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# A PLATEAU PROBLEM FOR COMPLETE SURFACES IN THE DE-SITTER THREE-SPACE

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**Abstract.** In this paper we establish some existence and uniqueness theorems for a Plateau problem at infinity for complete spacelike surfaces in $\mathbb{S}_1^3$ whose mean and Gauss–Kronecker curvatures verify the linear relationship $2\varepsilon(H-1)-(\varepsilon+1)(K-1)=0$ for $-\varepsilon \in \mathbb{R}^+$.

## 1. Introduction

The global approach to surfaces with a constant curvature is a subject of many studies in Submanifolds Geometry, especially of that ones whose structure equations are integrable in terms of holomorphic data, because it represent a powerful tool in the study of these surfaces. Some representative examples are the Enneper–Weierstrass representation for minimal surfaces in $\mathbb{R}^3$ [13] and the McNertney–Kobayashi one for maximal surfaces in $\mathbb{L}^3$ presented in [9].

In this paper we will deal with spacelike surfaces in $\mathbb{S}_1^3$, a topic developed in the recent years. For instance, in the compact case Ramanathan [14] proved that every compact spacelike surface in $\mathbb{S}_1^3$ with constant mean curvature is totally umbilical. On the other hand, Li [10] showed that every compact spacelike surface in $\mathbb{S}_1^3$ with constant Gaussian curvature is totally umbilical.

As a natural generalization of Ramanathan and Li results, Aledo and Gálvez [2] characterized the totally umbilical round spheres of $\mathbb{S}_1^3$ as the only compact linear Weingarten spacelike surfaces.

In this work we study a special case of linear Weingarten surfaces of Bianchi type, in short BLW-surfaces, studied in [3]. We center our attention on BLW-surfaces whose mean and Gauss–Kronecker curvatures verify the linear relationship

$$2\varepsilon(H-1)-(1+\varepsilon)(K-1)=0, \qquad -\varepsilon \in \mathbb{R}^+. \tag{1}$$

The paper is organized as follows. In Section 2 we introduce the notation and the main concepts, as well as some results exposed in [3] for BLW-surfaces, but rewritten in our context. These preliminary results show the existence of a special Riemannian metric $\sigma$ on any BLW-surface and ensure that its hyperbolic Gauss map is conformal for the conformal structure induced by $\sigma$. This fact will be the key to obtain a Weierstrass representation for such a surface in terms of meromorphic data which generalizes the one given by Aiyama and Akutagawa [1] for $H = 1$ and by Gálvez, Martínez and Milán [6] for flat surfaces.

Using these results for the case of complete surfaces, we conclude that such surfaces are conformally equivalent to the unit disk $\mathbb{D}$ and, up to a conformal transformation of $\mathbb{D}$, they are in correspondence with the set of meromorphic maps $G : \mathbb{D} \longrightarrow \mathbb{C} \cup \{\infty\}$ with bounded Schwarzian derivative. Also we see that the representation formula can be written in terms of its hyperbolic Gauss maps. This fact allows us to prove that every complete BLW-surface satisfying (1) whose hyperbolic Gauss map has no poles is embedded.

Finally, we will give the existence and uniqueness results (see Proposition 1 and Theorem 4) for the following Plateau problem:

*Given $\varepsilon_0 < 0$ and a Jordan curve $\Gamma$ on $\mathbb{S}^2_\infty \equiv \Pi \cup \{\infty\}$, find a complete BLW-surface $\psi : S \longrightarrow \mathbb{S}_1^3$ verifying*

$$2\varepsilon_0(H-1) - (\varepsilon_0+1)(K-1) = 0$$

*and such that $\Gamma$ is its asymptotic boundary.*

## 2. Preliminaries

Let us denote by $\mathbb{L}^4$ the four-dimensional **Lorentz–Minkowski space** given as the vector space $\mathbb{R}^4$ with the Lorentzian metric $\langle \cdot, \cdot \rangle$ induced by the quadratic form $-x_0^2 + x_1^2 + x_2^2 + x_3^2$, and consider the **de Sitter space** realized as the Lorentzian submanifold

$$\mathbb{S}_1^3 = \left\{(x_0, x_1, x_2, x_3) \in \mathbb{L}^4;\ -x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1\right\}.$$

It is well known that $\mathbb{S}_1^3$ inherits from $\mathbb{L}^4$ a time-orientable Lorentzian metric which makes it the standard model of a Lorentzian space of constant sectional curvature one.

We will also consider the **hyperbolic space**

$$\mathbb{H}^3 = \left\{(x_0, x_1, x_2, x_3) \in \mathbb{L}^4;\ -x_0^2 + x_1^2 + x_2^2 + x_3^2 = -1,\ x_0 > 0\right\}$$

and the **positive null cone** given by

$$\mathbb{N}_+^3 = \left\{(x_0, x_1, x_2, x_3) \in \mathbb{L}^4;\ -x_0^2 + x_1^2 + x_2^2 + x_3^2 = 0,\ x_0 > 0\right\}.$$

In addition, $\mathbb{L}^4$ will be considered as the space of $2 \times 2$ Hermitian matrices, $\mathrm{Herm}(2)$, in the following way

$$(x_0, x_1, x_2, x_3) \longmapsto \begin{pmatrix} x_0 - x_3 & x_1 + \mathrm{i}x_2 \\ x_1 - \mathrm{i}x_2 & x_0 + x_3 \end{pmatrix}$$

where $\langle m, m\rangle = -\det(m)$ for all $m \in \mathrm{Herm}(2)$. Thus, $\mathbb{S}^3_1$ corresponds to the set of matrices with a determinant $-1$. Moreover, the action of $\mathrm{SL}(2,\mathbb{C})$ on $\mathrm{Herm}(2)$

$$g \cdot m = gmg^\dagger, \qquad g \in \mathrm{Herm}(2) \text{ and } g^\dagger = {}^t\overline{g}$$

preserves the inner product, orientations and, therefore, $\mathbb{S}^3_1$ remains unchanged.

