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NEW INTEGRABLE MULTI-COMPONENT NLS TYPE EQUATIONS ON SYMMETRIC SPACES: $\mathbb{Z}_4$ AND $\mathbb{Z}_6$ REDUCTIONS

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Abstract. The reductions of the multi-component nonlinear Schrödinger models related to **C.I** and **D.III** type symmetric spaces are studied. We pay special attention to the MNLS related to the $\mathfrak{sp}(4)$, $\mathfrak{so}(10)$ and $\mathfrak{so}(12)$ Lie algebras. The MNLS related to $\mathfrak{sp}(4)$ is a three-component MNLS which finds applications to Bose–Einstein condensates. The MNLS related to $\mathfrak{so}(12)$ and $\mathfrak{so}(10)$ Lie algebras after convenient $\mathbb{Z}_6$ or $\mathbb{Z}_4$ reductions reduce to three and four-component MNLS showing new types of $\chi^{(3)}$-interactions that are integrable. We briefly explain how these new types of MNLS can be integrated by the inverse scattering method. The spectral properties of the Lax operators L and the corresponding recursion operator Λ are outlined. Applications to spinor model of Bose–Einstein condensates are discussed.

1. Introduction

When spinor **Bose–Einstein condensates** (BEC's) are trapped in magnetic potential, the spin degree of freedom is frozen. However, in the condensate trapped by an optical potential, the spin is free. We consider BEC's of alcali atoms in the $F = 1$ hyperfine state, elongated in x direction and confined in the transverse directions y, z by purely optical means. Then, in the absence of external magnetic fields is characterized by the magnetic quantum number which has three allowed values $m_F = 1, 0, -1$.Thus the assembly of atoms in the $F = 1$ hyperfine state can be described by a normalized spinor wave function $\mathbf{\Phi}(x,t) = (\Phi_1(x,t), \Phi_0(x,t), \Phi_{-1}(x,t))^T$ whose components are labelled by the values of m_F. In short the dynamics of

such BEC's is described by a three-component Gross–Pitaevskii (GP) system of equations. In the one-dimensional approximation described above the GP system goes into the following three-component nonlinear Schrödinger equation in (1D) x-space [18]

$$\begin{aligned}
&\mathrm{i}\partial_t\Phi_1 + \partial_x^2\Phi_1 + 2(|\Phi_1|^2 + 2|\Phi_0|^2)\Phi_1 + 2\Phi_{-1}^*\Phi_0^2 = 0\\
&\mathrm{i}\partial_t\Phi_0 + \partial_x^2\Phi_0 + 2(|\Phi_{-1}|^2 + |\Phi_0|^2 + |\Phi_1|^2)\Phi_0 + 2\Phi_0^*\Phi_1\Phi_{-1} = 0\\
&\mathrm{i}\partial_t\Phi_{-1} + \partial_x^2\Phi_{-1} + 2(|\Phi_{-1}|^2 + 2|\Phi_0|^2)\Phi_{-1} + 2\Phi_1^*\Phi_0^2 = 0.
\end{aligned} \tag{1}$$

This model is integrable by means of inverse scattering transform method [18]. It also allows an exact description of the dynamics and interaction of bright solitons with spin degrees of freedom. Matter-wave solitons are expected to be useful in atom laser, atom interferometry and coherent atom transport. It could contribute to the realization of quantum information processing or computation, as a part of new field of atom optics. Lax pairs and geometric interpretation of the model (1) are given in [8]. Darboux transformation for this special integrable model is developed in [21]. We will show that the system (1) is related to symmetric space $\mathbf{C.I} \simeq \mathrm{Sp}(4)/\mathrm{U}(2)$ with canonical $\mathbb{Z}_2$-reduction and has natural Lie algebraic interpretation.

The applications of the differential geometric and Lie algebraic methods to soliton type equations lead to the discovery of close relationship between the MNLS equations and the symmetric spaces [8]. It was shown that these MNLS systems have Lax representation with the generalized Zakharov–Shabat system as the Lax operator

$$L\psi(x,t,\lambda) \equiv \mathrm{i}\frac{\mathrm{d}\psi}{\mathrm{d}x} + (Q(x,t) - \lambda J)\psi(x,t,\lambda) = 0 \tag{2}$$

where J is a constant element of the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ of the simple Lie algebra $\mathfrak{g}$ and $Q(x,t) \equiv [J, \widetilde{Q}(x,t)]$. In other words, $Q(x,t)$ belongs to the coadjoint orbit $\mathcal{M}_J$ of $\mathfrak{g}$ passing through J.

The choice of J determines the dimension of $\mathcal{M}_J$ which can be viewed as the phase space of the relevant **nonlinear evolution equations** (NLEE). It is equal to the number of roots of $\mathfrak{g}$ such that $\alpha(J) \neq 0$. Taking into account that if α is a root, then $-\alpha$ is also a root of $\mathfrak{g}$; therefore $\dim \mathcal{M}_J$ is always even.

We concentrate on those most degenerate choices for J for which ad_J has just two non-vanishing eigenvalues $\pm 2a$; in this case $J^2 = a^2\mathbb{1}$. Such choices of J are compatible with several types of symmetric spaces in Cartan classification: $\mathbf{A.III} \simeq \mathrm{SU}(p+q)/\mathrm{S}(\mathrm{U}(p) \otimes \mathrm{U}(q))$, $\mathbf{C.I} \simeq \mathrm{Sp}(2p)/\mathrm{U}(p)$ and $\mathbf{D.III} \simeq \mathrm{SO}(2p)/\mathrm{U}(p)$ [8, 17]. The classification of the symmetric spaces related to a given simple Lie algebra $\mathfrak{g}$ is directly related to the classification of the Cartan involutions (i.e., to the classifications of the real forms) that the algebra admits. For more details see e.g. [17, 24].

The interpretation of the **Inverse Scattering Method** (ISM) as a generalized Fourier transforms and the expansion over the so-called "squared" solutions (see [20, 16] for regular and [10, 15, 12] for non-regular J) allow one to study all the fundamental properties of the corresponding NLEE's. These include: i) the description of the class of NLEE related to a given Lax operator $L(\lambda)$ and solvable by the ISM; ii) derivation of the infinite family of integrals of motion; and iii) their hierarchy of Hamiltonian structures.

The degeneracy of J means that the subalgebra $\mathfrak{g}_J \subset \mathfrak{g}$ of elements commuting with J (i.e., the kernel of the operator ad_J) is non-commutative which makes more difficult the derivation of the fundamental analytic solutions (FAS) of the Lax operator (2) and the construction of the corresponding (generating) recursion operator Λ. Here we continue our studies in [16, 10] finding new algebraic reductions of MNLS equations related to **C.I** and **D.III** type symmetric spaces. Some of them like equation (1) find applications to Bose–Einstein condensates and nonlinear optics. The derived reduced MNLS system seem to be new to the best of our knowledge.

