Seventh International Conference on
Geometry, Integrability and Quantization
June 2–10, 2005, Varna, Bulgaria
Ivailo M. Mladenov and Manuel de León, Editors
SOFTEX, Sofia 2005, pp 280–291

PAINLEVÉ ANALYSIS AND EXACT SOLUTIONS OF NONINTEGRABLE SYSTEMS

SERGEY YU. VERNOV

Skobeltsyn Institute of Nuclear Physics, Moscow State University
119992, Moscow, Russia

Abstract. Here we consider the cubic complex Ginzburg–Landau equation. Applying the Hone's method, based on the use of the Laurent-series solutions and the residue theorem, we have proved that this equation has no elliptic standing wave solutions. This result supplements Hone's result, that this equation has no elliptic travelling wave solutions. It has been shown that the Hone's method can be applied to a system of polynomial differential equations more effectively than to an equivalent differential equation.

1. Introduction

Nonlinear dynamic systems and evolution equations actively used in physics are often nonintegrable in the sense that it is impossible to find their general solutions using known methods. At the same time knowledge of special solutions with some given properties is sufficient for construction of physical models. At present time methods for construction of special solutions in terms of elementary (degenerated elliptic) and elliptic functions are well developed [3,8–11,19–21,24,26,29–31,34, 36] (see also [22] and references therein). Some of these methods are intended for the search of elliptic solutions only [3,20,29], others allow to find either solutions in terms of elementary functions [8,9,36] or both types of solutions [10,11,19, 21,24,26,30,31,34]. Note that both elliptic and degenerate elliptic functions are solutions of some first order polynomial differential equations.

Elliptic and degenerate elliptic solutions of some differential equation can exist only if there exist formal Laurent series solutions of them. One can construct such formal solutions using the Ablowitz–Ramani–Segur algorithm of the Painlevé test [1] (see also [32]). In [24] the method of construction of analytic special solutions of nonintegrable systems with the help of the Laurent series solutions has been proposed. The Laurent series solutions can be used to prove nonexistence of elliptic solutions as well [18].

In this paper we show that in order to prove the nonexistence of elliptic solutions the analysis of the system of differential equations may be more useful than the analysis of the equivalent differential equation. In two of the above mentioned papers ([18,24]) the cubic complex Ginzburg–Landau equation [13] has been investigated. We also consider this equation. Using the Hone's method we will prove that this equation has no elliptic standing wave solutions. This result supplements Hone's result, that this equation has no elliptic travelling wave solutions.

2. The Cubic Complex Ginzburg–Landau Equation

The one-dimensional cubic **complex Ginzburg–Landau equation** (CGLE) [13] is one of the most-studied nonlinear equations (see [4] and references therein). It is a generic equation which describes many physical phenomena, such as pattern formation near a supercritical Hopf bifurcation [4,12], the propagation of a signal in optical fibers [2], spatiotemporal intermittency in spatially extended dissipative systems [16,23]. The CGLE

$$\mathrm{i}A_t + pA_{xx} + q|A|^2A - \mathrm{i}\gamma A = 0 \tag{1}$$

where $p \in \mathbb{C}$, $q \in \mathbb{C}$ and $\gamma \in \mathbb{R}$ is not integrable if $pq\gamma \neq 0$. In the case $q/p \in \mathbb{R}$, $\gamma = 0$ the CGLE is integrable and coincides with the well-known nonlinear Schrödinger equation [15,25].

One of the most important direction in the study of the CGLE is the consideration of its travelling and standing wave solutions [3–6,9,17–19,24,27,28]. Substituting

$$A(x,t) = \sqrt{M(\xi)}\mathrm{e}^{\mathrm{i}(\varphi(\xi)-\omega t)}, \qquad \xi = x - ct, \qquad c \in \mathbb{R}, \quad \omega \in \mathbb{R} \tag{2}$$

in (1) we obtain the following third order system of ordinary differential equations

$$\begin{aligned} &\frac{M''}{2M} - \frac{M'^2}{4M^2} - \left(\psi - \frac{cs_r}{2}\right)^2 - \frac{cs_iM'}{2M} + d_rM + g_i = 0 \\ &\psi' + \left(\psi - \frac{cs_r}{2}\right)\left(\frac{M'}{M} - cs_i\right) + d_iM - g_r = 0 \end{aligned} \tag{3}$$