In this model the positive null cone can be regarded as the set of positive semi-definite Hermitian matrices with vanishing determinant and its elements can be written as $w\,{}^t\overline{w}$, where ${}^t w = (w_1, w_2)$ is a non zero vector in $\mathbb{C}^2$ uniquely determined, up to multiplication, by a unimodular complex number. Moreover, the map $w\,{}^t\overline{w} \to [(w_1, w_2)] \in \mathbb{CP}^1$ induces one from $\mathbb{N}^3/\mathbb{R}^+$ which identifies $\mathbb{S}^2_\infty$ with $\mathbb{CP}^1$. Thereby, the natural action of $\mathrm{SL}(2,\mathbb{C})$ on $\mathbb{S}^2_{+\infty}$ is the action of $\mathrm{SL}(2,\mathbb{C})$ on $\mathbb{CP}^1$ by Möbius transformations.

A smooth immersion $\psi : S \longrightarrow \mathbb{S}^3_1$ of a two-dimensional connected manifold $S$ is said to be a **spacelike surface** if the induced metric via $\psi$ is a Riemannian metric on $S$, which, as usual, is also denoted by $\langle\cdot,\cdot\rangle$. The time-orientation of $\mathbb{S}^3_1$ allows us to choose a timelike unit normal field $\eta$ globally defined on $S$, tangent to $\mathbb{S}^3_1$, and hence we may assume that $S$ is oriented by $\eta$.

Associated to $\psi$, let us consider the map $\phi : S \longrightarrow \mathbb{N}^3_+$ given by $\phi := \psi + \eta = (\phi_0, \phi_1, \phi_2, \phi_3)$. Then the **hyperbolic Gauss map** of $\psi$ is defined as the map

$$G = \frac{\phi_1 + \mathrm{i}\phi_2}{\phi_0 + \phi_3} \in \mathbb{C} \cup \{\infty\}$$

or equivalently

$$G \equiv \left(\frac{\phi_1}{\phi_0}, \frac{\phi_2}{\phi_0}, \frac{\phi_2}{\phi_0}\right) \in \mathbb{S}^2 \subseteq \mathbb{R}^3$$

(see [4]). In addition, by considering the natural inclusion $\mathbb{R}^3 \longrightarrow \mathbb{L}^4$ given by $(x_1, x_2, x_3) \longmapsto (1, x_1, x_2, x_3)$ we can also identify

$$G \equiv \frac{1}{\phi_0}(\phi_0, \phi_1, \phi_2, \phi_3) \in \mathbb{N}^3_+.$$

Also, it is easy to check that $\phi$ can be written in terms of $G$ and $\rho = \phi_0 + \phi_3$ as

$$\phi = \frac{\rho}{2}(1 - |G|^2, G + \overline{G}, -\mathrm{i}(G - \overline{G}), 1 - |G|^2) \in \mathbb{N}^3_+. \tag{2}$$

Given a spacelike immersion $\psi : S \longrightarrow \mathbb{S}^3_1$, we will denote by $I = \langle \mathrm{d}\psi, \mathrm{d}\psi\rangle$, $II = \langle \mathrm{d}\psi, -\mathrm{d}\eta\rangle$ and $III = \langle \mathrm{d}\eta, \mathrm{d}\eta\rangle$ its first, second and third fundamental forms, respectively.

Let $A$ be a Riemannian metric on $S$ and let $z$ be some conformal parameter for the metric $A$. Given a two-form $B = L\mathrm{d}z^2 + 2M|\mathrm{d}z|^2 + N\mathrm{d}\bar{z}^2$, we define $Q(B, A)$ as the two-form $Q(B, A) = L\mathrm{d}z^2$. Observe that, in particular, $Q(II, I)$ is nothing but the Hopf differential of the immersion $\psi$.

In this paper we will deal with the family of linear Weingarten surfaces in $\mathbb{S}^3_1$ whose mean and Gauss–Kronecker curvatures, $H$ and $K$ respectively, satisfy a linear relation of the type

$$2\varepsilon(H-1) - (\varepsilon+1)(K-1) = 0.$$

We will refer to these surfaces as **$\varepsilon$-surfaces**.

First, we will establish some results for $\varepsilon$-surfaces which are included in a forthcoming paper [3]. These results work for a wide class of linear Weingarten surfaces containing the $\varepsilon$-surfaces. The first one states that the hyperbolic Gauss map $G$ is conformal for the structure given by the metric $\sigma = \varepsilon\, I - (1+\varepsilon)\, II$ and it shows the relationship between the metric of $\phi$ on $\mathbb{N}^3_+$ and $\sigma$.

