[y_{i_1}, \underbrace{x, \dots, x}_{l}], [y_{i_2}, \underbrace{x, \dots, x}_{k-l}]]$, where (i_1, i_2) is any permutation of $\{1, 2\}$ and $l = 1, \dots, k-1$.

The left normed commutators are written as elements of the free associative algebra $K\langle X\rangle$ and thus the commutative polynomials are uniquely defined.

For example for

$$\begin{aligned} f = [y_1, x, y_2] &= y_1 x y_2 - x y_1 y_2 - y_2 y_1 x + y_2 x y_1 \\ &= v(g_1)(x, y_1, y_2) + v(g_2)(x, y_2, y_1) \end{aligned}$$

we have $g_1 = t_2 - t_1$ and $g_2 = -(t_3 - t_2)$.

For

$$\begin{aligned} f = [[y_1, x], [y_2, x, x]] &= y_1 x y_2 x^2 - 2 y_1 x^2 y_2 x + y_1 x^3 y_2 \\ &- x y_1 y_2 x^2 + 2 x y_1 x y_2 x - x y_1 x^2 y_2 - y_2 x^2 y_1 x + y_2 x^3 y_1 \\ &+ 2 x y_2 x y_1 x - 2 x y_2 x^2 y_1 - x^2 y_2 y_1 x + x^2 y_2 x y_1 \\ &= v(g_1)(x, y_1, y_2) + v(g_2)(x, y_2, y_1) \end{aligned}$$

one gets $g_1 = (t_2 - t_1)(t_3 - t_2)^2$ and $g_2 = -(t_2 - t_1)^2 (t_3 - t_2)$.

We denote the Lie algebra of the skew-symmetric due to the simplectic involution variables of $M_{2n}(K, *)$ as $\mathfrak{so}(2n, K, *)$.

Applying Proposition 3 and some technical manipulations we get the following result:

Proposition 6 ([7, Theorem 1]). *No Bergman polynomials are Lie identities for the Lie algebra* $\mathfrak{so}(4, K, *)$.

There was a comment made during the Conference on the possibility of getting the above result from more general considerations connected with the Lie structure of the algebra. But the pattern of proof of Proposition 6 introduced in [7] gives the possibility for an analogous result concerning $\mathfrak{so}(6, K, *)$ as well.

Investigating the identities of minimal degree for the Lie algebras $\mathfrak{so}(4)$, $\mathfrak{so}(3,1)$, $\mathfrak{so}(2,2)$ and $\mathfrak{sp}(4, R)$ considered in [3] is a natural step in research. For the physical reason of considering the above algebras we will mention that the Lie algebra $\mathfrak{so}(3,1)$ is the algebra of the Lorentz group SO$(3,1)$. The motion of a charged particle in a constant electromagnetic field can be described by a system of four linear differential equations, the so called **Lorentz equations** [2]. The time independent electromagnetic field is represented by a second order tensor

$$A = \begin{pmatrix} 0 & B_3 & -B_2 & E_1 \\ -B_3 & 0 & B_1 & E_2 \\ B_2 & -B_1 & 0 & E_3 \\ E_1 & E_2 & E_3 & 0 \end{pmatrix}$$

where E_1, E_2, E_3 and B_1, B_2, B_3 are the components of the electric respectively magnetic fields (A is an arbitrary element of the Lie algebra $\mathfrak{so}(3,1)$).

Theorem 2. *For the Lie algebras* $\mathfrak{so}(4)$, $\mathfrak{so}(3,1)$ *and* $\mathfrak{so}(2,2)$ *there exists a Bergman polynomial of type* (1) *of minimal degree five, which is a Lie identity. All Bergman type Lie identities in these algebras are consequences of the minimal identity.*

Proof: Elements of all considered algebras are matrices of the type

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

where A and B are some matrices of second order. Thus, we could apply the well known result that an identity for the considered subalgebras of $M_4(K)$ is an ordinary identity for $M_2(K)$. Thus due to the Amitsur–Levitski theorem the degree of the polynomial f has to be at least four.

We use the Hall basis of the free Lie algebra $L(X)$ on the set $X = \{x, y_2, y_1\}$, namely the elements of type $[y_{i_1}, x, \ldots, x, y_{i_2}]$ and $[[y_{i_1}, x, \ldots, x], [y_{i_2}, x, \ldots, x]]$. Thus, the polynomial $f = f_4$ of degree four has the form

$$f_4 = \alpha[[y_1, x], [y_2, x]] + \beta[y_1, x, x, y_2] + \gamma[y_2, x, x, y_1].$$

Writing $f_4 = v(g_1)(x, y_1, y_2) + v(g_2)(x, y_2, y_1)$ we get

$$g_1 = \alpha(t_2 - t_1)(t_3 - t_2) + \beta(t_2 - t_1)^2 - \gamma(t_3 - t_2)^2.$$

Applying Proposition 1 we get $g_1(t_1 = t_2) = 0$ and therefore $\gamma = 0$. As $g_1(t_2 = t_3) = 0$ we have also $\beta = 0$. And $g_1(t_1 = t_3) = 0$ implies $\alpha = 0$.

