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ON THE SYMMETRIES OF THE MANEV PROBLEM AND ITS REAL HAMILTONIAN FORM

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Abstract. The Manev model and its real form dynamics are known to possess Ermanno–Bernoulli type invariants similar to the Laplace–Runge–Lenz vector of the Kepler model. Using these additional invariants, we demonstrate here that both Manev model and its real Hamiltonian form posses the same $\mathfrak{so}(3)$ or $\mathfrak{so}(2,1)$ symmetry algebras (in addition to the angular momentum algebra) on angular momentum level sets. Thus Kepler and Manev models are shown to have identical symmetry algebras and hence sharing more features than previously expected.

1. Introduction

Since Kepler and Newton the elliptical trajectories became the new archetype of the (bounded) planetary motion and the circular orbit nowadays is viewed upon rather as a degenerate ellipse than as an embodiment of perfection. The advent of Einstein's theory did not produce a new archetype of heavenly motions, apart from the exceptional case of a collapse into the black holes. Nevertheless, among the variety of relativistic effects the perihelion shift of the inner planets and the light deflection in a gravity field are definitely the best recognizable effects in the Solar system. Maybe it is time to accept a new archetype of heavenly motions: *precessing ellipse* (or more generally, precessing conics). If precessing conics give us "the typical" motion of planets it is tempting to ask which central force field produces them. Surprisingly or not, the answer is: *the Manev model* – see equation (1) (and [2] for the precise formulation of the statement).

One may easily get the suspicion that Manev problem actually offers a larger natural family of models with common properties, among which Kepler model is a kind of degenerate case (just like the circle is a degenerate case among the conics). There are several types of arguments supporting this view:

- First, as we already said, it is a sensible generalization of Newton's gravitation law
- Second, there are *stability* arguments as Kepler-type motion is generally not preserved by small perturbations and any sort of "real world" interactions like Solar pressure, drag, etc., would destroy "fixed ellipse" motion; while in the Manev model we have persistent KAM tori and cylinders for a large class of even non-Hamiltonian perturbations [16]
- Third, Kepler problem is famous as one of archetypes of *superintegrable* systems – kind of property which is also assumed to be easily destroyed by small perturbations. Recently we reported [15] that Manev model has an additional independent globally defined constant of motion, albeit not for all initial data. Let us remark that for a generic central potential we could have disjoint set of initial data corresponding to closed orbits but in our case *all* points on certain level sets of the angular momentum lie on closed orbits which are intersections with the level sets of the additional invariant
- Fourth, Kepler and Manev problems share a common $\mathfrak{so}(2,1) \simeq \mathfrak{su}(1,1)$ algebra associated with the radial motion and among the possible realizations of this algebra the Manev model forms a class of its own [3,5].

Here we want to add one more argument supporting the view that the Manev model is the natural generalization of the Kepler model.

A celebrated feature of the Kepler problem is its *large symmetry algebras* $\mathfrak{so}(3)$ (or $\mathfrak{so}(2,1)$ for positive energies) in addition to the angular momentum algebra. Here we demonstrate that the Manev model has *exactly the same symmetry algebras* on angular momentum level sets provided we choose the "right" Hamiltonian description of its dynamics. Even more, the same symmetry algebras are also possessed by the real form dynamics of the Manev model – a closely connected dynamical model which we had already shown to be superintegrable for *all* initial data. This also stresses the importance of $\mathfrak{so}(3)$ or $\mathfrak{so}(2,1)$ dynamical symmetries due to their relevance for a wider class of models.

2. The Manev Problem Basics

By **Manev model** [20] we mean here the dynamics given by the Hamiltonian

$$H = \frac{1}{2}(p_x^2 + p_y^2 + p_z^2) - \frac{A}{r} - \frac{B}{r^2}, \qquad r = \sqrt{x^2 + y^2 + z^2}. \tag{1}$$

Here A and B are assumed to be arbitrary real constants whose positive values correspond to attractive forces. The genuine model proposed by George Manev was not invented as an approximation of relativity theory but as a consequence of (more general) Planck's action–reaction principle and he had derived a specific value for the constant $B = \frac{3G}{2c^2}A$, where G is the Newton universal constant and c is the light velocity. (As Manev was the first to derive it from the first principles his name has been attached to this model despite the fact that it is known since Newton.) The Manev model offers a surprisingly good practical approximation to Einstein's relativistic dynamics – at least at a Solar system level – capable to describe both the perihelion advance of the inner planets and the Moon's perigee motion. It was also argued in [10] that in a certain approximative regime it is the natural classical analog of the Schwarzschild problem. In the last decade it had enjoyed an increased interest either as a very suitable approximation from astronomers' point of view or as a toy model for applying different techniques of the modern dynamics (see, e.g., [4, 8, 10, 22, 23]).