The present article is organized as follows: In Section 2 we give some preliminaries about the simple Lie algebras and the general form of the MNLS models and the relevant recursion operators. Section 3 is devoted to the spectral properties of the Lax operator L. In Section 4 we discuss the Hamiltonian properties of the MNLS systems using the classical R-matrix method. In Section 5 we apply the approach in [27, 15] and derive the dressing factors and the soliton solutions for symmetric spaces related to $\mathbf{D}_5$ and $\mathbf{D}_6$ Lie algebras. Section 6 is devoted to the analysis of the reductions of the MNLS equations by applying the reduction group [25] method. The relation between reductions and scattering data for L operator are outlined in Section 7.

2. Preliminaries

2.1. Simple Lie Algebras

Here we fix the notations and the normalization conditions for the Cartan–Weyl generators $\{h_k, E_\alpha\}$ of $\mathfrak{g}$ ($r = \operatorname{rank}\mathfrak{g}$) with root system Δ. We introduce $h_k \in \mathfrak{h}$, $k = 1, \ldots, r$ as the Cartan elements dual to the orthonormal basis $\{e_k\}$ in the root space $\mathbb{E}^r$ and the **Weyl generators** E_α, $\alpha \in \Delta$. Their commutation relations

$$
\begin{aligned}
&[h_k, E_\alpha] = (\alpha, e_k)E_\alpha, \quad [E_\alpha, E_{-\alpha}] = \frac{2}{(\alpha,\alpha)}\sum_{k=1}^{r}(\alpha, e_k)h_k \\
&[E_\alpha, E_\beta] = \begin{cases} N_{\alpha,\beta}E_{\alpha+\beta} & \text{for } \alpha+\beta \in \Delta \\ 0 & \text{for } \alpha+\beta \notin \Delta \cup \{0\}. \end{cases}
\end{aligned} \tag{3}
$$

Here $\vec{a} = \sum_{k=1}^{r} a_k e_k$ is a r-dimensional vector dual to $J \in \mathfrak{h}$ and $(\cdot,\cdot)$ is the scalar product in $\mathbb{E}^r$. The normalization of the basis is determined by

$$E_{-\alpha} = E_\alpha^T, \qquad \langle E_{-\alpha}, E_\alpha \rangle = \frac{2}{(\alpha,\alpha)}, \qquad N_{-\alpha,-\beta} = -N_{\alpha,\beta}$$

where $N_{\alpha,\beta} = \pm(p+1)$ and the integer $p \geq 0$ is such that $\alpha + s\beta \in \Delta$ for all $s = 1, \ldots, p$, $\alpha + (p+1)\beta \notin \Delta$ and $\langle \cdot, \cdot \rangle$ is the **Killing form** of $\mathfrak{g}$, see [17]. The root system Δ of $\mathfrak{g}$ is invariant with respect to the group $W_\mathfrak{g}$ of **Weyl reflections** S_α

$$S_\alpha \vec{y} = \vec{y} - \frac{2(\alpha,\vec{y})}{(\alpha,\alpha)}\alpha, \qquad \alpha \in \Delta.$$

With each reflection S_α one can relate an internal automorphism of the algebra $\mathrm{Ad}_{A_\alpha} \in \mathrm{Aut}_0\,\mathfrak{g}$ which act in a natural way on the Cartan–Weyl basis, namely

$$S_\alpha(H_\beta) \equiv A_\alpha H_\beta A_\alpha^{-1} = H_{\beta'}, \qquad \beta' = S_\alpha \beta$$
$$S_\alpha(E_\beta) \equiv A_\alpha E_\beta A_\alpha^{-1} = n_{\alpha,\beta} E_{\beta'}, \qquad n_{\alpha,\beta} = \pm 1.$$

Since $S_\alpha^2 = \mathbb{1}$ we must have $A_\alpha^2 = \pm\mathbb{1}$.

As we already mentioned in the Introduction the MNLS equations correspond to Lax operator (2) with non-regular (constant) Cartan elements $J \in \mathfrak{h}$. If J is a regular element of the Cartan subalgebra of $\mathfrak{g}$ then ad_J has as many different eigenvalues as is the number of the roots of the algebra and they are given by $a_j = \alpha_j(J)$, $\alpha_j \in \Delta$. Such J's can be used to introduce ordering in the root system by assuming that $\alpha > 0$ if $\alpha(J) > 0$. In what follows we will assume that all roots for which $\alpha(J) > 0$ are positive.

Obviously we can consider the eigensubspaces of ad_J as grading of the algebra $\mathfrak{g}$. In what follows we will consider symmetric spaces related to maximally degenerated J, i.e., ad_J has only two non-vanishing eigenvalues $\pm 2a$. Then $\mathfrak{g}$ is split into a direct sum of the subalgebra $\mathfrak{g}_0$ and the linear subspaces $\mathfrak{g}_\pm$

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_+ \oplus \mathfrak{g}_-, \qquad \mathfrak{g}_\pm = \text{l.c.}\,\{X_{\pm j}\,;\, [J, X_{\pm}] = \pm 2a X_\pm\}\,.$$

The subalgebra $\mathfrak{g}_0$ contains the Cartan subalgebra $\mathfrak{h}$ and also all root vectors $E_{\pm\alpha} \in \mathfrak{g}$ corresponding to the roots α such that $(\vec{a}, \alpha) = 0$. The root system Δ splits into subsets of roots $\Delta = \theta_0 \cup \theta_+ \cup (-\theta_+)$, where

$$\theta_0 = \{\alpha \in \Delta\,;\, \alpha(J) = 0\}, \quad \theta_+ = \{\alpha \in \Delta\,;\, \alpha(J) = a > 0\}.$$

We can use the gauge transformation commuting with J to simplify Q; in particular we can remove all components of Q in $\mathfrak{g}_0$; effectively this means that our $Q(x,t) = Q_+(x,t) + Q_-(x,t) \in \mathfrak{g}_+ \cup \mathfrak{g}_-$ can be viewed as a local coordinate in the co-adjoint orbit $\mathcal{M}_J \simeq \mathfrak{g}/\mathfrak{g}_0$

$$Q_+(x,t) = \sum_{\alpha\in\theta_+} q_\alpha(x,t) E_\alpha, \qquad Q_-(x,t) = \sum_{\alpha\in\theta_-} p_\alpha(x,t) E_\alpha. \tag{4}$$

Obviously $Q_\pm \in \mathfrak{g}_\pm$ and

$$\mathrm{ad}_J Q \equiv [J,Q] = 2a(Q_+ - Q_-), \qquad (\mathrm{ad}_J)^{-1} Q = \frac{1}{2a}(Q_+ - Q_-)$$

besides $[E_\alpha, E_\beta] = 0$ for any pair of roots $\alpha, \beta \in \theta_+$. This simplifies solving the recursion relations and the explicit calculation of the recursion operator Λ.