It means that there is no Bergman type polynomial of degree four, which is a Lie identity.

Considering degree five, we get

$$\begin{aligned} f = f_5 &= a[[y_1, x], [y_2, x, x]] + b[[y_2, x], [y_1, x, x]] \\ &\quad + c[y_1, x, x, x, y_2] + d[y_2, x, x, x, y_1] \\ &= v(g_1)(x, y_1, y_2) + v(g_2)(x, y_2, y_1). \end{aligned}$$

Thus $g_1 = c(t_2 - t_1)^3 - d(t_3 - t_2)^3 + a(t_2 - t_1)(t_3 - t_2)^2 - b(t_2 - t_1)^2(t_3 - t_2)$. As f_5 is an identity for $M_2(K)$ we apply Proposition 1 again and get

$$g_1(t_1 = t_2) = 0 \Rightarrow d = 0$$
$$g_1(t_2 = t_3) = 0 \Rightarrow c = 0$$

$g_1(t_1 = t_3) = 0$ from which we get $(t_2 - t_1)^3(a + b) = 0,$ hence $a + b = 0$.

Considering the linearization of f_5, we could evaluate it only for the basic elements of the three considered algebras.

For the algebra $\mathfrak{so}(4)$ these are the elements

$$\begin{aligned} a_1 &= e_{21} - e_{12}, & a_4 &= e_{32} - e_{23} \\ a_2 &= e_{13} - e_{31}, & a_5 &= e_{42} - e_{24} \\ a_3 &= e_{41} - e_{14}, & a_6 &= e_{43} - e_{34}. \end{aligned}$$

For the algebra $\mathfrak{so}(3, 1)$ they are

$$\begin{aligned} b_1 &= e_{21} - e_{12}, & b_4 &= e_{32} - e_{23} \\ b_2 &= e_{13} - e_{31}, & b_5 &= e_{42} + e_{24} \\ b_3 &= e_{41} + e_{14}, & b_6 &= e_{43} + e_{34}. \end{aligned}$$

For the algebra $\mathfrak{so}(2,2)$ the basis are the elements

$$\begin{aligned} c_1 &= e_{12} - e_{21}, & c_4 &= e_{32} + e_{23} \\ c_2 &= e_{13} + e_{31}, & c_5 &= e_{42} + e_{24} \\ c_3 &= e_{41} + e_{14}, & c_6 &= e_{34} - e_{43}. \end{aligned}$$

We consider all the possibilities for the arguments of f_5 and make the corresponding evaluations. This procedure is done by `Mathematica` and it gives either trivial results or $a + b = 0$.

Thus f_5 is the minimal Bergman polynomial of degree five being a Lie identity for these algebras.

It is easy to be realized that the consequences of f_5 for higher degrees are

$$f_6 = \alpha f_5(x, [y_1, x], y_2) + \beta f_5(x, y_1, [y_2, x])$$

and

$$f_7 = \alpha f_5(x, [y_1, x], [y_2, x]) + \beta f_6(x, [y_1, x], y_2) + \gamma f_6(x, y_1, [y_2, x]).$$

They have only factors related to f_5 commutative polynomials g_1 and g_2.

Thus all Bergman type Lie identities for the algebras $\mathfrak{so}(4)$, $\mathfrak{so}(3,1)$ and $\mathfrak{so}(2,2)$ are consequences of the identity $f_5 = 0$ and this finishes the proof of the theorem. $\square$

Remark 1. *For the Lie algebra* $\mathfrak{sp}(4, R)$ *does not exist a Bergman polynomial of type* (1)*, which is a Lie identity.*

We point that the algebra $\mathfrak{sp}(4, R)$ is the algebra $\mathfrak{so}(4, R, *)$ and for it Proposition 6 is valid.

Acknowledgements

The author is grateful to Dr. Ivaïlo Mladenov from the Institute of Biophysics of the Bulgarian Academy of Sciences for giving the author the opportunity to understand that knowing more science not only mathematics is a good perspective.

This work was partially supported by the National Science Fund of the Bulgarian Ministry of Education and Science under Grant MM1503/2005.

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