Due to the rotational invariance each component of the angular momentum

$$L_j = \varepsilon_{jkm}\, p_k x_m \quad \text{with} \quad (x_1, x_2, x_3) = (x, y, z)$$

is an obvious first integral $\{H,\, L_j\} = 0$ and the model is Liouville integrable since $H,\, \vec{L}^2$ and any component, say L_z, of the angular momentum are in involution. The components themselves are not in involution but span an $\mathfrak{so}(3)$ algebra with respect to the Poisson bracket

$$\{L_j\,, L_k\} = \varepsilon_{jkm}\, L_m \tag{2}$$

and if we approach the question of the integrability solely in L_j terms, we obtain the most simple example of non-commutative integrability [24, 25].

The dynamics is confined on a plane which we assume to be XOY and is separable in polar coordinates r and $\theta = \arctan(y/x)$. On the reduced phase space (see, e.g., [14] for the generalities of the reduction procedure) obtained by fixing the angular momentum $L_z \equiv L$ to a certain value ℓ, the motion is governed by the reduced Hamiltonian

$$H_{\text{red}} = \frac{1}{2}\left(p_r^2 + \frac{\ell^2 - 2B}{r^2}\right) - \frac{A}{r}$$

and the "standard" symplectic structure

$$\omega = \mathrm{d}p_r \wedge \mathrm{d}r + \mathrm{d}L \wedge \mathrm{d}\theta. \tag{3}$$

The dynamics behave like radial motion of Kepler dynamics with angular momentum squared $\ell^2 - 2B$, while the case $2B > \ell^2$ corresponds to overall centripetal effect. On the other hand, the angular equation of motion $\dot{\theta} = \ell/r^2$ is still governed by the "authentic" angular momentum ℓ (and r is as just described). Consequently, the remarkable properties of Kepler dynamics that all negative energy orbits are closed and the frequencies of radial and angular motions coincide (for any initial conditions) are no more true. Thus we may have not only purely classical perihelion shifts we may have collapsing trajectories which are spirals but also if $2B \geq \ell^2 \neq 0$, even though in phase space they are symplectic transformations. Notice that in the Kepler dynamics the only allowed fall down is along straight lines. For this reason the set of initial data leading to collision has a positive measure and this may offer an explanation why collisions in the Solar system are estimated to happen more often than Newton theory predicts [9].

3. The Kepler Problem Invariants

In the case of Kepler problem, corresponding to $B = 0$, we have more first integrals (for details and historical notes see, e.g., [6, 17, 18, 26])

$$J_x = p_y L - \frac{A}{r} x\,, \qquad J_y = -p_x L - \frac{A}{r} y\,, \qquad \{H_K, \vec{J}\} = 0$$

where H_K is the Kepler Hamiltonian and J_x and J_y are the components of the Laplace–Runge–Lenz vector. They are not independent since

$$J^2 = 2H_K L^2 + A^2.$$

Together with the Hamiltonian and angular momentum they close the Lie algebra (with respect to the Poisson bracket)

$$\{H_K, L\} = 0\,, \quad \{L, J_x\} = J_y\,, \quad \{L, J_y\} = -J_x\,, \quad \{J_x, J_y\} = -2H_K L.$$

After redefining $\vec{E} = \vec{J}/\sqrt{|-2h_K|}$ *on each* $H_K = h_K$ *level set* we get

$$\{L, E_x\} = E_y\,, \qquad \{L, E_y\} = -E_x\,, \qquad \{E_x, E_y\} = -\operatorname{sign}(h_K)\, L$$

which makes obvious the fact that we have an $\mathfrak{so}(3)$ algebra for negative energies and $\mathfrak{so}(2,1)$ for positive ones. In the case of the three-dimensional Kepler problem the components of the angular momentum give us another copy of $\mathfrak{so}(3)$, see equation (2), so that the full symmetry algebra is $\mathfrak{so}(4)$ or $\mathfrak{so}(3,1)$ depending on the sign of h_K.

