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ON SPECIAL TYPES OF MINIMAL AND TOTALLY GEODESIC UNIT VECTOR FIELDS

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Abstract. We present a new equation with respect to a unit vector field on Riemannian manifold M^n such that its solution defines a totally geodesic submanifold in the unit tangent bundle with Sasakian metric and apply it to some classes of unit vector fields. We introduce a class of covariantly normal unit vector fields and prove that within this class the Hopf vector field is a unique global one with totally geodesic property. For the wider class of geodesic unit vector fields on a sphere we give a new necessary and sufficient condition to generate a totally geodesic submanifold in T_1S^n.

1. Introduction

This paper is organized as follows. In Section 2 we give definitions of harmonic and minimal unit vector fields, rough Hessian and harmonicity tensor for the unit vector field. In Section 3 we give definition of a totally geodesic unit vector field and prove a basic Lemma 2 which gives a necessary and sufficient condition for the unit vector field to be totaly geodesic. Theorem 2 contains a necessary and sufficient condition on strongly normal unit vector field to be minimal. In Section 4 we apply Lemma 2 to the case of a unit sphere (Lemma 4) and describe the geodesic unit vector fields on the sphere with totally geodesic property (Theorem 5). We also introduce a notion of covariantly normal unit vector field and prove that within this class the Hopf vector field is a unique one with a totally geodesic property (Theorem 3). This theorem is a revised and simplified version of Theorem 2.1 in [27]. Section 5 contains an observation that the Hopf vector field on a unit sphere provides an example of global imbedding of Sasakian space form into Sasakian manifold as a Sasakian space form with a specific φ-curvature (Theorem 6).

2. Preliminaries

2.1. Sasakian Metric

Let (M, g) be n-dimensional Riemannian manifold with metric g. Denote by $\langle\cdot,\cdot\rangle$ a scalar product with respect to g. A natural Riemannian metric on the tangent bundle has been defined by S. Sasaki [20]. We describe it briefly in terms of the *connection map*.

At each point $Q = (q, \xi) \in TM$ the tangent space $T_Q TM$ can be split into the so-called *vertical* and *horizontal* parts

$$T_Q TM = \mathcal{H}_Q TM \oplus \mathcal{V}_Q TM.$$

The vertical part $\mathcal{V}_Q TM$ is tangent to the fiber, while the horizontal part is transversal to it. If $(u^1, \dots, u^n; \xi^1, \dots, \xi^n)$ form the natural induced local coordinate system on TM, then for $\tilde X \in T_Q TM^n$ we have

$$\tilde X = \tilde X^i \partial/\partial u^i + \tilde X^{n+i} \partial/\partial \xi^i$$

with respect to the natural frame $\{\partial/\partial u^i, \partial/\partial \xi^i\}$ on TM.

Denote by $\pi : TM \to M$ the tangent bundle projection map. Then its differential $\pi_* : T_Q TM \to T_q M$ acts on $\tilde X$ as $\pi_* \tilde X = \tilde X^i \partial/\partial x^i$ and defines a linear isomorphism between $\mathcal{V}_Q TM$ and $T_q M$.

The so-called **connection map** $K : T_Q TM \to T_q M$ acts on $\tilde X$ by the rule $K\tilde X = (\tilde X^{n+i} + \Gamma^i_{jk} \xi^j \tilde X^k)\partial/\partial u^i$ and defines a linear isomorphism between $\mathcal{H}_Q TM$ and $T_q M$. The images $\pi_* \tilde X$ and $K\tilde X$ are called *horizontal* and *vertical* projections of $\tilde X$, respectively. It is easy to see that $\mathcal{V}_Q = \ker \pi_*|_Q$, $\mathcal{H}_Q = \ker K|_Q$.

Let $\tilde X, \tilde Y \in T_Q TM$. The *Sasakian metric* on TM is defined by the following scalar product

$$\langle\langle \tilde X, \tilde Y \rangle\rangle\big|_Q = \langle \pi_* \tilde X, \pi_* \tilde Y \rangle\big|_q + \langle K\tilde X, K\tilde Y \rangle\big|_q$$

at each point $Q = (q, \xi)$. Horizontal and vertical subspaces are mutually orthogonal with respect to Sasakian metric.

The operations inverse to projections are called *lifts*. Namely, if $X \in T_q M^n$, then $X^h = X^i \partial/\partial u^i - \Gamma^i_{jk} \xi^j X^k \partial/\partial \xi^i$ is in $\mathcal{H}_Q TM$ and it is called a **horizontal lift** of X, while $X^v = X^i \partial/\partial \xi^i$, which is in $\mathcal{V}_Q TM$, is called a **vertical lift** of X.