2.2. Lax Representation of the MNLS Type Models

The operator (2) together with the corresponding operator $M(\lambda)$

$$M(\lambda)\psi \equiv \left(\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} - [Q, \mathrm{ad}_J^{-1} Q] + 2\mathrm{i}\,\mathrm{ad}_J^{-1} Q_x + 2\lambda Q - 2\lambda^2 J\right)\psi(x,t,\lambda) = 0 \tag{5}$$

where $Q = Q(x,t)$, provide the Lax representation for the MNLS type systems. The compatibility condition $[L(\lambda), M(\lambda)] = 0$ of (2) and (5) gives the general form of the MNLS equations on symmetric spaces

$$\frac{\mathrm{i}}{2}\left[J, \frac{\partial Q}{\partial t}\right] + \frac{\partial^2 Q}{\partial x^2} - 2a^2[\mathrm{ad}_J^{-1} Q, [\mathrm{ad}_J^{-1} Q, Q]] = 0. \tag{6}$$

Following [1] one can consider more general M-operators of the form

$$M(\lambda)\Psi \equiv \mathrm{i}\frac{\mathrm{d}\Psi}{\mathrm{d}t} + \left(\sum_{k=1}^{N} V_k(x,t)\lambda^k\right)\Psi(x,t,\lambda) = 0, \quad f(\lambda) = \lim_{x\to\pm\infty} V(x,t,\lambda).$$

The Lax representation $[L(\lambda), M(\lambda)] = 0$ leads to a recurrent relations between $V_k(x,t) = V_k^{\mathrm{f}} + V_k^{\mathrm{d}}$

$$V_{k+1}^{\mathrm{f}}(x,t) \equiv \pi_J(V_{k+1}) = \Lambda_\pm V_k^{\mathrm{f}}(x,t) - \mathrm{ad}_J^{-1}[C_k, Q(x,t)], \qquad k = 1,\dots,N$$
$$V_k^{\mathrm{d}}(x,t) \equiv (\mathbb{1} - \pi_J)(V_k) = C_k + \mathrm{i}\int_{\pm\infty}^{x} \mathrm{d}y\,[Q(y,t), V_k^{\mathrm{f}}(y,t)]$$

where $\pi_J = \mathrm{ad}_J^{-1} \circ \mathrm{ad}_J$ and $C_k = (\mathbb{1} - \pi_J)C_k$ are block-diagonal integration constants, for details see, e.g. [1, 8]. These relations are resolved by the recursion operators

$$\Lambda_\pm Z = \frac{\mathrm{ad}_J}{4a^2}\left(\mathrm{i}\frac{\mathrm{d}Z}{\mathrm{d}x} + \mathrm{i}\left[Q(x), \int_{\pm\infty}^{x} \mathrm{d}y\,[Q(y), Z(y)]\right]\right) \tag{7}$$

where we assume that $Z \equiv \pi_J Z \in \mathcal{M}_J$. As a result we obtain that the class of (generically nonlocal) NLEE solvable by the ISM have the form

$$\mathrm{i}\,\mathrm{ad}_J^{-1}\frac{\partial Q}{\partial t} + \sum_{k=0}^{N} \Lambda_\pm^{N-k}\left[C_k, \mathrm{ad}_J^{-1} Q(x,t)\right] = 0, \quad f(\lambda) = \begin{pmatrix} f_+(\lambda) & 0 \\ 0 & f_-(\lambda) \end{pmatrix} \tag{8}$$

where $f(\lambda) = \sum_{k=0}^{N} C_k \lambda^{N-k}$ determines their dispersion law. The NLEE (8) become local if $f(\lambda) = f_0(\lambda)J$, where $f_0(\lambda)$ is a scalar function. In particular, if $f(\lambda) = -2\lambda^2 J$ we get the MNLS equation (6).

2.3. Basic Physical Example: C.I Type Symmetric Space $\mathrm{Sp}(2p)/\mathrm{U}(p)$

We choose $\mathfrak{g} \equiv \mathbf{C}_2 = \mathfrak{sp}(4)$ algebra; it has 2 simple roots, namely $\alpha_1 = e_1 - e_2$, $\alpha_2 = 2e_2$. We fix up the Cartan element

$$J = \mathrm{diag}(a, a, -a, -a), \qquad J^2 = a^2 \mathbb{1}.$$

Then the corresponding potential $Q(x,t)$ (4) takes the form

$$Q(x,t) = \begin{pmatrix} 0 & 0 & q_{12} & q_1 \\ 0 & 0 & q_2 & -q_{12} \\ p_{12} & p_2 & 0 & 0 \\ p_1 & -p_{12} & 0 & 0 \end{pmatrix}.$$

Imposing the involution $p_k = q_k^*$, $k = 1, 2$ and $p_{12} = q_{12}^*$ we obtain the following three-component MNLS system for the independent fields $q_{12}(x,t)$, $q_1(x,t)$ and $q_2(x,t)$

$$\begin{aligned} &\mathrm{i}a\frac{\partial q_{12}}{\partial t} + \frac{\partial^2 q_{12}}{\partial x^2} + 2q_{12}(|q_{12}|^2 + |q_1|^2 + |q_2|^2) - 2q_1 q_2 q_{12}^* = 0 \\ &\mathrm{i}a\frac{\partial q_1}{\partial t} + \frac{\partial^2 q_1}{\partial x^2} + 2q_1(|q_1|^2 + 2|q_{12}|^2) - 2q_{12}^2 q_2^* = 0 \\ &\mathrm{i}a\frac{\partial q_2}{\partial t} + \frac{\partial^2 q_2}{\partial x^2} + 2q_2(|q_2|^2 + 2|q_{12}|^2) - 2q_{12}^2 q_1^* = 0. \end{aligned} \tag{9}$$

If we identify the physical quantities of the system (1) with

$$\Phi_0 = q_{12}, \qquad \Phi_1 = -q_1, \qquad \Phi_{-1} = q_2$$

then equation (9) will coincide with (1). The system can be written in a Hamiltonian form by introducing the Poisson brackets

$$\{q_j(x), p_k(y)\} = 2\mathrm{i}\delta_{kj}\delta(x-y), \qquad \{q_{12}(x), p_{12}(y)\} = \mathrm{i}\delta(x-y)$$

and the Hamiltonian

$$\begin{aligned} H &= H_{\mathrm{kin}} + H_{\mathrm{int}} \\ H_{\mathrm{kin}} &= \frac{1}{a}\int_{-\infty}^{\infty} \mathrm{d}x \left[\frac{\partial\Phi_0}{\partial x}\frac{\partial\Phi_0^*}{\partial x} + \frac{1}{2}\left(\frac{\partial\Phi_1}{\partial x}\frac{\partial\Phi_1^*}{\partial x} + \frac{\partial\Phi_{-1}}{\partial x}\frac{\partial\Phi_{-1}^*}{\partial x}\right)\right] \\ H_{\mathrm{int}} &= -\frac{1}{2a}\int_{-\infty}^{\infty} \mathrm{d}x \left[(|\Phi_0|^2 + |\Phi_1|^2)^2 + (|\Phi_0|^2 + |\Phi_{-1}|^2)^2\right] \\ &\quad - \frac{1}{a}\int_{-\infty}^{\infty} \mathrm{d}x \left(|\Phi_0\Phi_{-1}^* + \Phi_1\Phi_0^*|^2\right). \end{aligned}$$

The soliton solutions of the $\mathfrak{sp}(4)$ MNLS system of equations (1) were derived independently in [18, 19].