According to [18], the first use of these first integrals was made by J. Hermann (= J. Ermanno) in 1710 (in order to find *all* possible orbits under an inverse square law force) in the disguise of **Ermanno–Bernoulli constants**

$$J_\pm = J_x \pm \mathrm{i} J_y = \left(\frac{L^2}{r} - A \mp \mathrm{i}\, L p_r \right) \mathrm{e}^{\pm \mathrm{i}\theta}$$

which satisfy

$$\{H_K, J_\pm\} = 0\,, \qquad \{L, J_\pm\} = \pm \mathrm{i} J_\pm\,, \qquad \{J_+, J_-\} = -4\mathrm{i} H_K L\,. \tag{4}$$

4. The Manev Problem Invariants and Symmetries

Let us start with the observation that Manev Hamiltonian could be expressed as

$$H = D_+ D_- - \frac{A^2}{\nu^2 L^2}$$

where

$$D_\pm = p_r \pm \mathrm{i}\left(\nu p_\perp - \frac{A}{\nu L}\right), \quad p_\perp = \frac{L}{r} \quad \text{and} \quad \nu^2 = \frac{\ell^2 - 2B}{\ell^2}\,.$$

$D_\pm$ have the property $\frac{\mathrm{d}}{\mathrm{d}\theta} D_\pm = \mp \mathrm{i}\nu D_\pm$ and hence $\frac{\mathrm{d}}{\mathrm{d}\theta} D_\pm \mathrm{e}^{\pm \mathrm{i}\nu\theta} = 0$.

Noting that $\mathrm{d}t = \frac{r^2}{\ell}\,\mathrm{d}\theta$ and multiplying by the first integral L (in order to achieve more similarity to "Ermanno–Bernoulli" constants) we obtain the invariance of

$$\mathcal{J}_\pm = L D_\pm \mathrm{e}^{\pm \mathrm{i}\nu\theta} = L\left[p_r \pm \mathrm{i}\left(\nu p_\perp - \frac{A}{\nu L}\right)\right] \mathrm{e}^{\pm \mathrm{i}\nu\theta}\,, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{J}_\pm = 0\,. \tag{5}$$

Of course, $\mathcal{J}_+$ and $\mathcal{J}_-$ are not independent as

$$\mathcal{J}_+ \mathcal{J}_- = 2HL^2 + \frac{A^2}{\nu^2}\,.$$

Obviously in the Kepler case ν equals one and (up to a multiplication by i) we recover the "Ermanno–Bernoulli" constants. When $A = 0$ we obtain "$1/r^2$ model" which attracted a lot of interest in the 1970's due to its conformal symmetry providing it with an algebra of explicitly time-dependent quantities [7]. The corresponding restriction $\mathcal{J}_\pm\big|_{A=0}$ seems to give previously unknown first integrals for "$1/r^2$ model" as well. One must note, however, some connection with the results of Feher [11] who uses the perihelion vector in the case of repulsive interaction (hence noncompact motion) to construct time-independent first integrals with large symmetry algebra. Actually the perihelion vectors were promoted as first integrals for any central field in [12] and utilized for some time afterwards before it was realized that they experience jumps in their direction when we have compact motion and correspondingly they are not genuine first integrals in this case.

4.1. Compact Motion Case

In the case when $\ell \neq 0$, $\ell^2 > 2B$, $H < 0$ and $A > 0$ the motion is on a two-dimensional torus. In order to have globally defined constants of motion in this case we have to require that the real valued ν's be rational, i.e.,

$$\nu = \sqrt{\ell^2 - 2B} : \ell = m : k \tag{6}$$

with m and k mutually prime integers. Then due to equation (5) the quantities

$$\mathcal{J}_\pm = \mathcal{J}_\mp^* = L\left[p_r \pm \mathrm{i}\left(\frac{m}{k}p_\perp - \frac{kA}{mL}\right)\right]\mathrm{e}^{\pm\mathrm{i}\nu\theta}$$