where $\psi \equiv \varphi' \equiv \mathrm{d}\varphi/\mathrm{d}\xi$, $M' \equiv \mathrm{d}M/\mathrm{d}\xi$. Six real parameters d_r, d_i, g_r, g_i, s_r and s_i are given in terms of c, p, q, γ and ω as follows

$$d_r + \mathrm{i}d_i = \frac{q}{p}, \quad s_r - \mathrm{i}s_i = \frac{1}{p}, \quad g_r + \mathrm{i}g_i = \frac{\gamma + \mathrm{i}\omega}{p} + \frac{1}{2}c^2s_is_r + \frac{\mathrm{i}}{4}c^2s_r^2. \tag{4}$$

Using (3) one can express ψ in terms of M and its derivatives

$$\psi = \frac{cs_r}{2} + \frac{G' - 2cs_iG}{2M^2(g_r - d_iM)} \tag{5}$$

where

$$G \equiv \frac{1}{2}MM'' - \frac{1}{4}M'^2 - \frac{cs_i}{2}MM' + d_r M^3 + g_i M^2 \qquad (6)$$

and in this way to obtain the third order differential equation in M

$$(G' - 2cs_i G)^2 - 4GM^2(d_i M - g_r)^2 = 0. \qquad (7)$$

We will consider the case in which p/q is not a real number. This condition is equivalent to the restriction $d_i \neq 0$. In this case the equation (7) is not integrable [7, 9]. Using the Painlevé analysis [7] or topological arguments [28] it has been shown that single-valued solutions depend on one arbitrary parameter. Equation (7) is autonomous, so this arbitrary parameter is ξ_0: if $M = f(\xi)$ is a solution, then $M = f(\xi - \xi_0)$, where $\xi_0 \in \mathbb{C}$ has to be a solution. Special solutions in terms of elementary functions have been found in [5,9,19,27]. All known exact solutions of the CGLE are degenerate elliptic (rational, trigonometric or hyperbolic) functions. The full list of these solutions is presented in [18,24].

3. Elliptic Functions

The function $\varrho(z)$ of the complex variable z is a doubly-periodic function if there exist two numbers ω_1 and ω_2 which ratio ω_1/ω_2 is not a real number, and such that for all $z \in \mathbb{C}$

$$\varrho(z) = \varrho(z + \omega_1) = \varrho(z + \omega_2). \qquad (8)$$

By definition a double-periodic meromorphic function is called an elliptic function [14]. The periods define the period parallelograms with vertices z_0, $z_0 + N_1\omega_1$, $z_0 + N_2\omega_2$ and $z_0 + N_1\omega_1 + N_2\omega_2$, where N_1 and N_2 are arbitrary natural numbers and z_0 is an arbitrary complex number. The classical theorems for elliptic functions (see, for example [14]) ensure that

- If an elliptic function has no poles then it is a constant.
- The number of elliptic function poles within any finite period parallelogram is finite.
- The sum of residues within any finite period parallelogram is equal to zero (the residue theorem).
- If $\varrho(z)$ is an elliptic function then any rational function of $\varrho(z)$ and its derivatives is an elliptic function as well.

From (5) it follows that if M is an elliptic function then ψ has to be an elliptic function. Therefore, if we prove that ψ can not be an elliptic function, we will prove that M can not be an elliptic function as well. To prove this we construct the Laurent-series solutions for system (3) and apply the residue theorem to the function ψ and its degrees.

4. Nonexistence of the Standing Wave Elliptic Solutions

4.1. The Laurent-Series Solutions and the Residue Theorem

To prove the non-existence of elliptic solutions to (3) we will use its solutions in the form of the Laurent series, which can be easily found due to the Ablowitz–Ramani–Segur algorithm of the Painleve test [1]. In such a way we obtain solutions only as a formal series, but really we will use only a finite number of these series coefficients, so we do not need actually the convergence of these series. It is known [9, 18, 24] that there are only two types of the Laurent-series solutions of (3) or (7). These solutions depend on the arbitrary parameter ξ_0, which determines the location of the singular point. At singular points ψ and M tend to infinity as $1/t$ and $1/t^2$, respectively. We denote the different Laurent series for the function ψ as