**Theorem 1.** *Let $\psi : S \longrightarrow \mathbb{S}^3_1$ be an $\varepsilon$-surface, with normal $\eta : S \longrightarrow \mathbb{H}^3$ and hyperbolic Gauss map $G$. Then $\sigma = \varepsilon\, I - (1+\varepsilon)\, II$ is a Riemannian metric on the chosen surface $S$. In addition, the hyperbolic Gauss map is conformal for $\sigma$. Moreover, $I_\phi = \langle \mathrm{d}\phi, \mathrm{d}\phi\rangle$ is a metric which is conformal to $\sigma$.*

**Remark 1.** In this setting, let $z$ be a local conformal parameter for $\sigma$. Then, using (2), the first fundamental form of $\phi$ is given by

$$I_\phi = 2\langle \phi_z, \phi_{\bar{z}}\rangle |\mathrm{d}z|^2 = \rho^2\, |G_z|^2\, |\mathrm{d}z|^2.$$

The second one establishes a conformal type representation for $\varepsilon$-surfaces in terms of its Gauss map and a solution of a *Liouville-type equation*.

**Theorem 2.** *Let $\psi : S \longrightarrow \mathbb{S}^3_1$ be a non totally umbilical $\varepsilon$-surface satisfying* (1) *with a normal $\eta : S \longrightarrow \mathbb{H}^3$ and hyperbolic Gauss map $G$. Let us take $\phi := \psi + \eta = (\phi_0, \phi_1, \phi_2, \phi_3)$ and $\rho = \phi_0 + \phi_3$. Given a local conformal parameter $z$ for the metric $\sigma$ on $S$, $\psi$ and $\eta$ can be recovered as*

$$\begin{aligned}
\psi_0 &= -\frac{1}{\rho}\left(1 - (\rho/2)^2(1+|G|^2)\right) - \frac{2}{\rho^2}\Re\left(\frac{G\rho_z}{G_z}\right) - \frac{|\rho_z|^2\,(1+|G|^2)}{|G_z|^2\,\rho^3}\\
\psi_1 + \mathrm{i}\psi_2 &= \frac{\rho G}{2} - 2\frac{\rho_{\bar{z}}}{\rho^2 \overline{G_z}} - 2\frac{|\rho_z|^2\, G}{|G_z|^2\,\rho^3} \qquad (3)\\
\psi_3 &= -\frac{1}{\rho}\left(-1 - (\rho/2)^2(1-|G|^2)\right) + \frac{2}{\rho^2}\Re\left(\frac{G\rho_z}{G_z}\right) - \frac{|\rho_z|^2\,(1-|G|^2)}{|G_z|^2\,\rho^3}
\end{aligned}$$

*and*

$$\begin{aligned}
\eta_0 &= \frac{1}{\rho}\left(1+(\rho/2)^2(1+|G|^2)\right)+\frac{2}{\rho^2}\Re\left(\frac{G\rho_z}{G_z}\right)+\frac{|\rho_z|^2(1+|G|^2)}{|G_z|^2\rho^3}\\
\eta_1+\mathrm{i}\eta_2 &= \frac{\rho G}{2}+2\frac{\rho_{\bar z}}{\rho^2\overline{G}_z}+2\frac{|\rho_z|^2 G}{|G_z|^2\rho^3} \qquad (4)\\
\eta_3 &= \frac{1}{\rho}\left(-1+(\rho/2)^2(1-|G|^2)\right)-\frac{2}{\rho^2}\Re\left(\frac{G\rho_z}{G_z}\right)+\frac{|\rho_z|^2(1-|G|^2)}{|G_z|^2\rho^3}
\end{aligned}$$

*where $\Re(w)$ stands for the real part of $w \in \mathbb{C}$. Moreover, the pseudo-metric $I_\phi = \langle \mathrm{d}\phi, \mathrm{d}\phi\rangle$ on $S$ has a constant curvature*

$$K_\phi = \varepsilon. \qquad (5)$$

*Conversely, let $S$ be a simply-connected Riemannian surface, $G : S \longrightarrow \mathbb{S}^2$ a meromorphic map, $\varepsilon \in \mathbb{R}$ and $\rho$ a solution of the Liouville-type equation*

$$(\ln\rho)_{z\bar z} = \varepsilon\frac{|G_z|^2}{4}\rho^2.$$

*Then the immersion given by* (3) *is a $\varepsilon$-surface with normal* (4) *and whose hyperbolic Gauss map coincides with $G$. Moreover, the conformal structure of $S$ as a Riemann surface coincides with that one induced by $\sigma$.*

**Remark 2.** Note that (5) says that $I_\phi$ has a constant Gaussian curvature $\varepsilon < 0$. This fact ensures the existence of a holomorphic function $h$ on $S$ for the structure given by $\sigma$ satisfying $1+\varepsilon|h|^2 > 0$, such that

$$I_\phi = \rho^2|\mathrm{d}G|^2 = \frac{4|\mathrm{d}h|^2}{(1+\varepsilon|h|^2)^2}.$$

Thus $\rho$ can be rewritten in terms of $G$ and $h$ as

$$\rho = \frac{4|\mathrm{d}h|}{|\mathrm{d}G|(1+\varepsilon|h|^2)}. \qquad (6)$$

Observe that in this way, $\rho$ is well defined.