The Sasakian metric can be completely defined by scalar product of combinations of lifts of vector fields from M to TM as

$$\langle\langle X^h, Y^h \rangle\rangle\big|_Q = \langle X, Y \rangle\big|_q, \quad \langle\langle X^h, Y^v \rangle\rangle\big|_Q = 0, \quad \langle\langle X^v, Y^v \rangle\rangle\big|_Q = \langle X, Y \rangle\big|_q.$$

2.2. Harmonic and Minimal Unit Vector Fields

Suppose, as above, that $u := (u^1, \dots, u^n)$ are the local coordinates on M^n. Denote by $(u, \xi) := (u^1, \dots, u^n; \xi^1, \dots, \xi^n)$ the natural local coordinates in the tangent bundle TM^n. If $\xi(u)$ is a (unit) vector field on M^n, then it defines a mapping

$$\xi : M^n \to TM^n \quad \text{or} \quad \xi : M^n \to T_1M^n, \quad \text{when} \quad |\xi| = 1$$

given by $\xi(u) = (u, \xi(u))$.

For the mappings $f : (M, g) \to (N, h)$ between Riemannian manifolds the *energy* of f is defined as

$$E(f) := \frac{1}{2} \int_M |\mathrm{d}\, f|^2 \,\mathrm{d}\,\mathrm{vol}_M$$

where $|\mathrm{d}\, f|$ is a norm of 1-form $\mathrm{d}\, f$ in the co-tangent bundle T^*M. Supposing on T_1M the Sasakian metric, the following definition becomes natural.

Definition 1. *A unit vector field is called **harmonic**, if it is a critical point of energy functional of mapping $\xi : M^n \to T_1M^n$.*

Up to an additive constant, the energy functional of the mapping is a total bending of a unit vector field [24]

$$B(\xi) := c_n \int_M |\nabla\xi\,|^2 \,\mathrm{d}\,\mathrm{vol}_M$$

where c_n is some normalizing constant and $|\nabla\xi|^2 = \sum_{i=1}^n |\nabla_{e_i}\xi|^2$ with respect to orthonormal frame $e_1, \dots, e_n$.

Introduce a point-wise linear operator $A_\xi : T_qM^n \to \xi_q^\perp$, acting as

$$A_\xi X = -\nabla_X \xi.$$

In case of integrable distribution $\xi^\perp$ the unit vector field ξ is called **holonomic** [1]. In this case the operator A_ξ is symmetric and is known as **Weingarten** or a **shape operator** for each hypersurface of the foliation. In general, A_ξ is not symmetric, but formally preserves the Codazzi equation. Namely, a covariant derivative of A_ξ is defined by

$$-(\nabla_X A_\xi)Y = \nabla_X\nabla_Y\xi - \nabla_{\nabla_X Y}\xi. \tag{1}$$

Then for the curvature operator of M^n we can write down the Codazzi-type equation

$$R(X, Y)\xi = (\nabla_Y A_\xi)X - (\nabla_X A_\xi)Y.$$

From this viewpoint, it is natural to call the operator A_ξ as *non-holonomic shape operator*. Remark, that the right hand side is, up to constant, a *skew symmetric part* of the covariant derivative of A_ξ.

Introduce a symmetric tensor field

$$\mathrm{Hess}_\xi(X, Y) = \frac{1}{2}[(\nabla_Y A_\xi)X + (\nabla_X A_\xi)Y] \tag{2}$$

which is the *symmetric part* of the covariant derivative of A_ξ. The trace

$$-\sum_{i=1}^{n} \mathrm{Hess}_\xi(e_i, e_i) := \Delta\xi$$

where $e_1, \dots, e_n$ is an orthonormal frame, is known as **rough Laplacian** [2] of the field ξ. Therefore, one can treat the tensor field (2) as a **rough Hessian** of the field ξ.

With respect to the above given notations, the unit vector field is harmonic if and only if [24]

$$\Delta\xi = -|\nabla\xi|^2\xi.$$

Introduce a tensor field

$$\mathrm{Hm}_\xi(X,Y) = \frac{1}{2}[R(\xi, A_\xi X)Y + R(\xi, A_\xi Y)X] \tag{3}$$

which is a symmetric part of the tensor field $R(\xi, A_\xi X)Y$. The trace

$$\mathrm{trace}\,\mathrm{Hm}_\xi := \sum_{i=1}^{n} \mathrm{Hm}_\xi(e_i, e_i)$$

is responsible for harmonicity of mapping $\xi : M^n \to T_1M^n$ in terms of general notion of harmonic maps [10]. Precisely, a *harmonic* unit vector field ξ defines a *harmonic mapping* $\xi : M^n \to T_1M^n$ if and only if [11]

$$\mathrm{trace}\,\mathrm{Hm}_\xi = 0.$$

From this viewpoint, it is natural to refer to the tensor field (3) as *harmonicity tensor* of the field ξ.