$$\psi_1 = \sum_{k=-1}^{\infty} C_k(\xi - \xi_0)^k \qquad \text{and} \qquad \psi_2 = \sum_{k=-1}^{\infty} D_k(\xi - \widetilde{\xi}_0)^k \tag{9}$$

with $C_{-1} \neq 0$ and $D_{-1} \neq 0$. A nonconstant elliptic function should have poles. Let us assume that in some parallelogram of periods $\psi(\xi)$ has $N_1 + N_2$ poles, its Laurent series expansions being ψ_1 in the neighbourhood of N_1 poles and ψ_2 in the neighbourhood of N_2 poles. If $\psi(\xi)$ is an elliptic function then the sum of its residues in some parallelogram of periods has to be zero, therefore, this function has both types of the Laurent series expansions (9) and

$$N_1 = -\frac{D_{-1}}{C_{-1}} N_2. \tag{10}$$

If $\psi(\xi)$ is an elliptic function then its powers ψ^k have to be elliptic functions as well, so they have N_1 Laurent series expansions ψ_1^k and N_2 Laurent series expansions ψ_2^k. To calculate residues of ψ_1^k (or ψ_2^k) we have to use only k leading terms in the Laurent series ψ_1 (ψ_2). The residue theorem for the functions ψ^k gives algebraic equations in the coefficients of ψ_1 and ψ_2 Laurent series. Coefficients of these series depend on numerical parameters of the system (3) and only on them (have no resonances), hence, we obtain a system of algebraic equations in coefficients of (3), at which (3) can have elliptic solutions.

For example, if we demand that the function ψ^2 is an elliptic function, then using (10) we obtain the following equation

$$C_0 = D_0 \tag{11}$$

analogously ψ^3, ψ^4 and ψ^5 can be elliptic function only if

$$\begin{aligned} & C_1C_{-1} + C_0^2 = D_1D_{-1} + D_0^2 \\ & C_2C_{-1}^2 + 3C_1C_0C_{-1} + C_0^3 = D_2D_{-1}^2 + 3D_1D_0D_{-1} + D_0^3 \\ & C_3C_{-1}^3 + 4C_2C_0C_{-1}^2 + 2C_1^2C_{-1}^2 + 6C_{-1}C_0^2C_1 + C_0^4 \\ & \qquad = D_3D_{-1}^3 + 4D_2D_0D_{-1}^2 + 2D_1^2D_{-1}^2 + 6D_1D_0^2D_{-1} + D_0^4. \end{aligned} \tag{12}$$

In Subsection 4.4 we also use the corresponding equation for ψ^7 under conditions $C_0 = 0,\ C_2 = 0,\ C_4 = 0,\ D_0 = 0,\ D_2 = 0$ and $D_4 = 0$

$$C_5C_{-1}^5 + 6C_3C_1C_{-1}^4 + 5C_1^3C_{-1}^3 = D_5D_{-1}^5 + 6D_{-1}^4D_3D_1 + 5D_1^3D_{-1}^3. \tag{13}$$

We have calculated the residues of powers of ψ with the help of the procedure **ydegree** from our package of Maple procedures.

Note that using the residue theorem in the Laurent series solutions for the function M we obtain more complex system of algebraic equations, because the function M tends to infinity at singular points as $1/\xi^2$, but not as $1/\xi$.

4.2. The Number of Essential Numerical Parameters of System (3)

The system (3) includes seven arbitrary constants, some of which can be fixed without loss of generality. First of all one can fix s_r and s_i. From the condition $p \notin \mathbb{R}$ (the case of real p will be considered separately) follows that $s_i \neq 0$. Using the following transformations

$$\widetilde{c} = \varpi c, \qquad \widetilde{s}_i = \frac{s_i}{\varpi}, \qquad \widetilde{s}_r = \tau s_r, \qquad \widetilde{\psi} = \psi - \frac{cs_r}{2}(1 - \tau\varpi) \tag{14}$$

one can put

$$\widetilde{s}_r = -\frac{1}{10} \qquad \text{and} \qquad \widetilde{s}_i = -\frac{3}{10}. \tag{15}$$