2.4. D.III Type Symmetric Space

We choose $\mathfrak{g} \equiv \mathbf{D}_6 \simeq \mathfrak{so}(12)$; it has 6 simple roots, namely $\alpha_1 = e_1 - e_2$, $\alpha_2 = e_2 - e_3$, $\alpha_3 = e_3 - e_4$, $\alpha_4 = e_4 - e_5$, $\alpha_5 = e_5 - e_6$ and $\alpha_6 = e_5 + e_6$. We fix up the Cartan element

$$J = \mathrm{diag}(a, a, a, a, a, a, -a, -a, -a, -a, -a, -a), \qquad J^2 = a^2 \mathbb{1}$$

which means that the subset $\theta_+ = \{e_i + e_j\}$ with $1 \leq i < j \leq 6$. Then the corresponding potential $Q(x,t)$ (4) takes the form

$$Q = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & q_{16} & q_{15} & q_{14} & q_{13} & q_{12} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & q_{26} & q_{25} & q_{24} & q_{23} & 0 & q_{12} \\
0 & 0 & 0 & 0 & 0 & 0 & q_{36} & q_{35} & q_{34} & 0 & q_{23} & -q_{13} \\
0 & 0 & 0 & 0 & 0 & 0 & q_{46} & q_{45} & 0 & q_{34} & -q_{24} & q_{14} \\
0 & 0 & 0 & 0 & 0 & 0 & q_{56} & 0 & q_{45} & -q_{35} & q_{25} & -q_{15} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & q_{56} & -q_{46} & q_{36} & -q_{26} & q_{16} \\
p_{16} & p_{26} & p_{36} & p_{46} & p_{56} & 0 & 0 & 0 & 0 & 0 & 0 & \\
p_{15} & p_{25} & p_{35} & p_{45} & 0 & p_{56} & 0 & 0 & 0 & 0 & 0 & 0 \\
p_{14} & p_{24} & p_{34} & 0 & p_{45} & -p_{46} & 0 & 0 & 0 & 0 & 0 & 0 \\
p_{13} & p_{23} & 0 & p_{34} & -p_{35} & p_{36} & 0 & 0 & 0 & 0 & 0 & 0 \\
p_{12} & 0 & p_{23} & -p_{24} & p_{25} & -p_{26} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & p_{12} & -p_{13} & p_{14} & -p_{15} & p_{16} & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}. \tag{10}$$

Here by $q_{ij}(x,t)$ and $p_{ij}(x,t)$ where i, j belong to the set of indices $\mathcal{J} = \{(ij);\ 1 \leq i < j \leq 6\}$ we denote the coefficients of the generators E_α and $E_{-\alpha}$ with $\alpha = e_i + e_j$. Then the generic NLEE (6) becomes a system of 30 equations. Imposing the natural involution $Q = Q^\dagger$, i.e., $p_{ij} = q_{ij}^*$ we obtain a MNLS for the 15 independent functions $q_{ij}(x,t)$.

Similarly we consider also the $\mathbf{D}_5 \simeq \mathfrak{so}(10)$ algebra. The corresponding $Q(x,t)$ will be a 10×10 matrix-valued function which can be obtained from the equation for (10) by removing the first and the last rows and columns. The generic NLEE will be a system of 20 equations, the involution $Q = Q^\dagger$ will reduce them to a 10-component MNLS.

3. Spectral Data and Generalized Exponents

Here we will start with a brief sketch of the direct scattering problem for (2). It is based on the Jost solutions [7, 26] defined by their asymptotics

$$\lim_{x\to\infty}\psi(x,t,\lambda)\mathrm{e}^{\mathrm{i}\lambda Jx}=\mathbb{1},\qquad \lim_{x\to-\infty}\phi(x,t,\lambda)\mathrm{e}^{\mathrm{i}\lambda Jx}=\mathbb{1}$$

and the scattering matrix $T(t,\lambda)=(\psi(x,t,\lambda))^{-1}\phi(x,t,\lambda)$ and its inverse $\hat{T}(\lambda,t)$

$$T(t,\lambda)=\begin{pmatrix}a^+(t,\lambda) & -b^-(t,\lambda)\\ b^+(t,\lambda) & a^-(t,\lambda)\end{pmatrix},\quad \hat{T}(t,\lambda)=\begin{pmatrix}c^-(t,\lambda) & d^-(t,\lambda)\\ -d^+(t,\lambda) & c^+(t,\lambda)\end{pmatrix} \tag{11}$$

where $a^\pm(t,\lambda)$ and $b^\pm(t,\lambda)$ are $r\times r$ block matrices. The blocks $a^\pm$, $b^\pm$, $c^\pm$ and $d^\pm$ satisfy a number of relations coming from the fact that $T(\lambda)\hat{T(\lambda)}=\mathbb{1}$, for example

$$a^+(\lambda)c^-(\lambda)+b^-(\lambda)d^+(\lambda)=\mathbb{1},\qquad a^+(\lambda)d^-(\lambda)-b^-(\lambda)c^+(\lambda)=0$$

etc.

The fundamental analytic solutions (FAS) $\chi^\pm(x,t,\lambda)$ of $L(\lambda)$ are analytic functions of λ for $\operatorname{Im}\lambda\gtrless 0$ and are related to the Jost solutions by

$$\chi^\pm(x,t,\lambda)=\phi(x,t,\lambda)S_J^\pm(t,\lambda)=\psi(x,t,\lambda)T_J^\mp(t,\lambda).$$

Here $S_J^\pm$, $T_J^\pm$ upper- and lower- block-triangular matrices

$$S_J^+(t,\lambda)=\begin{pmatrix}\mathbb{1} & d^-(t,\lambda)\\ 0 & c^+(t,\lambda)\end{pmatrix},\qquad S_J^-(t,\lambda)=\begin{pmatrix}c^-(t,\lambda) & 0\\ -d^+(t,\lambda) & \mathbb{1}\end{pmatrix}$$

$$T_J^+(t,\lambda)=\begin{pmatrix}\mathbb{1} & -b^-(t,\lambda)\\ 0 & a^-(t,\lambda)\end{pmatrix},\qquad T_J^-(t,\lambda)=\begin{pmatrix}a^+(t,\lambda) & 0\\ b^+(t,\lambda) & \mathbb{1}\end{pmatrix}$$

satisfy $T_J^\pm(t,\lambda)\hat{S}_J^\pm(t,\lambda)=T(t,\lambda)$ and can be viewed as the factors of a generalized Gauss decompositions of $T(t,\lambda)$ [10]. If $Q(x,t)$ evolves according to (8) then

$$\mathrm{i}\frac{\mathrm{d}b^\pm}{\mathrm{d}t}+f_\mp(\lambda)b^\pm(t,\lambda)-b^\pm(t,\lambda)f_\pm(\lambda)=0,\quad \mathrm{i}\frac{\mathrm{d}a^\pm}{\mathrm{d}t}+[f_\pm(\lambda),a^\pm(t,\lambda)]=0. \tag{12}$$