Finally, we can rewrite Theorem 2 in terms of $h$ and a holomorphic one-form on $S$ given by

$$\alpha = \frac{\rho|\mathrm{d}G|}{|\mathrm{d}h|}\left(\frac{\mathrm{d}\rho}{\rho^2\mathrm{d}G}\right)_z \mathrm{d}z \qquad (7)$$

because the expressions of the first and the second fundamental form of $\psi$ can be calculated in a simple way.

**Corollary 1.** *Let $S$ be a non compact simply-connected surface and the map $\psi : S \longrightarrow \mathbb{S}^3_1$ is an $\varepsilon$-immersion satisfying* (1). *Then there exists a pair $(h, \alpha)$, where $h$ is a holomorphic function and $\alpha$ a holomorphic one-form on $S$, such that its first and second fundamental forms are given by*

$$I = -(1+\varepsilon)\alpha \mathrm{d}h + \left( \frac{(1+\varepsilon)^2 |\mathrm{d}h|^2}{(1+\varepsilon|h|^2)^2} + (1+\varepsilon|h|^2)^2|\alpha|^2 \right) - (1+\varepsilon)\bar{\alpha}\mathrm{d}\bar{h} \quad (8)$$

$$II = -\varepsilon\alpha \mathrm{d}h + \left( \frac{(\varepsilon^2-1)|\mathrm{d}h|^2}{(1+\varepsilon|h|^2)^2} + (1+\varepsilon|h|^2)^2|\alpha|^2 \right) - \varepsilon\bar{\alpha}\mathrm{d}\bar{h} \quad (9)$$

*respectively. In addition the metric $\sigma$ becomes*

$$\sigma = -\left( (1+\varepsilon|h|^2)^2|\alpha|^2 - \frac{(1+\varepsilon)^2|\mathrm{d}h|^2}{(1+\varepsilon|h|^2)^2} \right) \quad (10)$$

*and the Gauss–Kronecker and the mean curvatures of $\psi$ are given by*

$$K = 1 - \frac{4\varepsilon|\mathrm{d}h|^2}{(1+\varepsilon)^2|\mathrm{d}h|^2 - (1+\varepsilon|h|^2)^4|\alpha|^2}$$

$$H = 1 + \frac{2(1+\varepsilon)|\mathrm{d}h|^2}{(1+\varepsilon|h|^2)^4|\alpha|^2 - (1+\varepsilon)^2|\mathrm{d}h|^2}.$$

*Conversely, given a simply-connected Riemann surface $S$, $-\varepsilon \in \mathbb{R}^+$ and a pair $(h, \alpha)$ as above such that* (10) *is a positive definite metric, then there exists a map $\psi : S \longrightarrow \mathbb{S}^3_1$ which is an $\varepsilon$-immersion, unique up to isometries of $\mathbb{S}^3_1$, with $I$, $II$ and $\sigma$ given by* (8), (9) *and* (10), *respectively.*

**Remark 3.** The data $(h, \alpha)$ will be called the **Weierstrass data** for the immersion.

## 3. Complete $\varepsilon$-Surfaces

In this section we establish some results about complete $\varepsilon$-surfaces. First of all, note that this kind of surfaces cannot be compact. Then, we have the following:

**Lemma 1.** *Let $\psi : S \longrightarrow \mathbb{S}^3_1$ be a complete $\varepsilon$-surface with Weierstrass data $(h, \omega)$. Then $S$ is conformally equivalent to $\mathbb{D}$ and $h$ is a global diffeomorphism onto $\mathbb{D}_\varepsilon = \{z \in \mathbb{C};\, |z|^2 < -1/\varepsilon\}$.*

**Proof:** Since $I$ is complete and $\sigma$ is definite positive, we get from (8) and (10) that

$$\frac{1}{2} I \leq \frac{(1+\varepsilon)^2|\mathrm{d}h|^2}{(1+\varepsilon|h|^2)^2} + (1+\varepsilon|h|^2)^2|\omega|^2 \leq 2\frac{(1+\varepsilon)^2|\mathrm{d}h|^2}{(1+\varepsilon|h|^2)^2}$$

and so the metric $4|\mathrm{d}h|^2/(1+\varepsilon|h|^2)^2$ is also complete. Therefore, $h : S \longrightarrow \mathbb{D}_\varepsilon$ is bijective and $S$ is conformally equivalent to $\mathbb{D}$. □

From this lemma it follows that, given a complete $\varepsilon$-surface with the Weierstrass data $(h, \alpha)$, we can consider, up to a change of the parameter that $S = \mathbb{D}$ and $h(z) = z/\sqrt{-\varepsilon}$.