Using the transformations

$$\widetilde{M} = \mu M, \qquad \widetilde{d}_i = \frac{d_i}{\mu}, \qquad \widetilde{d}_r = \frac{d_r}{\mu} \tag{16}$$

we can fix d_r or d_i. Following [18] we will fix the value of d_r. Our restriction on parameters p and q gives no information about d_r, so we have to consider two cases: $d_r = 0$ and $d_r \neq 0$ separately. Using scaling transformations of the independent variable ξ it is possible to fix g_i or g_r, but, following [18, 24] we leave them arbitrary to consider zero and nonzero values of these parameters at once.

From the second equation in (3) it follows that if ψ is a constant then M can not be an elliptic function, so to obtain nontrivial elliptic solutions we have to assume that ψ has poles. We do not restrict ourself to the case $c = 0$ and prove the non-existence of either travelling or standing wave solutions. It has been noted in [10] that one does not need to transform a system of differential equations into one equation to obtain the Laurent-series solutions.

4.3. The Case $d_r = 0$

Let us consider the system (3) with

$$d_r = 0, \qquad s_r = -\frac{1}{10} \qquad \text{and} \qquad s_i = -\frac{3}{10}. \tag{17}$$

There exist two different Laurent-series solutions (we put $\xi_0 = \widetilde{\xi}_0 = 0$) of (3)

$$\breve{\psi}_1 = \frac{\sqrt{2}}{\xi} - \frac{c(\sqrt{2}+1)}{20} + \mathcal{O}(\xi), \qquad \breve{M}_1 = \frac{3\sqrt{2}}{d_i}\left(\frac{1}{\xi^2} - \frac{1}{10\xi}\right) + \mathcal{O}(1) \tag{18}$$

and

$$\breve{\psi}_2 = -\frac{\sqrt{2}}{\xi} + \frac{c(\sqrt{2}-1)}{20} + \mathcal{O}(\xi), \qquad \breve{M}_2 = -\frac{3\sqrt{2}}{d_i}\left(\frac{1}{\xi^2} - \frac{1}{10\xi}\right) + \mathcal{O}(1). \tag{19}$$

From (10) it follows that $N_1 = N_2$, that is to say, if the function $\psi(\xi)$ has N poles with residues, which are equal to $\sqrt{2}$, within some finite period parallelogram, then in this domain the number of poles, which residues are equal to $-\sqrt{2}$, has to be equal to N as well.

Residues of $\breve{\psi}_1^2$ are equal to $-2\sqrt{2}c(\sqrt{2}+1)/20$, whereas residues of $\breve{\psi}_2^2$ are $-2\sqrt{2}c(\sqrt{2}-1)/20$. From equation (11) we obtain that the sum of residues of the function ψ^2 is equal to zero if and only if $c = 0$. So, we prove the absence of the travelling wave solutions. Note that to obtain this result we have used only two coefficients of the Laurent series ψ_1 and ψ_2. In the case $c = 0$ we have to apply the residue theorem for ψ^3 and ψ^4, so, we have to calculate four coefficients in these series (two of them are zero at $c = 0$)

$$\breve{\psi}_1 = \frac{\sqrt{2}}{\xi} + \frac{0}{\xi} + \frac{1}{21}\left(5\sqrt{2}g_i - g_r\right)\xi + 0\xi^2 + \mathcal{O}(\xi^3) \tag{20}$$

and

$$\breve{\psi}_2 = -\frac{\sqrt{2}}{\xi} + \frac{0}{\xi} - \frac{1}{21}\left(5\sqrt{2}g_i + g_r\right)\xi + 0\xi^2 + \mathcal{O}(\xi^3). \tag{21}$$

From (12) we obtain that the functions ψ^3 and ψ^4 satisfy the residue theorem if and only if

$$g_i = 0 \qquad \text{and} \qquad g_r = 0. \tag{22}$$

In this case the Laurent-series solutions give

$$\breve{\psi}_1(\xi) = \frac{\sqrt{2}}{\xi}, \qquad \breve{M}_1(\xi) = \frac{3\sqrt{2}}{d_i\xi^2} \tag{23}$$

and

$$\breve{\psi}_2(\xi) = -\frac{\sqrt{2}}{\xi}, \qquad \breve{M}_2(\xi) = -\frac{3\sqrt{2}}{d_i\xi^2}. \tag{24}$$