On the real axis in the complex λ-plane both FAS $\chi^\pm(x,t,\lambda)$ are linearly dependent

$$\chi^+(x,t,\lambda)=\chi^-(x,t,\lambda)G_0(t,\lambda) \tag{13}$$

and $G_0(t,\lambda)$ can be considered as a minimal set of scattering data in the case of absence of discrete eigenvalues for the Lax operator (2), see [12].

The mapping between the potential of the Lax operator and the scattering data is based on the Wronskian relations [2]. As an example of one we write down

$$b_\alpha^+(t,\lambda)=\frac{\mathrm{i}}{2}\int_{-\infty}^{\infty}\mathrm{d}x\,\langle[Q(x,t),J]\chi^+(x,t,\lambda)E_\alpha(\chi^+(x,t,\lambda))^{-1}\rangle$$

where E_α is the root vector corresponding to the root $\alpha \in \theta_+$. Thus the "squared" solutions that appeared first in [1, 20] were later generalized in [16, 10] to Lax operators of the type (2) as follows

$$e^{\pm}_{\alpha}(x,t,\lambda) = \pi_J \left(\chi^{\pm}(x,t,\lambda) E_\alpha (\chi^{\pm}(x,t,\lambda))^{-1} \right).$$

They can also be viewed as natural generalizations of the usual exponentials and their completeness relations in $\mathcal{M}_J$ [16, 10] provide us the spectral decompositions of the recursion operators $\Lambda_\pm$ for which $e^{\pm}_{\pm\alpha}(x,\lambda)$ are eigenfunctions

$$\Lambda_+ e^{\pm}_{\mp\alpha}(x,\lambda) = \lambda e^{\pm}_{\mp\alpha}(x,\lambda), \quad \Lambda_- e^{\pm}_{\pm\alpha}(x,\lambda) = \lambda e^{\pm}_{\pm\alpha}(x,\lambda), \quad \alpha \in \theta_+.$$

The (generating) recursion operators $\Lambda_\pm$ appeared first in the AKNS-approach [1] as a tool to generate the class of all M-operators as well as the NLEE related to the given Lax operator. Next Gel'fand and Dickey [9] discovered that the class of these M-operators is contained in the diagonal of the resolvent of L. The kernel of the resolvent of L can be explicitly defined in terms of the fundamental analytic solutions $\chi^{\pm}(x,\lambda)$ of (2), see [11, 16, 10].

4. Hamiltonian Properties of the MNLS Models

It is well known that the MNLS equations possess hierarchies of Hamiltonian structures. The phase space $\mathcal{M}_J$ of the MNLS equations is the co-adjoint orbit of the $\mathfrak{g} \simeq \mathbf{D}_r$ determined by J; in addition we assume that the matrix elements of $Q(x,t)$ are smooth functions tending to zero fast enough for $|x| \to \infty$.

On the **D.III**-type symmetric spaces the Hamiltonian of (6) is given by

$$H^{(0)}_{\mathrm{MNLS}} = a \int_{-\infty}^{\infty} \mathrm{d}x \left\{ 2\langle \mathrm{ad}_J^{-1} Q_x, \mathrm{ad}_J^{-1} Q_x \rangle + \frac{1}{2} \langle [\mathrm{ad}_J^{-1} Q, Q], [\mathrm{ad}_J^{-1} Q, Q] \rangle \right\}. \tag{14}$$

Direct calculation shows that this Hamiltonian is proportional to the third coefficient I_3 of the expansion of the generating functional of the principal series of motion $\ln \det \boldsymbol{a}^+$ with respect to λ

$$\ln \det a^+(t,\lambda) = \sum_{k=1}^{\infty} I_k \lambda^{-k}$$

i.e., $H^{(0)}_{\mathrm{MNLS}} = 8\mathrm{i} I_3$. The equation of motion (6) that Q satisfies is generated by the canonical symplectic structure

$$\Omega^{(0)}_{\mathrm{MNLS}} = -2\mathrm{i}a \int_{-\infty}^{\infty} \mathrm{d}x \left\langle \delta Q(x,t) \wedge \mathrm{ad}_J^{-1} \delta Q(x,t) \right\rangle.$$

The hierarchies of symplectic structures defined on $\mathcal{M}_J$ are generated by the corresponding recursion operators $\Lambda_\pm$ (7) and are given by the following families of

compatible two-forms

$$\Omega^{(k)}_{\mathrm{MNLS}} = -2\mathrm{i}a\int_{-\infty}^{\infty}\mathrm{d}x\left\langle \delta Q(x,t)\underset{,}{\wedge}\Lambda^k\,\mathrm{ad}_J^{-1}\,\delta Q(x,t)\right\rangle,\quad \Lambda=\frac{1}{2}\left(\Lambda_+ + \Lambda_-\right).$$

The corresponding Hamiltonians for the higher MNLS are $H^{(k)}_{\mathrm{MNLS}} = 8\mathrm{i}I_{3+k}$. For $f(\lambda) = -2\lambda^2 J$ equation (12) gives $\mathrm{d}a^{\pm}/\mathrm{d}t = 0$ and $a^{\pm}(\lambda)$ can be viewed as generating functionals of integrals of motion whose number r^2 is larger than the rank r of $\mathfrak{g}$. This is obviously due to the degeneracy of the dispersion law. For generic $f(\lambda)$ from (12) there follows that only functions of the eigenvalues of $a^{\pm}(\lambda)$ will be conserved. Indeed, it follows from the classical R-matrix approach [7]. One of the definitions of the classical R-matrix is based on the Lax representation of the MNLS (2). In particular, if $U(x,\lambda)$ has the form

$$U(x,\lambda) = Q(x,t) - \lambda J$$

and the matrix elements of $Q(x,\lambda)$ satisfy the Poisson brackets $\{\cdot,\cdot\}$

$$\{p_{\alpha_2}(x,t), q_{\alpha_1}(y,t)\} = \mathrm{i}\frac{(\alpha_1,\alpha_1)}{2}\delta_{\alpha_1\alpha_2}\delta(x-y)$$

then the classical R-matrix can be defined through the relation

$$\{U(x,\lambda)\underset{,}{\otimes}U(y,\mu)\} = [R(x-y), U(x,\lambda)\otimes\mathbb{1} + \mathbb{1}\otimes U(y,\mu)]\delta(x-y). \quad (15)$$