Consider now the image $\xi(M^n) \subset T_1M^n$ with a pull-back Sasakian metric.

Definition 2. *A unit vector field ξ on Riemannian manifold M^n is called minimal if the image of (local) imbedding $\xi : M^n \to T_1M^n$ is minimal submanifold in the unit tangent bundle T_1M^n with Sasakian metric.*

A number of results on minimal unit vector fields one can find in [4, 5, 6, 8, 12, 13, 14, 15, 16, 17, 19, 21, 22, 23]. In [25], the author has found explicitly the second fundamental form of $\xi(M^n)$ and presented some examples of unit vector fields of *constant mean curvature.*

3. Totally Geodesic Unit Vector Fields

Definition 3. *A unit vector field ξ on Riemannian manifold M^n is called totally geodesic if the image of (local) imbedding $\xi : M^n \to T_1M^n$ is totally geodesic submanifold in the unit tangent bundle T_1M^n with Sasakian metric.*

Using the explicit expression for the second fundamental form [25], the author gave a full description of the totally geodesic (local) unit vector fields on two-dimensional Riemannian manifold.

Theorem 1 ([28]). *Let (M^2, g) be a Riemannian manifold with a sign-preserving Gaussian curvature K. Then M admits a totally geodesic unit vector field ξ if and only if there is a local parametrization of M with respect to which the metric g is of the form*

$$ds^2 = du^2 + \sin^2\alpha(u)\, dv^2$$

where $\alpha(u)$ solves the differential equation $\dfrac{d\alpha}{du} = 1 - \dfrac{a+1}{\cos\alpha}$. The corresponding local unit vector field ξ is of the form

$$\xi = \cos(av + \omega_0)\partial_u + \frac{\sin(av+\omega_0)}{\sin\alpha(u)}\partial_v$$

where a and ω_0 are constants.

For the case of *flat* Riemannian two-manifold, the totally geodesic unit vector field is either parallel or moves helically along a pencil of parallel straight lines on a plane with a constant angle speed [26]. It is easy to see that the following corollary is true.

Corollary 1. *Integral trajectories of a totally geodesic (local) unit vector field on the non-flat Riemannian manifold M^2 are locally conformally equivalent to the integral trajectories of totally geodesic unit vector field on a plane. Moreover, with respect to Cartesian coordinates (x, y) on the plane, these integral trajectories are*

$$x = c \qquad \textit{for } a = 0$$

$$y(x) = -\frac{1}{a}\ln|\sin(ax)| + c \qquad \textit{for } a \neq 0$$

where c is a parameter.

In what follows, we present a new differential equation with respect to a unit vector field such that its solution generates a totally geodesic submanifold in T_1M^n.

In terms of horizontal and vertical lifts of vector fields from the base to its tangent bundle, the differential of mapping $\xi : M^n \to TM^n$ is acting as

$$\xi_* X = X^h + (\nabla_X \xi)^v = X^h - (A_\xi X)^v \tag{4}$$

where ∇ means Levi-Civita connection on M^n and the lifts are considered to points of $\xi(M^n)$.

It is well known that if ξ is a *unit* vector field on M^n, then the vertical lift ξ^v is a *unit normal* vector field on a hypersurface $T_1M^n \subset TM^n$. Since ξ is of unit length, $\xi_* X \perp \xi^v$ and hence in this case $\xi_* : TM^n \to T(T_1M^n)$.

Denote by $A^t_\xi : \xi^\perp_q \to T_qM^n$ a formal adjoint operator

$$\langle A_\xi X, Y\rangle_q = \langle X, A^t_\xi Y\rangle_q.$$

Denote by $\xi^\perp$ a distribution on M^n with ξ as its normal unit vector field. Then for each vector field $N \in \xi^\perp$, the vector field

$$\tilde N = (A^t_\xi N)^h + N^v \tag{5}$$

is normal to $\xi(M^n)$. Thus, (5) presents the normal distribution on $\xi(M^n)$.