The straightforward substitution of these functions in system (3) along with $c = 0$, $d_r = 0$, $g_r = 0$ and $g_i = 0$ proves that they are exact solutions. Using the Ablowitz–Ramani–Segur algorithm [1] it is easy to prove that the coefficients of the Laurent-series solutions does not include arbitrary parameters, so the obtained solutions are unique single-valued solutions and the CGLE has no elliptic solution for these values of parameters as well. Thus we have proved the non-existence of both travelling and standing wave elliptic solutions for $d_r = 0$.

4.4. The Case $d_r \neq 0$

In this case we can use the following values of parameters without loss of generality

$$d_r = \frac{1}{2}, \qquad s_r = -\frac{1}{10} \qquad \text{and} \qquad s_i = -\frac{3}{10}. \tag{25}$$

To simplify calculations, and following [18], we will express d_i through a new real parameter

$$d_i = \pm\frac{3\sqrt{\beta^2-1}}{4\sqrt{2}}, \qquad \beta > 1. \tag{26}$$

If $d_i > 0$ (for the $+$ sign in equation (26)), the system (3) has the following Laurent series solutions

$$\widetilde{\psi}_1 = \frac{\sqrt{2}(\beta+1)}{\sqrt{\beta^2-1}}\xi^{-1} - \frac{c}{10}\left(\frac{\sqrt{\beta^2-1}}{\sqrt{2}(\beta-1)} + \frac{1}{2}\right) + \mathcal{O}(\xi) \tag{27}$$

$$\widetilde{\psi}_2 = -\frac{\sqrt{2}(\beta-1)}{\sqrt{\beta^2-1}}\xi^{-1} + \frac{c}{10}\left(\frac{\sqrt{\beta^2-1}}{\sqrt{2}(\beta+1)} - \frac{1}{2}\right) + \mathcal{O}(\xi) \tag{28}$$

If there are N_1 Laurent series of type $\widetilde{\psi}_1$ and N_2 Laurent series of type $\widetilde{\psi}_2$ then equation (10) gives

$$N_1 = \frac{\beta-1}{\beta+1}N_2. \tag{29}$$

The residues theorem for ψ^2 gives $c\beta = 0$. Using condition $\beta > 1$, we derive that c has to be equal to zero, so we rederive the main result of [18] that the CGLE has no elliptic travelling wave solutions for non-zero values of parameters. Note that the use of Laurent series of $\psi(\xi)$ instead of the Laurent series of $M(\xi)$ allows to simplify the calculations.

Let us consider the standing wave solutions ($c = 0$) of the CGLE. The Laurent series solutions are

$$\begin{aligned}\widetilde{\psi}_1 = & \frac{\sqrt{2}(\beta+1)}{\sqrt{\beta^2-1}}\xi^{-1} - \frac{\left(\beta g_r - 5g_r - 5\sqrt{2(\beta^2-1)}g_i\right)}{3(7\beta+5)}\xi \\ & + \frac{1}{90(\beta+1)(7\beta+5)^2}\Big\{32(\beta^2-1)(\beta-8)g_i g_r \\ & + \sqrt{2(\beta^2-1)}\left(122(\beta^2-1)g_i^2 + (11\beta^2-34\beta+61)g_r^2\right)\Big\}\xi^3 \\ & + \frac{\beta-1}{1890(\beta+1)(3\beta+1)(7\beta+5)^3}\Big\{24g_i^2 g_r(\beta^2-1)(147\beta^2 \\ & + 934\beta+775) - 4g_r^3(231\beta^4+656\beta^3-18\beta^2-552\beta-445) \\ & + g_i\sqrt{2(\beta^2-1)}(20g_i^2(\beta^2-1)(399\beta+349) \\ & - 3g_r^2(483\beta^3-473\beta^2-2823\beta-2435))\Big\}\xi^5 + \mathcal{O}(\xi^7)\end{aligned} \tag{30}$$