Our system of equations (15) allows R-matrix given by [8]

$$R(\lambda-\mu) = \frac{\mathrm{i}}{2}\frac{1}{\lambda-\mu}P$$

where

$$P = \left(\sum_{k=1}^{r} h_k\otimes h_k + \sum_{\alpha\in\Delta}\frac{E_\alpha\otimes E_{-\alpha}}{\langle E_\alpha, E_{-\alpha}\rangle}\right).$$

Here h_k are the Cartan elements introduced in (3) which are properly normalized to $\langle h_i, h_k\rangle = \delta_{ik}$. P is the second Casimir endomorphism of the algebra and posses special properties concerning its action onto the matrices of the corresponding group

$$P(A\otimes B) = (B\otimes A)P.$$

Using these properties of the P-matrix and the commutation relations of the algebra (3) we obtain

$$[P, Q(x)\otimes\mathbb{1} + \mathbb{1}\otimes Q(x)] = 0$$
$$[P, \lambda J\otimes\mathbb{1} + \mu\mathbb{1}\otimes J] = 2(\lambda-\mu)\{U(x,\lambda)\underset{,}{\otimes}U(x,\mu)\}.$$

These relations are true for any $Q(x)$ taking value in the algebra and the relation (15) seems most natural and its right hand side does not contain $Q(x,t)$.

Let us now show, that the classical R-matrix is a very effective tool for calculating the Poisson brackets between the matrix elements of $T(\lambda)$. It will be more convenient here to consider periodic boundary conditions on the interval $[-L, L]$, i.e., $Q(x-L) = Q(x+L)$ and to introduce the fundamental solution $T(x, y, \lambda)$ [7]

$$\mathrm{i}\frac{\mathrm{d}T(x,y,\lambda)}{\mathrm{d}x} + U(x,\lambda)T(x,y,\lambda) = 0, \qquad T(x,x,\lambda) = \mathbb{1}.$$

Skipping the details we just formulate the following relation for the Poisson brackets between the matrix elements of $T(x, y, \lambda)$

$$\left\{T(x,y,\lambda) \underset{,}{\otimes} T(x,y,\mu)\right\} = [R(\lambda-\mu), T(x,y,\lambda)\otimes T(x,y,\mu)]\,. \tag{16}$$

The corresponding monodromy matrix $T_L(\lambda)$ describes the transition from $-L$ to L and $T_L(\lambda) = T(-L, L, \lambda)$. The Poisson brackets between the matrix elements of $T_L(\lambda)$ follow directly from equation (16)) and are given by

$$\left\{T_L(\lambda) \underset{,}{\otimes} T_L(\mu)\right\} = [R(\lambda-\mu), T_L(\lambda)\otimes T_L(\mu)]\,. \tag{17}$$

We could also write the Poisson brackets between the matrix elements of the inverse of the monodromy matrix $\hat{T}_L(\lambda)$

$$\left\{\hat{T}_L(\lambda) \underset{,}{\otimes} \hat{T}_L(\mu)\right\} = \left[\hat{T}_L(\lambda)\otimes \hat{T}_L(\mu), R(\lambda-\mu)\right]. \tag{18}$$

An elementary consequence of this result is the involutivity of the integrals of motion $I_{L,k}$ and $J_{L,k}$ from the principal series which appear in the expansions of

$$\ln\det \boldsymbol{a}_L^+(\lambda) = \sum_{k=1}^{\infty} I_{L,k}\lambda^{-k}, \qquad -\ln\det \boldsymbol{c}_L^-(\lambda) = \sum_{k=1}^{\infty} J_{L,k}\lambda^{-k} \tag{19}$$

$$\ln\det \boldsymbol{c}_L^+(\lambda) = \sum_{k=1}^{\infty} J_{L,k}\lambda^{-k}, \qquad -\ln\det \boldsymbol{a}_L^-(\lambda) = \sum_{k=1}^{\infty} I_{L,k}\lambda^{-k}. \tag{20}$$

As a result of the reduction conditions (35) that we impose, the generating functional of the principal series of integrals of motion is only one, i.e., $\ln\det \boldsymbol{a}_L^+(\lambda)$.

Another important property of the integrals $I_{L,k}$ and $J_{L,k}$ is their locality, i.e., their densities depend only on Q and its x-derivatives.

The simplest consequence of the relations (17) and (18) is the involutivity of $I_{L,k}$ and $J_{L,k}$. Indeed, taking the trace of both sides of (17) and (18) shows that $\{\mathrm{tr}\, T_L(\lambda), \mathrm{tr}\, T_L(\mu)\} = 0$ and $\{\mathrm{tr}\, \hat{T}_L(\lambda), \mathrm{tr}\, \hat{T}_L(\mu)\} = 0$. We can also multiply both sides of (17) and (18) by $C\otimes C$ and then take the trace. This proves

$$\{\mathrm{tr}\, T_L(\lambda)C, \mathrm{tr}\, T_L(\mu)C\} = 0, \qquad \left\{\mathrm{tr}\, \hat{T}_L(\lambda)C, \mathrm{tr}\, \hat{T}_L(\mu)C\right\} = 0.$$

In particular, for $C = \mathbb{1} + J$ and $C = \mathbb{1} - J$ we get the involutivity of

$$\left\{\operatorname{tr} \boldsymbol{a}_L^{\pm}(\lambda), \operatorname{tr} \boldsymbol{a}_L^{\pm}(\mu)\right\} = 0, \qquad \left\{\operatorname{tr} \boldsymbol{a}_L^{+}(\lambda), \operatorname{tr} \boldsymbol{a}_L^{-}(\mu)\right\} = 0$$
$$\left\{\operatorname{tr} \boldsymbol{c}_L^{\pm}(\lambda), \operatorname{tr} \boldsymbol{c}_L^{\pm}(\mu)\right\} = 0, \qquad \left\{\operatorname{tr} \boldsymbol{c}_L^{+}(\lambda), \operatorname{tr} \boldsymbol{c}_L^{-}(\mu)\right\} = 0.$$

Equations (17) and (18) were derived for the typical representation $V^{(1)}$ of the corresponding group G, but they hold true also for any other finite-dimensional representation of G. Let us denote by $V^{(r)}$ the r-th fundamental representation of G; then the element $T_L(\lambda)$ will have representation in $V^{(r)}$, see [17]. In particular, if we consider equations (17) and (18) in the representation $V^{(r)}$ and sandwich them between the highest and lowest weight vectors in $V^{(r)}$ we get

$$\{\det \boldsymbol{a}_L^{+}(\lambda), \det \boldsymbol{a}_L^{+}(\mu)\} = 0, \qquad \{\det \boldsymbol{c}_L^{-}(\lambda), \det \boldsymbol{c}_L^{-}(\mu)\} = 0. \tag{21}$$