Lemma 1. *Let M^n be Riemannian manifold and T_1M^n its unit tangent bundle with Sasakian metric. Let ξ a smooth (local) unit vector field on M^n. The second fundamental form $\tilde\Omega_{\tilde N}$ of $\xi(M^n) \subset T_1M^n$ with respect to the normal vector field* (5) *is of the form*

$$\tilde\Omega_{\tilde N}(\xi_*X, \xi_*Y) = -\langle \mathrm{Hess}_\xi(X,Y) + A_\xi \mathrm{Hm}_\xi(X,Y), N\rangle \tag{6}$$

where X and Y are arbitrary vector fields on M^n.

Proof: By definition, we have

$$\tilde\Omega_{\tilde N}(\xi_*X, \xi_*Y) = \langle\langle \tilde\nabla_{\xi_*X}\, \xi_*Y, \tilde N\rangle\rangle_{(q,\xi(q))}$$

where $\tilde\nabla$ is the Levi-Civita connection of Sasakian metric on TM^n. To calculate $\tilde\nabla_{\xi_*X}\xi_*Y$, we can use the formulas [18]

$$\tilde\nabla_{X^h}Y^h = (\nabla_XY)^h - \frac{1}{2}(R(X,Y)\xi)^v, \qquad \tilde\nabla_{X^v}Y^h = \frac{1}{2}(R(\xi,X)Y)^h$$

$$\tilde\nabla_{X^h}Y^v = (\nabla_XY)^v + \frac{1}{2}(R(\xi,Y)X)^h, \qquad \tilde\nabla_{X^v}Y^v = 0.$$

A direct calculation yields

$$\begin{aligned}\tilde\nabla_{\xi_*X}\, \xi_*Y = \Big(\nabla_XY + \frac{1}{2}R(\xi,\nabla_X\xi)Y + \frac{1}{2}R(\xi,\nabla_Y\xi)X\Big)^h \\ + \Big(\nabla_X\nabla_Y\xi - \frac{1}{2}R(X,Y)\xi\Big)^v.\end{aligned}$$

The derivative above is not tangent to $\xi(M^n)$. It contains a projection on “external” normal vector field, i.e. on ξ^v which is a unit normal of T_1M^n inside TM^n. To correct the situation, we should subtract this projection, namely $-\langle\nabla_X\xi, \nabla_Y\xi\rangle\xi$, from the vertical part of the derivative.

Therefore, we have

$$\begin{aligned}\tilde\Omega_{\tilde N}(\xi_*X, \xi_*Y) = \langle \nabla_X\nabla_Y\xi + \langle\nabla_X\xi, \nabla_Y\xi\rangle\xi - \frac{1}{2}R(X,Y)\xi, N\rangle \\ + \langle \nabla_XY + \frac{1}{2}R(\xi,\nabla_X\xi)Y + \frac{1}{2}R(\xi,\nabla_Y\xi)X, A^t_\xi N\rangle\end{aligned}$$

or, equivalently,

$$\tilde{\Omega}_{\tilde{N}}(\xi_* X, \xi_* Y) = \langle \nabla_X \nabla_Y \xi + \langle \nabla_X \xi, \nabla_Y \xi \rangle \xi - \frac{1}{2} R(X, Y)\xi$$
$$+ A_\xi (\nabla_X Y + \frac{1}{2} R(\xi, \nabla_X \xi) Y + \frac{1}{2} R(\xi, \nabla_Y \xi) X), N \rangle.$$

Taking into account (1), (2), (3) and (5), and also

$$R(X, Y)\xi = \nabla_X \nabla_Y \xi - \nabla_Y \nabla_X \xi - \nabla_{[X,Y]}\xi$$

we can write

$$\tilde{\Omega}_{\tilde{N}}(\xi_* X, \xi_* Y) = -\langle \mathrm{Hess}_\xi(X, Y) + A_\xi \mathrm{Hm}_\xi(X, Y), N \rangle$$

which completes the proof. □

Lemma 2. *Let M^n be Riemannian manifold and $T_1 M^n$ its unit tangent bundle with Sasakian metric. Let ξ be a smooth (local) unit vector field on M^n. The vector field ξ generates a totally geodesic submanifold $\xi(M^n) \subset T_1 M^n$ if and only if ξ satisfies*

$$\mathrm{Hess}_\xi(X, Y) + A_\xi \mathrm{Hm}_\xi(X, Y) - \langle A_\xi X, A_\xi Y \rangle \xi = 0 \qquad (7)$$

for all (local) vector fields X, Y on M^n.