$$\begin{aligned}\widetilde{\psi}_2 = & -\frac{\sqrt{2}(\beta-1)}{\sqrt{\beta^2-1}}\xi^{-1} - \frac{\left(\beta g_r + 5g_r + 5\sqrt{2(\beta^2-1)}g_i\right)}{3(7\beta-5)}\xi \\ & + \frac{1}{90(\beta+1)(7\beta+5)^2}\Big\{32(\beta^2-1)(\beta+8)g_i g_r \\ & + \sqrt{2(\beta^2-1)}\left(122(\beta^2-1)g_i^2 - (11\beta^2-34\beta-61)g_r^2\right)\Big\}\xi^3 \\ & + \frac{\beta+1}{1890(3\beta-1)(\beta-1)(7\beta-5)^3}\Big\{24g_i^2 g_r(\beta^2-1)(147\beta^2 \\ & - 934\beta+775) - 4g_r^3(231\beta^4-656\beta^3-18\beta^2+552\beta-445) \\ & - g_i\sqrt{2(\beta^2-1)}(20g_i^2(\beta^2-1)(399\beta-349) \\ & - 3g_r^2(483\beta^3+473\beta^2-2823\beta+2435)\Big\}\xi^5 + \mathcal{O}(\xi^7).\end{aligned} \tag{31}$$

The residues of ψ^2, ψ^4 and ψ^6 are equal to zero at $c = 0$. Substituting the coefficients of the Laurent series $\widetilde{\psi}_1$ and $\widetilde{\psi}_2$, we transform system (12) and equation (13) into the algebraic system in β, g_i and g_r. This system is too cumbersome to be presented here, but can be easily solved via computer algebra system like Maple or REDUCE. The condition $\beta > 1$ leaves only one solution of this system

$$g_r = 0, \qquad g_i = 0. \tag{32}$$

In the case $d_i < 0$ we also obtain that the residue theorem for powers of ψ can be satisfied only if $g_r = 0$ and $g_i = 0$. Let us consider system (3) with zero values of c, g_i and g_r, $d_r = 1/2$ and an arbitrary (nonzero) value of d_i

$$\begin{aligned} &2MM'' - M'^2 - 4M^2\psi^2 + 2M^3 = 0 \\ &M\psi' + M'\psi + d_i M^2 = 0. \end{aligned} \tag{33}$$

The straightforward substitution gives that the functions

$$\widetilde{\psi}_1(\xi) = \frac{3+\sqrt{9+32d_i^2}}{4d_i\xi} \qquad \widetilde{M}_1(\xi) = \frac{3(3+\sqrt{9+32d_i^2})}{4d_i^2\xi^2} \tag{34}$$

and

$$\widetilde{\psi}_2(\xi) = \frac{3-\sqrt{9+32d_i^2}}{4d_i\xi}, \qquad \widetilde{M}_2(\xi) = \frac{3(3-\sqrt{9+32d_i^2})}{4d_i^2\xi^2} \tag{35}$$

are exact solutions of the system (33). This system has no other single-valued solutions, so we have proved the non-existence of neither elliptic standing wave nor elliptic travelling wave solutions in the case $d_r \neq 0$ as well. In our calculations we assume that $s_i \neq 0$. At the same time our results prove the non-existence of elliptic solutions in the case $s_i = 0$ too. Indeed, if $c = 0$ then cases with $s_i = 0$ and $s_i \neq 0$ coincide, to transform the case $\{s_i = 0,\, c \neq 0\}$ into the considered case $\{s_i \neq 0,\, c = 0\}$ we have to add a constant to $\psi(\xi)$.

5. Conclusions

The Laurent-series solutions are useful not only to find elliptic solutions, but also to prove their non-existence. Using the Hone's method, based on residue theorem, we have amplified the Hone's result [18], that the CGLE with generic (non-zero) values of parameters has no elliptic travelling wave solution and have proved the non-existence of both standing and travelling wave elliptic solutions of the CGLE in the case when p/q is not a real number. The existence of similar elliptic solutions to Nonlinear Schrodinger Equation [9] and degenerate elliptic solutions to the CGLE one could interpret as indications that the CGLE has elliptic solutions, so the obtained result is unexpected.