Since equations (21) hold true for all values of λ and μ we can insert into them the expansions (19) with the result

$$\{I_{L,k}, I_{L,p}\} = 0, \qquad \{J_{L,k}, J_{L,p}\} = 0, \qquad k, p = 1, 2, \dots .$$

Lets go back to our basic physical example from Section 2.3 associated with the algebra $\mathfrak{g} \equiv \mathfrak{sp}(4)$ and consider the monodromy matrix $T_L(\lambda)$ in the typical representation. The matrix elements of the diagonal blocks of $T_{L\,ab}(\lambda)$ for $1 \le a \le 2$ and $1 \le b \le 2$ are denoted by $T_{L\,ab}(\lambda) = \boldsymbol{a}_{ab}^{+}$ according to (11). Now Poisson brackets between the scattering data $\boldsymbol{a}_L^{+}(\lambda)$ and $\boldsymbol{a}_L^{+}(\mu)$ are obtained in a straightforwar way from (17)

$$\left\{\boldsymbol{a}_{L\,11}^{+}(\lambda), \boldsymbol{a}_{L\,12}^{+}(\mu)\right\} = \frac{\mathrm{i}}{2(\lambda-\mu)}\left(\boldsymbol{a}_{L\,11}^{+}(\lambda)\boldsymbol{a}_{L\,12}^{+}(\mu) - \boldsymbol{a}_{L\,12}^{+}(\lambda)\boldsymbol{a}_{L\,11}^{+}(\mu)\right)$$
$$\left\{\boldsymbol{a}_{L\,11}^{+}(\lambda), \boldsymbol{a}_{L\,21}^{+}(\mu)\right\} = \frac{\mathrm{i}}{2(\lambda-\mu)}\left(\boldsymbol{a}_{L\,21}^{+}(\lambda)\boldsymbol{a}_{L\,11}^{+}(\mu) - \boldsymbol{a}_{L\,11}^{+}(\lambda)\boldsymbol{a}_{L\,21}^{+}(\mu)\right)$$
$$\left\{\boldsymbol{a}_{L\,11}^{+}(\lambda), \boldsymbol{a}_{L\,22}^{+}(\mu)\right\} = \frac{\mathrm{i}}{2(\lambda-\mu)}\left(\boldsymbol{a}_{L\,21}^{+}(\lambda)\boldsymbol{a}_{L\,12}^{+}(\mu) - \boldsymbol{a}_{L\,12}^{+}(\lambda)\boldsymbol{a}_{L\,21}^{+}(\mu)\right)$$
$$\left\{\boldsymbol{a}_{L\,22}^{+}(\lambda), \boldsymbol{a}_{L\,12}^{+}(\mu)\right\} = \frac{\mathrm{i}}{2(\lambda-\mu)}\left(\boldsymbol{a}_{L\,12}^{+}(\lambda)\boldsymbol{a}_{L\,22}^{+}(\mu) - \boldsymbol{a}_{L\,22}^{+}(\lambda)\boldsymbol{a}_{L\,12}^{+}(\mu)\right)$$
$$\left\{\boldsymbol{a}_{L\,22}^{+}(\lambda), \boldsymbol{a}_{L\,21}^{+}(\mu)\right\} = \frac{\mathrm{i}}{2(\lambda-\mu)}\left(\boldsymbol{a}_{L\,22}^{+}(\lambda)\boldsymbol{a}_{L\,21}^{+}(\mu) - \boldsymbol{a}_{L\,21}^{+}(\lambda)\boldsymbol{a}_{L\,22}^{+}(\mu)\right)$$
$$\left\{\boldsymbol{a}_{L\,21}^{+}(\lambda), \boldsymbol{a}_{L\,12}^{+}(\mu)\right\} = \frac{\mathrm{i}}{2(\lambda-\mu)}\left(\boldsymbol{a}_{L\,11}^{+}(\lambda)\boldsymbol{a}_{L\,22}^{+}(\mu) - \boldsymbol{a}_{L\,22}^{+}(\lambda)\boldsymbol{a}_{L\,11}^{+}(\mu)\right)$$
$$\left\{\boldsymbol{a}_{L\,11}^{+}(\lambda), \boldsymbol{a}_{L\,11}^{+}(\mu)\right\} = 0, \qquad \left\{\boldsymbol{a}_{L\,22}^{+}(\lambda), \boldsymbol{a}_{L\,22}^{+}(\mu)\right\} = 0$$
$$\left\{\boldsymbol{a}_{L\,12}^{+}(\lambda), \boldsymbol{a}_{L\,12}^{+}(\mu)\right\} = 0, \qquad \left\{\boldsymbol{a}_{L\,21}^{+}(\lambda), \boldsymbol{a}_{L\,21}^{+}(\mu)\right\} = 0.$$

Now that we know the Poisson brackets between the matrix elements of $\boldsymbol{a}^+$ it is not difficult to extend the above result over its invariants

$$\{\operatorname{tr} \boldsymbol{a}_L^+(\lambda), \boldsymbol{a}_{L\,ij}^+(\mu)\} = \frac{\mathrm{i}}{2}\frac{1}{\lambda-\mu}[\boldsymbol{a}_L^+(\lambda), \boldsymbol{a}_L^+(\mu)]_{ij}, \qquad i,j=1,2$$
$$\{\det \boldsymbol{a}_L^+(\lambda), \boldsymbol{a}_{L\,ij}^+(\mu)\} = 0, \qquad i,j=1,2.$$

This somewhat more concrete analysis allows one to see that only functions of the eigenvalues of $\boldsymbol{a}_L^+(\lambda)$ produce integrals of motion in involution.

We are able to transfer these results also for the case of potentials belonging to the co-adjoint orbit of the $\mathfrak{g} \simeq \mathbf{D}_r$ determined by J and with zero boundary conditions

$$\{T(\lambda,t) \underset{,}{\otimes} T(\mu,t)\} = [R(\lambda-\mu), T(\lambda,t)\otimes T(\mu,t)]. \tag{22}$$