Proof: Taking into account (6), the condition on ξ to be totally geodesic takes the form

$$-\mathrm{Hess}_\xi(X, Y) - A_\xi \mathrm{Hm}_\xi(X, Y) = \lambda \xi.$$

Multiplying the equation above by ξ, we can find easily $\lambda = -\langle A_\xi X, A_\xi Y \rangle$. □

Following [16], we call a unit vector field ξ **strongly normal** if

$$\langle (\nabla_X A_\xi) Y, Z \rangle = 0$$

for all $X, Y, Z \in \xi^\perp$. In other words, $(\nabla_X A_\xi) Y = \lambda \xi$ for all $X, Y \in \xi^\perp$. It is easy to find the function λ. Indeed, we have

$$\lambda = \langle (\nabla_X A_\xi) Y, \xi \rangle = \langle \nabla_{\nabla_X Y} \xi - \nabla_X \nabla_Y \xi, \xi \rangle$$
$$= -\langle \nabla_X \nabla_Y \xi, \xi \rangle = \langle \nabla_X \xi, \nabla_Y \xi \rangle.$$

Thus, the strongly normal unit vector field can be characterized by the equation

$$(\nabla_X A_\xi) Y = \langle A_\xi X, A_\xi Y \rangle \xi \qquad (8)$$

for all $X, Y \in \xi^\perp$.

The strong normality condition highly simplifies the second fundamental form of $\xi(M^n) \subset T_1 M^n$. An orthonormal frame $e_1, e_2, \ldots, e_n$ is called *adapted* to the field ξ if $e_1 = \xi$ and $e_2, \ldots, e_n \in \xi^\perp$.

Lemma 3. *Let ξ be a unit strongly normal vector field on Riemannian manifold M^n. With respect to the adapted frame, the matrical components of the second fundamental form of $\xi(M^n) \subset T_1(M^n)$ simultaneously take the form*

$$\tilde{\Omega}_{\tilde{N}} = \begin{pmatrix} * & * & \dots & * \\ * & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & 0 & \dots & 0 \end{pmatrix}.$$

Proof: Set $N_\sigma = e_\sigma$, $\sigma = 2, \dots, n$. The condition (8) implies

$$R(X,Y)\xi = 0, \qquad \mathrm{Hess}_\xi(X,Y) = \langle A_\xi X, A_\xi Y\rangle \xi, \qquad \mathrm{Hm}_\xi(X,Y) \sim \xi$$

for all $X, Y \in \xi^\perp$. Therefore, with respect to the adapted frame

$$\tilde{\Omega}_\sigma(\xi_* e_\alpha, \xi_* e_\beta) = 0, \qquad \alpha, \beta = 2, \dots, n$$

for all $\sigma = 2, \dots, n$. □

The following assertion is a natural corollary of the Lemma 3.

Theorem 2. *Let ξ be a unit strongly normal vector field. Denote by k the geodesic curvature of its integral trajectories and by ν the principal normal unit vector field of the trajectories. The field ξ is minimal if and only if*

$$k[\xi,\nu] + \xi(k)\nu - kA_\xi R(\nu,\xi)\xi + k^2\xi = 0$$

where $[\xi,\nu] = \nabla_\xi \nu - \nabla_\nu \xi$.

Proof: Indeed,

$$\tilde{\Omega}_\sigma(\xi_* e_1, \xi_* e_1) = -\langle \mathrm{Hess}_\xi(\xi,\xi) + A_\xi \mathrm{Hm}_\xi(\xi,\xi), e_\sigma\rangle.$$

Denote by ν a vector field of the principal normals of ξ-integral trajectories and by k their geodesic curvature function. Then

$$\mathrm{Hess}_\xi(\xi,\xi) = \nabla_{\nabla_\xi \xi} - \nabla_\xi \nabla_\xi \xi = k\nabla_\nu \xi - \nabla_\xi(k\nu) = k[\nu,\xi] - \xi(k)\nu$$

$$\mathrm{Hm}_\xi(\xi,\xi) = -R(\xi, \nabla_\xi \xi)\xi = -kR(\xi,\nu)\xi$$

and we get

$$\tilde{\Omega}_\sigma(\xi_* e_1, \xi_* e_1) = \langle k[\xi,\nu] + \xi(k)\nu - kA_\xi R(\nu,\xi)\xi, e_\sigma\rangle.$$

Finally, to be minimal, the field ξ should satisfy

$$k[\xi,\nu] + \xi(k)\nu - kA_\xi R(\nu,\xi)\xi = \lambda\,\xi.$$

Multiplying by ξ, we get

$$\lambda = k\langle [\xi,\nu], \xi\rangle = k\langle \nabla_\xi \nu, \xi\rangle = -k^2$$

which completes the proof. □

Thus, we get the following

Corollary 2 ([16]). *Every unit strongly normal geodesic vector field is minimal.*

Most of examples of minimal unit vector fields in [16] are based on this Corollary.