**Theorem 3.** *Let $S$ be a simply-connected surface and $\psi : S \longrightarrow \mathbb{S}^3_1$ a complete $\varepsilon$-surface. Then*

i) *$S$ can be identified with $\mathbb{D}$ and $h$ can be taken as $h(z) = z/\sqrt{-\varepsilon}$.*

ii) *The hyperbolic Gauss map $G : \mathbb{D} \longrightarrow \mathbb{C} \cup \{\infty\}$ is a local diffeomorphism for which*

$$|\{G, z\}| < \frac{2(1+\varepsilon)}{-\varepsilon} \frac{1}{(1-|z|^2)^2}, \qquad z \in \mathbb{D} \tag{11}$$

*where $\{G, z\} := \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{G_{zz}}{G_z}\right) - \frac{1}{2}\left(\frac{G_{zz}}{G_z}\right)^2$ is the Schwazian derivative of $G^+$.*

iii) *The immersion and the Gauss map can be recovered as*

$$\frac{1}{\sqrt{-\varepsilon}\,|G_z|(1-|z|^2)} \times \begin{pmatrix} |G|^2 + \varepsilon\left|G\mathcal{R} + G_z(1-|z|^2)\right|^2 & (1+\varepsilon|\mathcal{R}|^2)G + \varepsilon G_z(1-|z|^2)\overline{\mathcal{R}} \\ (1+\varepsilon|\mathcal{R}|^2)\overline{G} + \varepsilon\overline{G_z}(1-|z|^2)\mathcal{R} & 1+\varepsilon|\mathcal{R}|^2 \end{pmatrix} \tag{12}$$

*and*

$$\frac{1}{\sqrt{-\varepsilon}\,|G_z|(1-|z|^2)} \times \begin{pmatrix} |G|^2 - \varepsilon\left|G\mathcal{R} + G_z(1-|z|^2)\right|^2 & (1-\varepsilon|\mathcal{R}|^2)G - \varepsilon G_z(1-|z|^2)\overline{\mathcal{R}} \\ (1-\varepsilon|\mathcal{R}|^2)\overline{G} - \varepsilon\overline{G_z}(1-|z|^2)\mathcal{R} & 1-\varepsilon|\mathcal{R}|^2 \end{pmatrix} \tag{13}$$

*respectively, where*

$$\mathcal{R} = \bar{z} - \frac{G_{zz}}{2G_z}(1-|z|^2).$$

*Conversely, let $G : \mathbb{D} \longrightarrow \mathbb{C} \cup \{\infty\}$ be a meromorphic map. If $G$ verifies* (11) *then* (12) *is a $\varepsilon$-surface with hyperbolic Gauss map $G$ and Weierstrass data*

$$(z/\sqrt{-\varepsilon}, -\frac{1}{2}\sqrt{-\varepsilon}\,\{G, z\}\,\mathrm{d}z).$$

*Moreover, if*

$$|\{G, z\}| \leq \frac{b_0}{(1-|z|^2)^2}, \qquad z \in \mathbb{D} \tag{14}$$

*with $b_0 < -2(1+\varepsilon)/\varepsilon$, then the immersion is complete.*

**Proof:** Let $\psi : S \longrightarrow \mathbb{S}^3_1$ be a simply connected complete $\varepsilon$-surface. Then, from (10) it is clear that $\sigma$ is positive definite if, and only if,

$$|\alpha| < \frac{1+\varepsilon}{\sqrt{-\varepsilon}} \frac{|\mathrm{d}z|}{(1-|z|^2)^2}, \qquad z \in \mathbb{D}. \tag{15}$$

The representation formula follows directly as we can take $h(z) = z/\sqrt{-\varepsilon}$ and then (6) can be expressed as

$$\rho = \frac{4|\mathrm{d}z|}{\sqrt{-\varepsilon}\,|\mathrm{d}G|(1-|z|^2)}. \tag{16}$$

Thus, a straightforward computation gives the formulas replacing (16) in (3) and (4). Recall that the immersion is written in the Hermitian model as $2 \times 2$ matrix.

Also, (7) can be expressed as

$$\frac{-2}{\sqrt{-\varepsilon}} \frac{\alpha}{\mathrm{d}z} = \frac{2G_z G_{zzz} - 3(G_{zz})^2}{2(G_z)^2} = \{G, z\} \tag{17}$$

whence from (15) and (17)

$$|\{G, z\}| < \frac{2(1+\varepsilon)}{-\varepsilon} \frac{1}{(1-|z|^2)^2}, \qquad z \in \mathbb{D}.$$

Consequently, $G$ is a local diffeomorphism with bounded Schwarzian derivative. Otherwise there would exist a point $z_0$ such that $G$ is not a bijection and we could write $G_z$ in a neighborhood of $z_0$ as

$$G_z = (z - z_0)^k \sum_{n=0}^{\infty} c_n (z - z_0)^n$$

where $c_0 \neq 0$ and $k$ is a non-zero integer. But then

$$\{G, z\} = -\frac{k(k+2)}{2} \frac{1}{(z-z_0)^2} - \frac{kc_1}{2c_0} \frac{1}{z - z_0} + \widetilde{h}(z)$$

where $\widetilde{h}$ is a holomorphic function in a neighborhood of $z_0$, which implies that either $\{G, z\} = \infty$ at $z_0$ (which contradicts the above inequality) or $k = -2$ and $G$ has a pole of order one in $z_0$, that is, $G$ is locally bijective.