We have shown that the Hone's method can be applied to a system of polynomial differential equations more effectively than to an equivalent differential equation. In general when one makes use of the Conte–Musette and Hone's methods he can choose a function, which analytic form should be found. It could be any polynomial of unknown functions and their derivatives. The Hone's method is so effective in the case of the CGLE, because coefficients of the Laurent series solutions depend only on parameters of equations, i.e., they does not include additional arbitrary parameters (have no resonances). It is an important problem to generalize Hone's method on the Laurent series solutions for the cases with resonances.

Another field for future investigations is the improvement and generalization of the Conte–Musette method, future development of computer algebra realization of them [33,35]. On the one hand it should be generalized on the case of multivalued solutions (the first result in this direction has been obtained in [34]), on the other hand in the case of the search for elliptic solutions only it is important to use particular properties of elliptic functions, for example, those which follow from the residue theorem.

Acknowledgements

The author is grateful to the organizers of the Seventh International Conference on Geometry, Integrability and Quantization for hospitality and would like to thank R. Conte, who called his attention to [18], and A. Hone for useful comments. This work has been supported in part by RFBR grant 05–01–10620, by Russian Federation President's Grant NSh–1685.2003.2 and by the grant of the scientific Program "Universities of Russia" UR.02.02.503.

References

[1] Ablowitz M., Ramani A. and Segur H., *A Connection Between Nonlinear Evolution Equations and Ordinary Differential Equations of P-Type* I & II, J. Math. Phys. **21** (1980) 715–721 1006–1015.

[2] Agrawal G., *Nonlinear Fiber Optics*, Academic, Boston, 1989.

[3] Akhmediev N., Afanasjev V. and Soto-Crespo J., *Singularities and Spesial Soliton Solutions of the Cubic-Quintic Complex Ginzburg–Landau Equation*, Rev. Phys. E **53** (1996) 1190–1201.

[4] Aranson I. and Kramer L., *The World of the Complex Ginzburg–Landau Equation*, Rev. Mod. Phys. **74** (2002) 99–143, cond-mat/0106115.

[5] Bekki N. and Nozaki K., *Formations of Spatial Patterns and Holes in the Generalized Ginzburg–Landau Equation*, Phys. Lett. A **110** (1985) 133–135.

[6] Brusch L., Torcini A., van Hecke M., Zimmermann M. and Bär M., *Modulated Amplitude Waves and Defect Formation in the One-Dimensional Complex Ginzburg–Landau Equation*, Physica D **160** (2001) 127–148, nlin.CD/0104029.

[7] Cariello F. and Tabor M., *Painlevé Expansions for Nonintegrable Evolution Equations*, Physica D **39** (1989) 77–94.

[8] Conte R. and Musette M., *Link Between Solitary Waves and Projective Riccati Equations*, J. Phys. A **25** (1992) 5609–5623

[9] Conte R. and Musette M., *Linearity Inside Nonlinearity: Exact Solutions to the Complex Ginzburg-Landau Equation*, Physica D **69** (1993) 1–17.

[10] Conte R. and Musette M., *Solitary Waves of Nonlinear Nonintegrable Equations*, nlin.PS/0407026.

[11] Fan E., *An Algebraic Method for Finding a Series of Exact Solutions to Integrable and Nonintegrable Nonlinear Evolutions Equations*, J. Phys. A **36** (2003) 7009–7026.

[12] Cross M. and Hohenberg P., *Pattern Formation Outside of Equilibrium*, Rev. Mod. Phys. **65** (1993) 851–1112.

[13] Ginzburg V. and Landau L., *On the Theory of Superconductors*, Zh. Eksp. Teor. Fiz. **20** (1950) 1064–1082.

[14] Bateman H. and Erdélyi A., *Higher Transcendental Functions*, vol. 3, McGraw-Hill, New York, 1955.

[15] Faddeev L. and Takhtajan L., *Hamiltonian Methods in the Theory of Solitons*, Nauka, Moscow, 1986; English translation: Springer, Berlin, 1987.

[16] van Hecke M., Storm C. and van Saarlos W., *Sources, Sinks and Wavenumber Selection in Coupled CGL Equations and Experimental Implications for Counter-Propagating Wave Systems*, Physica D **133** (1999) 1–47, Patt-sol/9902005.