are conserved by the flow of equation (1) on any surface $L = \ell$ satisfying the rationality condition (6). Thus we have conditional constants of motion which exist only for disjoint but infinite set of values ℓ, cf. *the invariant relations of* [19]. The trajectory in the configuration space is a "rosette" with m petals and this is connected to the fact that $\mathcal{J}_\pm$ are invariant under the action of the cyclic group generated by rotations to the angle $\frac{2\pi k}{m}$

$$\theta \to \theta + 2\pi\frac{k}{m}n\,, \qquad n = 0, 1, \ldots, m-1\,.$$

While in the Kepler case we could unambiguously attach the Laplace–Runge–Lenz vector to Ermanno–Bernoulli invariants this is not possible now due to this finite symmetry. (It is intuitively clear that if the Laplace–Runge–Lenz vector points to the perihelion of the fixed Kepler ellipse, now we have m petals to choose between.)

4.2. Noncompact Motion Cases

- When $0 \neq \ell^2 > 2B$ and $H \geq 0$ the additional invariants are always globally defined and have the form just described with ν real.
- In the case when $0 \neq \ell^2 < 2B$ we may introduce $\upsilon = \mathrm{i}\nu$ with υ real and respectively

$$\mathcal{J}_\pm = L\left[p_r \pm \left(\upsilon p_\perp + \frac{A}{\upsilon L}\right)\right]\mathrm{e}^{\pm\upsilon\theta} \tag{7}$$

  will be first integrals for *any* ℓ.
- Finally, when $\ell^2 = 2B$ we have the first integral

$$j = Lp_r + A\theta$$

  satisfying $\{H, j\} = 0$, $\{L, j\} = A$.

4.3. Symmetry Algebra

We had already commented [15] that Poisson brackets of $\mathcal{J}_+$ and $\mathcal{J}_-$ does not close on a "nice" algebra. Fortunately, one may note that on any $L = \ell$ surface Manev's model dynamical vector field could be also described by the pair

$$H^{\diamondsuit} = \frac{1}{2}\left(p_r^2 + \nu^2\frac{L^2}{r^2}\right) - \frac{A}{r}\,, \qquad \omega^{\diamondsuit} = \mathrm{d}p_r \wedge \mathrm{d}r + \nu^2 \mathrm{d}L \wedge \mathrm{d}\theta \tag{8}$$

hence it is bi-Hamiltonian in this restricted sense. We shall denote the Poisson brackets corresponding to $\omega^{\diamondsuit}$ by $\{\ ,\ \}^{\diamondsuit}$ and it is easy to check that

$$\{\mathcal{J}_+, \mathcal{J}_-\}^{\diamondsuit} = -\frac{4\mathrm{i}}{\nu}HL$$

analogously to equation (4), thus obtaining a closed algebra together with

$$\{H, L\}^{\diamondsuit} = 0, \qquad \{H, \mathcal{J}_\pm\}^{\diamondsuit} = 0, \qquad \{L, \mathcal{J}_\pm\}^{\diamondsuit} = \pm\frac{\mathrm{i}}{\nu}\mathcal{J}_\pm\,.$$

Defining in the case of $0 \neq \ell^2 > 2B$

$$K_1 = \frac{\nu}{2\sqrt{|2H|}}(\mathcal{J}_+ + \mathcal{J}_-)\,, \qquad K_2 = \frac{\mathrm{i}\nu}{2\sqrt{|2H|}}(\mathcal{J}_+ - \mathcal{J}_-)\,, \qquad K_3 = \nu L$$

we obtain $\mathfrak{so}(3)$ or $\mathfrak{so}(2,1)$ algebra

$$\{K_1, K_2\}^{\diamondsuit} = \operatorname{sign}(-H)K_3\,, \qquad \{K_2, K_3\}^{\diamondsuit} = K_1\,, \qquad \{K_3, K_1\}^{\diamondsuit} = K_2$$

with Casimir invariant

$$K_1^2 + K_2^2 + \operatorname{sign}(-H)K_3^2 = \frac{A^2}{|2H|}$$

and so, the space of invariants is lying on a sphere or hyperboloid (which degenerate to a point or cone if $A = 0$).

Thus, the angular momentum and the analogues of the Laplace–Runge–Lenz vector components in Manev model have *exactly the same algebra* as in the Kepler model provided we choose the "right" Hamiltonian formulation of its dynamics – presented in equation (8).