The converse statement follows from Theorem 2. Moreover, if $G$ verifies (14), the induced metric $I$ can be estimated as follows

$$\begin{aligned} I &= \left| \frac{(1+\varepsilon)\mathrm{d}h}{1+\varepsilon|h|^2} - (1+\varepsilon|h|^2)\alpha \right|^2 \geq \left( \frac{(1+\varepsilon)|\mathrm{d}h|}{1+\varepsilon|h|^2} - (1+\varepsilon|h|^2)|\alpha| \right)^2 \\ &= \left( \frac{1+\varepsilon}{\sqrt{-\varepsilon}(1-|z|^2)} - (1-|z|^2)\frac{\sqrt{-\varepsilon}}{2}|\{G,z\}| \right)^2 |\mathrm{d}z|^2 \\ &\geq \left( \frac{1+\varepsilon}{\sqrt{-\varepsilon}} - \frac{\sqrt{-\varepsilon}}{2} b_0 \right)^2 \frac{|\mathrm{d}z|^2}{(1-|z|^2)^2}. \end{aligned}$$

Thus, since $b_0 < -2(1+\varepsilon)/\varepsilon$ and $|\mathrm{d}z|^2/(1-|z|^2)^2$ is a complete metric on $\mathbb{D}$, the immersion $\psi$ is complete. □

**Remark 4.** Let $\psi_0 : \mathbb{D} \longrightarrow \mathbb{S}^3_1$ be a $\varepsilon$-surface generated by the Weierstrass data $h_0(z) = z/\sqrt{-\varepsilon}$ and the hyperbolic Gauss map $G_0$. Let us consider a conformal equivalence $\varphi : \mathbb{D} \longrightarrow \mathbb{D}$,

$$\varphi(\zeta) = \mathrm{e}^{\mathrm{i}\theta} \frac{\zeta + \zeta_0}{\overline{\zeta_0}\zeta + 1}, \qquad \zeta \in \mathbb{D}$$

for certain $\zeta_0 \in \mathbb{D}$, $\theta \in \mathbb{R}$. Then

$$\frac{\varphi_\zeta(\zeta)}{1-|\varphi(\zeta)|^2} = \mathrm{e}^{\mathrm{i}\theta} \frac{1}{1-|\zeta|^2} \frac{1+\zeta_0\overline{\zeta}}{1+\overline{\zeta_0}\zeta}.$$

Thus, the immersion $\psi_1 : \mathbb{D} \longrightarrow \mathbb{S}^3_1$ with $h_1(\zeta) = \zeta/\sqrt{-\varepsilon}$ and $G_1(\zeta) = G_0(\varphi(\zeta))$ verifies

$$G_{1\zeta}(\zeta)(1-|\zeta|^2) = \mathrm{e}^{\mathrm{i}\theta} G_{0z}(\varphi(\zeta))(1-|\varphi(\zeta)|^2) \frac{1+\zeta_0\overline{\zeta}}{1+\overline{\zeta_0}\zeta}$$

$$\mathcal{R}_{\psi_1}(\zeta) = \mathrm{e}^{\mathrm{i}\theta} \mathcal{R}_{\psi_0}(\varphi(\zeta)) \frac{1+\zeta_0\overline{\zeta}}{1+\overline{\zeta_0}\zeta}$$

and we have, from (12), that $\psi_0(\varphi(\zeta)) = \psi_1(\zeta)$.

### 3.1. Consequences

Let $S$ be a simply-connected non-compact surface and $\psi : S \longrightarrow \mathbb{S}^3_1$ an $\varepsilon$-immersion. If we assume that $G$ has no poles, then from [8] and up to an isometry $\psi_0 + \psi_3 > 0$.

On the other hand, we can identify $(\mathbb{S}^3_1)^+ = \{(x_0, x_1, x_2, x_3) \in \mathbb{S}^3_1;\, x_0 + x_3 > 0\}$ and $(\mathbb{L}^3)^+ = \{(y_0, y_1, y_2) \in \mathbb{R}^3;\, y_0 > 0\}$ by means of the map

$$\begin{array}{cccc} \Phi : & (\mathbb{S}^3_1)^+ & \longrightarrow & (\mathbb{L}^3)^+ \\ & (x_0, x_1, x_2, x_3) & \longmapsto & \frac{1}{x_0+x_3}(1, x_1, x_2). \end{array} \tag{18}$$

Under this identification the induced metric by $\langle\cdot,\cdot\rangle$ on $(\mathbb{L}^3)^+$ is given by

$$ds^2 = \frac{1}{y_0^2}\left(-dy_0^2 + dy_1^2 + dy_2^2\right)$$

which is conformal to the usual metric in $\mathbb{L}^3$, and its ideal boundary except one point, $(\mathbb{S}^2_\infty)^*$, is identified with the plane $\Pi = \{y_0 = 0\}$. Thus, the asymptotic boundary of the set $\Sigma \subset (\mathbb{L}^3)^+$ is

$$\partial_\infty \Sigma = \mathrm{cl}(\Sigma) \cap \Pi$$

where $\mathrm{cl}(\Sigma)$ is the closure of $\Sigma$ in $\{(y_0, y_1, y_2) \in \mathbb{R}^3 \,;\, y_0 \geq 0\}$.

## 4. The Plateau Problem

For the study of these surfaces at infinity let us consider the following Plateau problem:

> *Given $\varepsilon_0 < 0$ and a Jordan curve $\Gamma$ on $\mathbb{S}^2_\infty \equiv \Pi \cup \{\infty\}$, find a complete $\varepsilon$-surface $\psi : S \longrightarrow \mathbb{S}_1^3$ such that $\Gamma$ is its asymptotic boundary.*

Since every isometry in $\mathbb{S}_1^3$ preserving the orientation induces a Möbius transformation in $\mathbb{S}^2_\infty$, we can suppose that the Jordan curve $\Gamma$ lies on $\Pi$.