4. The Case of a Unit Sphere

If the manifold is a unit sphere S^{n+1}, the equation (7) can be simplified essentially.

Lemma 4. *A unit (local) vector field ξ on a unit sphere S^{n+1} generates a totally geodesic submanifold $\xi(S^{n+1}) \subset T_1 S^{n+1}$ if and only if ξ satisfies*

$$\begin{aligned}(\nabla_X A_\xi)Y = \frac{1}{2}\Big[&(\mathcal{L}_\xi\, g)(X,Y)\, A_\xi \xi + \langle \xi, X\rangle (A_\xi^2 Y + Y) \\ &+ \langle \xi, Y\rangle (A_\xi^2 X - X)\Big] + \langle A_\xi X, A_\xi Y\rangle \xi\end{aligned} \tag{9}$$

where $(\mathcal{L}_\xi\, g)(X,Y) = \langle \nabla_X \xi, Y\rangle + \langle X, \nabla_Y \xi\rangle$ is a Lie derivative of metric tensor in a direction of ξ.

Proof: Indeed, on the unit sphere

$$(\nabla_Y A_\xi)X - (\nabla_X A_\xi)Y = R(X,Y)\xi = \langle \xi, Y\rangle X - \langle \xi, X\rangle Y.$$

Hence,

$$\mathrm{Hess}_\xi(X,Y) = (\nabla_X A_\xi)Y + \frac{1}{2}[\langle \xi, Y\rangle X - \langle \xi, X\rangle Y].$$

For $\mathrm{Hm}_\xi(X,Y)$ we have

$$\begin{aligned}\mathrm{Hm}_\xi(X,Y) &= \frac{1}{2}\Big[\langle \nabla_X \xi, Y\rangle \xi - \langle \xi, Y\rangle \nabla_X \xi + \langle \nabla_Y \xi, X\rangle \xi - \langle \xi, X\rangle \nabla_Y \xi\Big] \\ &= \frac{1}{2}(\mathcal{L}_\xi\, g)(X,Y)\xi + \frac{1}{2}\Big[\langle \xi, Y\rangle A_\xi X + \langle \xi, X\rangle A_\xi Y\Big].\end{aligned}$$

Finally, we find

$$\begin{aligned}(\nabla_X A_\xi)Y = \frac{1}{2}\Big[&(\mathcal{L}_\xi\, g)(X,Y) A_\xi \xi + \langle \xi, X\rangle (A_\xi^2 Y + Y) + \langle \xi, Y\rangle (A_\xi^2 X - X)\Big] \\ &+ \langle A_\xi X, A_\xi Y\rangle \xi.\end{aligned}$$

□

Remind that the operator A_ξ is symmetric if and only if the field ξ is holonomic, and is skew-symmetric if and only if the field ξ is a Killing vector field. Both types of these fields can be included into a class of **covariantly normal unit vector fields**.

Definition 4. *A regular unit vector field on Riemannian manifold is said to be covariantly normal if the operator* $A_\xi : TM \to \xi^\perp$ *defined by* $A_\xi X = -\nabla_X \xi$ *satisfies the normality condition*

$$A_\xi^t A_\xi = A_\xi A_\xi^t$$

with respect to some orthonormal frame.

The integral trajectories of holonomic and Killing unit vector fields are always geodesic. Every covariantly normal unit vector field possesses this property.

Lemma 5. *Integral trajectories of a covariantly normal unit vector field are geodesic lines.*

Proof: Suppose ξ is a unit covariantly normal vector field on a Riemannian manifold M^{n+1}. Find a unit vector field ν_1 such that

$$\nabla_\xi \xi = -k\nu_1.$$

Geometrically, the function k is a geodesic curvature of the integral trajectory of the field ξ.
Complete up the pair (ξ, ν_1) to the orthonormal frame $(\xi, \nu_1, \dots, \nu_n)$. Then we can set

$$\nabla_\xi \xi = -k\nu_1, \qquad \nabla_{\nu_\alpha} \xi = -a_\alpha^\beta \nu_\beta$$

where $\alpha, \beta = 1, \dots, n$. With respect to the frame $(\xi, \nu_1, \dots, \nu_n)$ the matrix A_ξ takes the form

$$-A_\xi = \begin{pmatrix} 0 & k & 0 & \dots & 0 \\ 0 & a_1^1 & a_2^1 & \dots & a_n^1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & a_1^n & a_2^n & \dots & a_n^n \end{pmatrix}$$

and, therefore,

$$A_\xi A_\xi^t = \begin{pmatrix} k^2 & ka_1^1 & \dots & ka_1^n \\ ka_1^1 & * & \dots & * \\ \vdots & \vdots & \ddots & \vdots \\ ka_1^n & * & \dots & * \end{pmatrix}, \qquad A_\xi^t A_\xi = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & * & \dots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & * & \dots & * \end{pmatrix}$$

which allows to conclude that $k = 0$. □

Now we can easily prove the following

Theorem 3. *Let* ξ *be a global covariantly normal unit vector field on a unit sphere* S^{n+1}. *Then* ξ *is a totally geodesic if and only if* $n = 2m$ *and* ξ *is a Hopf vector field.*