Similarly, in the case of $0 \neq \ell^2 < 2B$ (which has no direct analogue in the Kepler mechanics) the new Poisson brackets of the invariants in (7) are

$$\{\mathcal{J}_+, \mathcal{J}_-\}^{\diamondsuit} = \frac{4HL}{v}$$

and we can define

$$K_1 = \frac{v}{2\sqrt{|2H|}}(\mathcal{J}_+ + \mathcal{J}_-)\,, \qquad K_2 = \frac{v}{2\sqrt{|2H|}}(\mathcal{J}_+ - \mathcal{J}_-)\,, \qquad K_3 = vL$$

and in this way to obtain $\mathfrak{so}(2,1)$ algebra

$$\{K_1, K_2\}^{\diamondsuit} = \operatorname{sign}(-H) K_3\,, \qquad \{K_2, K_3\}^{\diamondsuit} = K_1\,, \qquad \{K_3, K_1\}^{\diamondsuit} = -K_2$$

for both choices of the sign of H. The Casimir invariant is

$$K_1^2 - K_2^2 + \operatorname{sign}(-H) K_3^2 = -\frac{A^2}{|2H|}\,.$$

The mere existence of an algebra of well defined first integrals does not presuppose suitable group action on the phase space. Here we have an immediate obstacle for the existence of group actions as $\mathcal{J}_\pm$ (or K_1, K_2) do not commute with L and hence do not preserve any $L = \ell$ surface.

If we would like to find a "nice" algebra having some chance to yield a group action in the phase space it should be an algebra of rotationally invariant functions (i.e., commuting with L). Such an $\mathfrak{so}(2,1)$ algebra had actually been obtained at the end of 1960's as a tool for determining the energy levels in the quantum Manev model (but without calling it so) [1]. It is worth noting that this algebra somehow distinguishes the Manev model as soon it was demonstrated [5] that this is the most general model (under some set of assumptions) with discrete and continuous spectrum having this algebra. A more recent survey [3] reported that the only explicit potentials realizing $\mathfrak{su}(1,1)$ (isomorphic to $\mathfrak{so}(2,1)$) algebra with discrete spectrum are Manev, Morse and **spiked oscillator** (i.e., $V = ar^2 + b/r^2$) ones. In the classical case the basis of the algebra is defined by

$$T_1 = \frac{1}{2}\left(rp^2 - \frac{2B}{r} - r\right), \qquad T_2 = \vec{p}\cdot\vec{r}, \qquad T_3 = \frac{1}{2}\left(rp^2 - \frac{2B}{r} + r\right)$$

so that

$$\{T_1, T_2\} = T_3\,, \qquad \{T_2, T_3\} = -T_1\,, \qquad \{T_3, T_1\} = -T_2$$

and its Casimir invariant is

$$T_3^2 - T_1^2 - T_2^2 = \ell^2 - 2B\,.$$

Thus the space of T_i-s is two- or one-sheeted hyperboloid for positive/negative values of $\ell^2 - 2B$, or a cone if $\ell^2 = 2B$. Let us note that this is not a symmetry algebra as its elements do not commute with the Hamiltonian, and as all its elements commute with L it does not matter whether we use Poisson brackets corresponding to ω – equation (3), or to $\omega^{\diamondsuit}$ – equation (8).

The Hamiltonian does not depend linearly on T_i-s but the combination

$$rH = \frac{1}{2}(T_1 + T_3) - A$$

does and hence through an appropriate reparametrization the radial motion could be represented as a dynamics on the algebra. To see this let us note that

$$\{r(H-h), T_i\} = r\{H, T_i\} + (H-h)\{r, T_i\}$$

and hence on each $H = h$ level set

$$\{r(H-h), T_i\} = r\{H, T_i\} = r\frac{\mathrm{d}T_i}{\mathrm{d}t}$$

so that the H-evolution is just the reparametrized $\left[\frac{1}{2}(T_1 + T_3) - h(T_3 - T_1) - A\right]$-evolution.