**Lemma 2.** *Let $\psi : S \longrightarrow \mathbb{S}_1^3$ be an $\varepsilon$-surface with holomorphic hyperbolic Gauss map $G$ and asymptotic boundary a curve on $\Pi$. Then $\psi$ is an embedding.*

**Proof:** Relying on Section 3.1 we have that $\Phi \circ \psi(S)$ is a spacelike surface immersed in $(\mathbb{L}^3)^+$ with boundary on $\Pi$. Hence, the surface is locally a graph on $\Pi$ and it is proper. Therefore, it is a global graph and $\psi$ is an embedding. □

**Proposition 1.** *Let $\Gamma$ be a Jordan curve on $\Pi$, $\mathrm{int}(\Gamma)$ the bounded component of $\mathbb{C} \setminus \Gamma$ and $G : \mathbb{D} \longrightarrow \mathbb{I}_\Gamma$ a conformal equivalence, where*

$$\mathbb{I}_\gamma = \left\{(0, y_1, y_2) \in \mathbb{R}^3 \,;\, y_1 + \mathrm{i}y_2 \in \mathrm{int}(\Gamma)\right\}.$$

*If $G$ verifies (11) for a fixed $\varepsilon < 0$, then the immersion $\psi : \mathbb{D} \longrightarrow \mathbb{S}_1^3$ associated with $G$ by means of Theorem 3 is an embedded solution of the Plateau problem for $\Gamma$. Moreover, $\psi$ can be extended continuously from $\mathbb{D}$ to $\overline{\mathbb{D}}$ in such a way that $\widetilde{\psi}_{\partial\overline{\mathbb{D}}} : \partial\overline{\mathbb{D}} \longrightarrow \Gamma$ is a homeomorphism. Even more, if $\Gamma$ is $C^\infty$, then $\psi$ and its derivatives have continuous extensions to $\overline{\mathbb{D}}$.*

**Proof:** Let $G : \mathbb{D} \longrightarrow \mathbb{I}_\Gamma$ be a conformal equivalence verifying (11) for a fixed $\varepsilon < 0$. Then, from Theorem 3 and (18) we get that

$$\psi(z) = \left(\frac{\sqrt{-\varepsilon}\,|G_z|(1-|z|^2)}{1+\varepsilon|\mathcal{R}|^2}, G + \varepsilon G_z(1-|z|^2)\frac{\overline{\mathcal{R}}}{1+\varepsilon|\mathcal{R}|^2}\right)$$

is a complete $\varepsilon$-surface with hyperbolic Gauss map $G$. Observe that $G$ can be continuously extended to a homomorphism $\widetilde{G}$ from $\overline{\mathbb{D}}$ to

$$\left\{(0, y_1, y_2) \in \mathbb{R}^3;\ y_1 + \mathrm{i}y_2 \in \Gamma\right\}$$

(see [5]). Moreover, if $\Gamma$ is differentiable, such extension is a diffeomorphism. Hence

$$\psi(z) \longrightarrow (0, \widetilde{G}) \text{ when } |z| \longrightarrow 1$$

and from [5, Lemma 14.2.8] it follows that $|G_z|(1-|z|^2) \longrightarrow 0$ when $|z| \longrightarrow 1$, and from [8] we have that $\mathcal{R}$ is bounded with $\alpha_0 \leq 1 + \varepsilon|\mathcal{R}|^2 \leq 1$ where $\alpha_0$ is a positive constant. Thus, taking $\widetilde{\psi}(z) = (0, \widetilde{G})$ when $|z| = 1$, we see that $\psi$ has a continuous extension $\widetilde{\psi}$ defined on $\overline{\mathbb{D}}$ whose asymptotic boundary is $\Gamma$. In particular, $\psi$ is a solution of the Plateau problem for the curve $\Gamma$. Moreover, if $\Gamma$ is differentiable then $\widetilde{G}$ is also differentiable and $\widetilde{G}_z \neq 0$ for $|z| = 1$. Therefore, $\mathcal{R}$ is well defined on $\overline{\mathbb{D}}$ and $\widetilde{\psi}$ is a differentiable extension of $\psi$. □

Note that the normal vector field to $\psi$ points toward the interior domain bounded by $\widetilde{\psi}(\overline{\mathbb{D}}) \cup \mathrm{int}(\Gamma)$ in the highest Euclidean point, $p_0$, because, in other case, the hyperbolic Gauss map cannot be bounded so that the vertical straight line are geodesics and so $G(p_0) = \infty$. Moreover, from Remark 4 we have that the solution to the Plateau problem does not depend on the chosen conformal equivalence.