[17] van Hecke M., *Coherent and Incoherent Structures in Systems Described by the 1D CGLE: Experiments and Identification*, Physica D **174** (2003) 134–151, cond-mat/01100068.

[18] Hone A., *Non-Existence of Elliptic Travelling Wave Solutions of the Complex Ginzburg–Landau Equation*, Phósica D 205 (2005) 292–306.

[19] Kudryashov N., *Exact Solutions of a Generalized Equation of Ginzburg–Landau*, Matematicheskoye modelirovanie **1** (1989) 151–158 (in Russian).

[20] Kudryashov N., *Nonlinear Differential Equations with Exact Solutions Expressed via the Weierstrass Function*, nlin.CD/0312035.

[21] Kudryashov N., *Simplest Equation Method to Look for Exact Solutions of Nonlinear Differential Equations*, nlin.SI/0406007.

[22] Kudryashov N., *Analytic Theory of Nonlinear Differential Equations*, Moscow-Izhevsk, RCD, 2004.

[23] Manneville P., *Dissipative Structures and Weak Turbulence*, Academic Press, Boston, 1990.

[24] Musette M. and Conte R., *Analytic Solitary Waves of Nonintegrable Equations*, Physica D **181** (2003) 70–76; nlin.PS/0302051

[25] Newell A., *Solitons in Mathematics and Physics*, Society for Industrial and Applied Mathematics, Filadelphia, 1985.

[26] Nikitin A. and Barannyk T., *Solitary Wave and Other Solutions for Nonlinear Heat Equations*, math-ph/0303004.

[27] Nozaki K. and Bekki N., *Exact Solutions of the Generalized Ginzburg–Landau Equation*, J. Phys. Soc. Jpn. **53** (1984) 1581–1582.

[28] van Saarloos W. and Hohenberg P., *Fronts, Pulses, Sources and Sinks in Generalized Complex Ginzburg–Landau Equations*, Physica D **56** (1992) 303–367, Erratum **69** (1993) 209.

[29] Santos G., *Application of Finite Expansion in Elliptic Functions to Solve Differential Eduations*, J. Phys. Soc. Jpn. **58** (1989) 4301.

[30] Timoshkova E., *A New Class of Trajectories of Motion in the Hénon–Heiles Potential Field*, Rus. Astron. J., **76** (1999) 470–475, Astron. Rep. **43** (1999) 406–411.

[31] Vernov S. and Timoshkova E., *On Two Nonintegrable Cases of the Generalized Hénon–Heiles System*, Physics of Atomic Nuclei (Yadernaya Fizika) **68** (2005) 1947–1955; math-ph/0402049.

[32] Vernov S., *Constructing Solutions for the Generalized Hénon–Heiles System through the Painlevé Test*, Theor. Math. Phys. **135** (2003) 792–801; math-ph/0209063.

[33] Vernov S., *Construction of Single-Valued Solutions for Nonintegrable Systems with the Help of the Painlevé Test*, In: Proc. of the International Conference "Computer Algebra in Scientific Computing" (CASC 2004, St. Petersburg, Russia), V. Ganzha, E. Mayr and E. Vorozhtsov (Eds.), Technische Universität, Munchen, Garching, 2004, pp. 457–465; nlin.SI/0407062

[34] Vernov S., *Single-Valued ànd Multivalued Solutions for the Generalized Henon–Heiles System with an Additional Nonpolynomial Term*, In: Geometry, Integrability and Quantization VI, I. Mladenov and A. Hirshfeld (Eds.), SOFTEX, Sofia, 2005, pp. 297–310.

[35] Vernov S., *Interdependence between the Laurent-Series and Elliptic Solutions of Nonintegrable Systems*, In: the Lecture Notes of Computer Science (Springer) LNCS, V. 3718 (the Proceedings of International Conference "Computer Algebra in Scientific Computing" (CASC 2005, Kalamata, Greece)), V. Ganzha, E. Mayr and E. Vorozhtsov (Eds.), Springer, Berlin, 2005, pp. 457–468.

[36] Vladimirov V. and Kutafina E., *Exact Travelling Wave Solutions of Some Nonlinear Evolutions Equations*, Rep. Math. Phys. **54** (2004) 261–271.