Proof: Suppose ξ is covariantly normal and totally geodesic. Then

$$A_\xi \xi = -\nabla_\xi \xi = 0$$

by Lemma 5 and the equation (9) takes the form

$$(\nabla_X A_\xi)Y = \frac{1}{2}\Big[\langle \xi, X\rangle (A_\xi^2 Y + Y) + \langle \xi, Y\rangle (A_\xi^2 X - X)\Big] + \langle A_\xi X, A_\xi Y\rangle \xi. \quad (10)$$

Setting $X = Y = \xi$ we get an identity. Set $Y = \xi$ and take arbitrary unit $X \perp \xi$. Then we get

$$2(\nabla_X A_\xi)\xi + X = A_\xi^2 X.$$

On the other hand, directly

$$(\nabla_X A_\xi)\xi = -(\nabla_X \nabla_\xi \xi - \nabla_{\nabla_X \xi}\xi) = A_\xi^2 X.$$

Hence,

$$A_\xi^2|_{\xi^\perp} = -E.$$

Therefore, $n = 2m$. Since A_ξ is real normal linear operator, there exists an orthonormal frame such that

$$A_\xi = \begin{pmatrix} 0 & & & & & & \\ & 0 & 1 & & & & \\ & -1 & 0 & & & & \\ & & & \ddots & & & \\ & & & & & 0 & 1 \\ & & & & & -1 & 0 \end{pmatrix}$$

with zero all other entries. Therefore, $A_\xi + A_\xi^t = 0$ and ξ is a Killing vector field. Since ξ is supposed global, ξ is a Hopf vector field.

Finally, if we take $X, Y \perp \xi$, we get the equation

$$(\nabla_X A_\xi)Y = \langle A_\xi X, A_\xi Y\rangle \xi.$$

But for a Killing vector field ξ we have [16]

$$(\nabla_X A_\xi)Y = R(\xi, X)Y = \langle X, Y\rangle \xi.$$

Since ξ is a Hopf vector field, $\langle A_\xi X, A_\xi Y\rangle = \langle X, Y\rangle$. So, in this case we have an identity.

If we suppose now that ξ is a Hopf vector field on a unit sphere, then ξ is covariantly normal as a Killing vector field and totally geodesic [27] as a characteristic vector field of a standard contact metric structure on S^{2m+1}. □

Theorem 3 is a correct and simplified version of Theorem 2.1 [27], where the normality of the operator A_ξ was implicitly used in a proof.

In the case of a *weaker condition* on the field ξ to be only a *geodesic* one, the result is not so definite. We begin with some preparations.

The almost complex structure on TM^n is defined by

$$JX^h = X^v, \qquad JX^v = -X^h$$

for all vector field X on M^n. Thus, TM^n with Sasakian metric is an almost Kählerian manifold. It is Kählerian if and only if M^n is flat [9].

The unit tangent bundle T_1M^n is a hypersurface in TM^n with a unit normal vector ξ^v at each point $(q, \xi) \in T_1M^n$. Define a unit vector field $\bar\xi$, a 1-form $\bar\eta$ and a $(1,1)$ tensor field $\bar\varphi$ on T_1M^n by

$$\bar\xi = -J\xi^v = \xi^h, \qquad JX = \bar\varphi X + \bar\eta(X)\xi^v.$$

The triple $(\bar\xi, \bar\eta, \bar\varphi)$ form a standard almost contact structure on T_1M^n with Sasakian metric g_S. This structure is not almost contact *metric* one. By taking

$$\tilde\xi = 2\bar\xi = 2\xi^h, \qquad \tilde\eta = \frac{1}{2}\bar\eta, \qquad \tilde\varphi = \bar\varphi, \qquad g_{cm} = \frac{1}{4}g_S$$

at each point $(q, \xi) \in T_1M^n$, we get the *almost contact metric structure* $(\tilde\xi, \tilde\eta, \tilde\varphi)$ on (T_1M^n, g_{cm}).