Remark 1. *There exists also an algebra which is closed for the Poisson brackets corresponding to the initial symplectic structure equation* (3) *formed by* L *and*

$$\Sigma_\pm = rD_\pm \mathrm{e}^{\pm\frac{\mathrm{i}\theta}{\nu}}$$

as

$$\{L, \Sigma_\pm\} = \pm\frac{\mathrm{i}}{\nu}\Sigma_\pm\,, \qquad \{\Sigma_+, \Sigma_-\} = 2\mathrm{i}\nu L.$$

This algebra may be considered as a generalization of the algebra used in [21] *to obtain the energy levels of the Coulomb problem. One should note that* $\Sigma_\pm$ *are neither* H-, *nor* L-*invariant making this algebra less appealing for our purposes.*

5. Real Form Dynamics of the Manev Problem

We shall be dealing here with the "real form dynamics" of the Manev model. The notion of **real form dynamics** has been introduced very recently and we refer the interesting reader to [13] for its motivation, definition and a list of examples.

The Manev Hamiltonian (and the canonical symplectic form as well) is invariant under the involution $\mathcal{C}$ reflecting the y-degree of freedom

$$\mathcal{C}(x) = x, \qquad \mathcal{C}(y) = -y, \qquad \mathcal{C}(z) = z$$
$$\mathcal{C}(p_x) = p_x, \qquad \mathcal{C}(p_y) = -p_y, \qquad \mathcal{C}(p_z) = p_z.$$

Consequently, the "real form dynamics" of Manev model for this choice of involution will be given by

$$H_{\mathbb{R}} = \frac{1}{2}(p_x^2 - p_y^2 + p_z^2) - \frac{A}{\rho} - \frac{B}{\rho^2}\,, \qquad \omega_{\mathbb{R}} = \mathrm{d}p_x \wedge \mathrm{d}x - \mathrm{d}p_y \wedge \mathrm{d}y + \mathrm{d}p_z \wedge \mathrm{d}z$$

where $\rho = \sqrt{x^2 - y^2 + z^2}$ is the "radius" of the pseudo-sphere. This is not an ordinary central field dynamics but rather an "indefinite metric central field" as $H_{\mathbb{R}}$ depends on indefinite metric distance ρ. The real form Hamiltonian $H_{\mathbb{R}}$ and the appropriate "angular momentum" $\tilde{L}_j$ are still commuting first integrals and the model is integrable. The involution acts on $\tilde{L}_j$ according to

$$\mathcal{C}(\tilde{L}_j) = (-1)^j \tilde{L}_j \quad \text{and} \quad \{\tilde{L}_j, \tilde{L}_k\} = \varepsilon_{jki}(-1)^{i+1}\tilde{L}_i$$

instead of equation (2). The resulting algebra is $\mathfrak{so}(2,1)$ which is the real form of $\mathfrak{so}(3)$ corresponding to C taking the rôle of Cartan involution.

We shall assume again that the motion is on the XOY-plane and in order to avoid the question of the behavior of trajectories on the singularities we restrict our attention on the C-invariant configuration space

$$\{(x, y, z) \in \mathbb{R}^2\,;\, z = 0, x^2 > y^2, x > 0\}\,.$$

Then the dynamics is separable in pseudo-polar coordinates $\rho \in (0, \infty)$ and $\vartheta = \operatorname{artanh}(y/x) \in (-\infty, \infty)$

$$H = \frac{1}{2}\left(p_\rho^2 - \frac{\tilde{L}^2}{\rho^2}\right) - \frac{A}{\rho} - \frac{B}{\rho^2}\,, \qquad \omega = \mathrm{d}p_\rho \wedge \mathrm{d}\rho + \mathrm{d}\tilde{L} \wedge \mathrm{d}\vartheta$$

with $\tilde{L} \equiv \tilde{L}_z = \pi_\vartheta$, hence $\dot{\pi}_\vartheta = 0$ and $\dot{\vartheta} = -\tilde{L}/\rho^2$.

The analysis of the resulting trajectories could be found in [15] and we shall not reproduce it here but we shall concentrate on the symmetry properties of the model.