**Theorem 4.** *Let $\Gamma$ be some Jordan curve on $\{y_0 = 0\} \subset \mathbb{R}^3$. Then, for every $\varepsilon \in (-1/4, 0)$, ($\varepsilon \in (-1/2, 0)$ if $\Gamma$ is convex), there exists an embedded solution for the corresponding Plateau problem. Moreover, it is the only solution contained in $(\mathbb{S}^3_1)^+$ whose normal vector field points towards the interior at the highest Euclidean point.*

**Proof:** Let us consider a conformal equivalence $G$ from $\mathbb{D}$ into

$$\mathbb{I}_\Gamma = \{x_1 + \mathrm{i}x_2 \in \mathbb{C};\ (x_1, x_2, 0) \in \mathrm{int}(\Gamma)\}.$$

Then, if $\Gamma$ is convex we get from [8, Corollary 3] that

$$|\{G, z\}| \leq 2/(1-|z|^2)^2 < (1+\varepsilon)/(-\varepsilon(1-|z|^2)^2) \text{ for all } \varepsilon \in (-1/2, 0).$$

On the other hand, if $\Gamma$ is not convex, from [11] we have

$$|\{G, z\}| \leq 6/(1-|z|^2)^2 < (1+\varepsilon)/(-\varepsilon(1-|z|^2)^2) \text{ for all } \varepsilon \in (-1/4, 0).$$

Hence, from Proposition 1, the immersion $\psi$ associated with $\varepsilon$ and $G$ is a solution of the Plateau problem for $\Gamma$.

Let us suppose that $\chi : S \longrightarrow (\mathbb{S}^3_1)^+$ is another solution for $\varepsilon_0 < 0$ and a Jordan curve $\Gamma$ whose normal vector field points toward the interior at the highest

Euclidean point. Without loss of generality we can suppose that the origin is contained in $\mathrm{int}(\Gamma)$. Let $C_0$ and $C_2$ be two circles on $\Pi$ centered at the origin and bounding a closed annulus $A$ in $\Pi$ containing $\Gamma$ in its interior. Choose $C_0$ and $C_2$ so that the totally umbilical $\varepsilon$-surfaces (with hyperbolic normal pointing downwards), $S_0$ and $S_2$ associated with $\varepsilon_0$ and asymptotic boundary $C_0$ and $C_2$, respectively, satisfy $S_0$ is below $\chi(S)$ and $\chi(S)$ is below $S_2$.

Let us consider $\Gamma_t$, $0 \le t \le 2$, a foliation of the annulus $A$ such that $\Gamma_t$ is convex for every $t \in [0,2]$, being $\Gamma_0 = C_0$, $\Gamma_1 = \Gamma$ and $\Gamma_2 = C_2$. Let $G_t$ be the unique conformal equivalence from $\mathbb{D}$ into $\{x_1 + \mathrm{i}x_2 \in \mathbb{C};\ (0, x_1, x_2) \in \mathrm{int}(\Gamma_t)\}$ such that $G_t(0) = 0$ and $(G_t)_z(0)$ is a positive real number. Since the curves $\Gamma_t$ are convex, we have that $|\{G_t, z\}| \le 2/(1-|z|^2)^2$ (see [12]) and, from Proposition 1, we can consider the corresponding $\varepsilon$-surface $\psi_t : \mathbb{D} \longrightarrow (\mathbb{S}^3_1)^+$ associated with $\varepsilon_0$ and the hyperbolic Gauss map $G_t$. In particular, $\psi_0(\mathbb{D}) = S_0$, $\psi_1(\mathbb{D}) = \psi(\mathbb{D})$ and $\psi_2(\mathbb{D}) = S_2$.

Our object is to prove that $\chi(S) = \psi_1(\mathbb{D})$. First, observe that if $t_n \in [0,2]$ and $\{t_n\}$ is a sequence converging to $t_0$, then $\{\widetilde{G}_{t_n}\}$ converges uniformly to $\widetilde{G}_{t_0}$ ([7, Theorem II.5.2]), where $\widetilde{G}_t$ denotes the extension of $G_t$ to $\overline{\mathbb{D}}$.

Let us see that if $0 \le t < 1$ then $\psi_t(\mathbb{D})$ is under $\chi(S)$. Observe also that if

$$J = \{t \in [0,1)\,;\ \psi_t(\mathbb{D}) \cap \chi(S) \ne \emptyset\}$$

is not empty, then $J$ has a minimum. In fact, given a sequence $\{t_n\} \subset J$ converging to the infimum $t_0$ of $J$, there exist $p_n \in \mathbb{D}$ such that $\psi_{t_n}(p_n) \in \mathrm{cl}(\chi(S))$ and a subsequence $\{p_m\}$ must converge to a point $p_0 \in \overline{\mathbb{D}}$. Thus, since $\{\psi_{t_m}\}$ converges uniformly to $\psi_{t_0}$, then $\psi_{t_0}(p_0) \in \mathrm{cl}(\chi(S))$. But $p_0 \notin \partial\mathbb{D}$ because $\Gamma_{t_0} \cap \Gamma_1 = \emptyset$, that is, $t_0 \in J$.

Let us suppose now that $J \ne \emptyset$. Then, from the uniform convergence we have that, for every $t \in [0, t_0)$, $\psi_t$ and $\psi_{t_0}$ are under $\chi(S)$. Hence, since $\psi_{t_0}(\mathbb{D})$ and $\chi(S)$ must be tangent at a point with the same normal vector and $\psi_0(S)$ is below $\chi(S)$, it follows that $\psi_{t_0}(\mathbb{D})$ and $\chi(S)$ agree, which is not possible because they do not have the same asymptotic boundary.

Therefore, it must be $J = \emptyset$ and using again the uniform convergence we have that $\psi_1(\mathbb{D})$ is under $\chi(S)$. Analogously, we can see that $\chi(S)$ is under $\psi_1(\mathbb{D})$ reasoning in the interval $[1,2]$, and consequently $\psi_1(\mathbb{D}) = \chi(S)$. □

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