In a case of a general almost contact metric manifold $(\tilde M, \tilde\xi, \tilde\eta, \tilde\varphi, \tilde g)$ the following definition is known [7].

Definition 5. *A submanifold N of a contact metric manifold $(\tilde M, \tilde\xi, \tilde\eta, \tilde\varphi, \tilde g)$ is called invariant if $\tilde\varphi(T_pN) \subset T_pN$ and anti-invariant if $\tilde\varphi(T_pN) \subset (T_pN)^\perp$ for every $p \in N$.*

If N is the invariant submanifold, then the characteristic vector field $\tilde\xi$ is *tangent* to N at each of its points.

After all mentioned above, the following definition is natural [3].

Definition 6. *A unit vector field ξ on a Riemannian manifold (M^n, g) is called invariant (anti-invariant) is the submanifold $\xi(M^n) \subset (T_1M^n, g_{cm})$ is invariant (anti-invariant).*

It is easy to see from (4) that the *invariant* unit vector field is always a geodesic one, i.e. its integral trajectories are geodesic lines.

Binh, Boeckx and Vanhecke [3] have considered this kind of unit vector fields and proved the following

Theorem 4. *A unit vector field ξ on (M^n, g) is invariant if and only if $(\tilde\xi = \xi,\ \tilde\eta = \langle\cdot, \xi\rangle_g, \tilde\varphi = A_\xi)$ is an almost contact structure on M^n. In particular, ξ is a geodesic vector field on M^n and $n = 2m + 1$.*

Now we can formulate the result.

Theorem 5. *A unit geodesic vector field ξ on S^{n+1} is totally geodesic if and only if $n = 2m$ and ξ is a strongly normal invariant unit vector field.*

Proof: Suppose ξ is a geodesic and totally geodesic unit vector field. Then $A_\xi \xi = 0$ and the equation (9) takes the form (10). Follow the proof of Theorem 3, we come to the following conditions on the field ξ

$$A_\xi^2 X = -X, \qquad (\nabla_X A_\xi)Y = \langle A_\xi X, A_\xi Y \rangle \xi \tag{11}$$

for all $X, Y \in \xi^\perp$. From the left equation in (11) we conclude that $n = 2m$. Comparing the right one with (8), we see that ξ is a strongly normal vector field.

Consider now a $(1,1)$ tensor field $\varphi = A_\xi = -\nabla\xi$ and a 1-form $\eta = \langle \cdot, \xi \rangle$. Taking into account the left equation in (11) and $A_\xi \xi = 0$, we see that

$$\varphi^2 X = -X + \eta(X)\xi, \qquad \varphi\xi = 0, \qquad \eta(\varphi X) = 0, \qquad \eta(X) = 1$$

for any vector field X on the sphere. Therefore, the triple

$$\tilde{\varphi} = A\xi, \qquad \tilde{\xi} = \xi, \qquad \tilde{\eta} = \langle \cdot, \xi \rangle$$

form an *almost contact structure* with the field ξ as a characteristic vector field of this structure. By Theorem 4, the field ξ is invariant.

Conversely, suppose ξ is strongly normal and invariant vector field on S^{n+1}. Then, by Theorem 4, ξ is geodesic and $n = 2m$. The rest of the proof is a direct checking of formula (10). □

5. A Remarkable Property of the Hopf Vector Field

It is well-known that for a unit sphere S^n the standard contact metric structure on $T_1 S^n$ is a Sasakian one. If ξ is a Hopf unit vector field on S^{2m+1}, then ξ is a characteristic vector field of a standard contact metric structure on the unit sphere S^{2m+1}. By Theorem 4, the submanifold $\xi(S^{2m+1})$ is invariant submanifold in $T_1 S^{2m+1}$. Therefore, $\xi(S^{2m+1})$ *is also Sasakian* with respect to the induced structure [29]. Since the Hopf vector field is strongly normal, by Theorem 5, the submanifold $\xi(S^{2m+1})$ is totally geodesic. The sectional curvature of the submanifold $\xi(S^{2m+1})$ was found in [27] and implies a remarkable corollary.

Theorem 6. *Let ξ be a Hopf vector field on the unit sphere S^{2m+1}. With respect to the induced structure, the manifold $\xi(S^{2m+1})$ is a Sasakian space form of φ-curvature* $5/4$.

In other words, the Hopf vector field provides an example of embedding of a *Sasakian space form* of φ-curvature 1 into *Sasakian manifold* such that the image is contact, totally geodesic *Sasakian space form* of φ-curvature $5/4$ with respect to the induced structure.

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