5.1. Symmetries of the Real Form Manev Model

Proceeding as before, one could easily find the additional first integrals. What is different is that since the motion is never on a two-torus the new integrals are always globally defined on each $\tilde{L} = \ell$ level set for *all* initial data. We have to consider the following cases:

Case 1. When $0 \neq \ell^2 > -2B$ the integrals take the form

$$\mathcal{J}_\pm = \tilde{L}\left[p_\rho \mp \left(\upsilon p_\perp + \frac{A}{\upsilon \tilde{L}}\right)\right] \mathrm{e}^{\pm \upsilon \vartheta}$$

with

$$\upsilon^2 = \frac{\ell^2 + 2B}{\ell^2}\,, \qquad p_\perp = \frac{\tilde{L}}{\rho}\cdot$$

$\mathcal{J}_\pm$ are not independent as

$$\mathcal{J}_+ \mathcal{J}_- = -2H\tilde{L}^2 + \frac{A^2}{\upsilon^2}\,.$$

The alternative Hamiltonian description

$$H^\diamondsuit = \frac{1}{2}\left(p_\rho^2 - \upsilon^2 \frac{\tilde{L}^2}{\rho^2}\right) - \frac{A}{\rho}\,, \quad \omega^\diamondsuit = \mathrm{d}p_\rho \wedge \mathrm{d}\rho + \upsilon^2 \mathrm{d}\tilde{L} \wedge \mathrm{d}\vartheta$$

again provides us with new Poisson brackets $\{\ ,\ \}^\diamondsuit$ corresponding to $\omega^\diamondsuit$ and it is easy to check that

$$\{\mathcal{J}_+, \mathcal{J}_-\}^\diamondsuit = -\frac{4}{\upsilon} H L\,.$$

Now we can define

$$K_1 = \frac{\upsilon}{2\sqrt{|2H|}}(\mathcal{J}_+ + \mathcal{J}_-)\,, \quad K_2 = \frac{\upsilon}{2\sqrt{|2H|}}(\mathcal{J}_+ - \mathcal{J}_-)\,, \quad K_3 = \upsilon L$$

to obtain $\mathfrak{so}(2,1)$ algebra

$$\{K_1, K_2\}^\diamondsuit = \operatorname{sign}(-H)K_3\,, \quad \{K_2, K_3\}^\diamondsuit = K_1\,, \quad \{K_3, K_1\}^\diamondsuit = -K_2$$

for both choices of the sign of H. Its Casimir invariant is

$$K_1^2 - K_2^2 + \operatorname{sign}(-H)K_3^2 = -\frac{A^2}{|2H|}.$$

Case 2. In the case when $0 \neq \ell^2 < -2B$ let $\nu^2 = \frac{-(\ell^2+2B)}{\ell^2}$ and we obtain new invariants which are globally defined for *any* ℓ

$$\mathcal{J}_\pm = \mathcal{J}_\mp^* = \tilde{L}\left[p_\rho \mp \mathrm{i}\left(\nu p_\perp - \frac{A}{\nu\tilde{L}}\right)\right]\mathrm{e}^{\pm\mathrm{i}\nu\vartheta}$$

with

$$\mathcal{J}_+\mathcal{J}_- = 2H\tilde{L}^2 + \frac{A^2}{\nu^2}\,.$$

Defining

$$K_1 = \frac{\nu}{2\sqrt{|2H|}}(\mathcal{J}_+ + \mathcal{J}_-)\,, \quad K_2 = \frac{\mathrm{i}\nu}{2\sqrt{|2H|}}(\mathcal{J}_+ - \mathcal{J}_-)\,, \quad K_3 = -\nu L$$

we obtain $\mathfrak{so}(3)$ or $\mathfrak{so}(2,1)$ algebra

$$\{K_1, K_2\}^\diamondsuit = \operatorname{sign}(-H)K_3\,, \quad \{K_2, K_3\}^\diamondsuit = K_1\,, \quad \{K_3, K_1\}^\diamondsuit = K_2$$

with Casimir invariant

$$K_1^2 + K_2^2 + \operatorname{sign}(-H)K_3^2 = \frac{A^2}{|2H|}.$$

Case 3. When $\ell^2 = -2B$ we have the first integral

$$j = Lp_\rho - A\vartheta$$

satisfying $\{H, j\} = 0$, $\{\tilde{L}, j\} = -A$.

6. Conclusions

We have shown that Manev model not only possesses Ermanno–Bernoulli type invariants but also has exactly the same symmetry algebras $\mathfrak{so}(3)$ or $\mathfrak{so}(2,1)$ as the Kepler model (in addition to the angular momentum algebra). This indicates that Manev model shares its most celebrated mathematical features and is the most natural candidate for its generalization.

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