Here $T(\lambda,t)$ is the scattering matrix, obtained after taking the limit $L\to\infty$ in the corresponding monodromy matrix $T_L(\lambda)$. Indeed, let us multiply (16) by $E(y,\lambda)\otimes E(y,\mu)$ on the right and by $E^{-1}(x,\lambda)\otimes E^{-1}(x,\mu)$ on the left, where $E(x,\lambda) = \exp(-\mathrm{i}\lambda J x)$ and take the limit for $x\to\infty$, $y\to-\infty$. Since

$$\lim_{x\to\pm\infty}\frac{\mathrm{e}^{\mathrm{i}x(\lambda-\mu)}}{\lambda-\mu} = \pm\mathrm{i}\pi\delta(\lambda-\mu)$$

we get

$$\begin{aligned}\left\{T(\lambda)\underset{,}{\otimes}T(\mu)\right\} &= R_+(\lambda-\mu)T(\lambda)\otimes T(\mu) - T(\lambda)\otimes T(\mu)R_-(\lambda-\mu)\\ R_\pm(\lambda-\mu) &= \frac{1}{2(\lambda-\mu)}\left(\sum_{k=1}^{r} h_k\otimes h_k + \Pi_{0J}\right) \pm \mathrm{i}\pi\delta(\lambda-\mu)\Pi_{1J}\end{aligned} \tag{23}$$

where Π_{0J} and Π_{1J} are defined as follows

$$\Pi_{0J} = \sum_{\alpha\in\theta_0}(E_\alpha\otimes E_{-\alpha} + E_{-\alpha}\otimes E_\alpha)$$
$$\Pi_{1J} = \sum_{\alpha\in\theta_+}(E_\alpha\otimes E_{-\alpha} - E_{-\alpha}\otimes E_\alpha).$$

Analogously we prove that:

i) there are integrals $I_k = \lim_{L\to\infty} I_{L,k}$ and $J_p = \lim_{L\to\infty} J_{L,p}$ that are in involution, i.e.,

$$\{I_k, I_p^\pm\} = \{J_k, J_p\} = 0$$

for some positive values of k and p;

ii) the principal series of integrals of motion I_k, generated by $\ln\det a^\pm(t,\lambda)$, i.e., the eigenvalues of $\boldsymbol{a}^\pm(\lambda)$ and $\boldsymbol{c}^\pm(\lambda)$ produce integrals of motion in involution.

So for MNLS we have extra integrals of motion that are not all in involution. Indeed, lets denote the matrix elements of the scattering matrix $T(\lambda,t)$ according to (11) and multiply 23 by $E_{ab}\otimes E_{cd}$ on the right, where $(E_{ab})_{ij}=\delta_{ai}\delta_{bj}$ and take the trace of the elements in the first and in the second position of the tensor product. Thus we obtain the Poisson brackets between the scattering data

$$\{T_{ba}(\lambda),T_{dc}(\mu)\}=\operatorname{tr}\left(T(\lambda)\otimes T(\mu)\left(E_{ab}\otimes E_{dc}R_{+}-R_{-}E_{ab}\otimes E_{dc}\right)\right).$$

Lets list the relations contained in the above equality and concerning the block-diagonal portions of the scattering matrix $\boldsymbol{a}^{+}(\lambda)$ and $\boldsymbol{a}^{-}(\lambda)$

$$\left\{\boldsymbol{a}^{\pm}_{ba}(\lambda),\boldsymbol{a}^{\pm}_{dc}(\mu)\right\}=\frac{\mathrm{i}}{2(\lambda-\mu)}\left(\boldsymbol{a}^{\pm}_{da}(\lambda)\boldsymbol{a}^{\pm}_{bc}(\mu)-\boldsymbol{a}^{\pm}_{bc}(\lambda)\boldsymbol{a}^{\pm}_{da}(\mu)\right). \tag{24}$$

The above result allows us to compute the Poisson brackets between the invariants of $\boldsymbol{a}^{\pm}(\lambda)$ and their matrix elements

$$\begin{aligned}&\left\{\operatorname{tr}(\boldsymbol{a}^{\pm}(\lambda))^{k},\boldsymbol{a}^{\pm}_{ab}(\mu)\right\}=\frac{\mathrm{i}}{2}\frac{k}{\lambda-\mu}\left[\left(\boldsymbol{a}^{\pm}(\lambda)\right)^{k},\boldsymbol{a}^{\pm}(\mu)\right]_{ab}\\&\{\operatorname{tr}\ln\boldsymbol{a}^{\pm}(\lambda),\boldsymbol{a}^{\pm}_{dc}(\mu)\}=0.\end{aligned} \tag{25}$$

This analysis allows reveals that only the eigenvalues of $\boldsymbol{a}^{\pm}(\lambda)$ and $\boldsymbol{c}^{\pm}(\lambda)$ produce integrals of motion in involution.

From (22) it follows that the first integrals I_k generated by $\ln\det a^{\pm}(t,\lambda)$ are in involution. Due to the special degenerate choice of the dispersion law $f(\lambda)=-2\lambda^2 J$, any matrix elements of the blocks $\boldsymbol{a}^{\pm}(\lambda)$ will generate integrals of motion, which, however, will not be in involution, see (24)–(25). The Hamiltonian for the MNLS models is proportional to I_3^{+}, i.e., belongs to the principal series. If we choose a generic (i.e., non-degenerate) dispersion law then the Hamiltonian of the corresponding NLEE will not be in involution with $I_k^{\pm}$. Such are the dispersion laws for the NLEE's that allow "boomeron" and "trappon" type solutions [3, 5, 4]. This is the reason why their velocities become time-dependent.

5. Dressing Factors and Soliton Solutions

The main idea of the dressing method is starting from a FAS $\chi^{\pm}_{(0)}(x,\lambda)$ of $L(\lambda)$ with potential $q_{(0)}$ to construct a new singular solution $\chi^{\pm}_{(1)}(x,\lambda)$ of the Riemann–Hilbert Problem (13) with singularities located at prescribed positions $\lambda_1^{\pm}$. Then the new solutions $\chi^{\pm}_{(1)}(x,\lambda)$ will correspond to a potential $q_{(1)}$ of $L(\lambda)$ with two discrete eigenvalues $\lambda_1^{\pm}$. It is related to the regular one by the dressing factors $u(x,\lambda)$

$$\chi^{\pm}_{(1)}(x,\lambda)=u(x,\lambda)\chi^{\pm}_{(0)}(x,\lambda)u_{-}^{-1}(\lambda),\qquad u_{-}(\lambda)=\lim_{x\to-\infty}u(x,\lambda).$$

If $\mathfrak{g} \simeq \mathbf{B}_r, \mathbf{D}_r$ the dressing factors take the form [13]

$$u(x,\lambda) = \mathbb{1} + (c_1(\lambda)-1)P_1(x) + (c_1^{-1}(\lambda)-1)P_{-1}(x), \quad P_{-1} = SP_1^T S^{-1} \tag{26}$$

where the rank 1 projector $P_1(x)$ and the function $c_1(\lambda)$ are given by

$$P_1(x) = \frac{|n(x)\rangle\langle m(x)|}{\langle m(x)|n(x)\rangle}, \qquad c_1(\lambda) = \frac{\lambda-\lambda^+}{\lambda-\lambda^-}$$
$$|n(x)\rangle = \chi_0^+(x,\lambda_1^+)|n_0\rangle, \qquad \langle m(x)| = \langle m_0|\hat{\chi}_0^-(x,\lambda_1^-).$$

$|n_0\rangle$ and $\langle m_0|$ are constant vectors and

$$S = \sum_{k=1}^{r}(-1)^{k+1}(E_{k\bar{k}} + E_{\bar{k}k}), \qquad \bar{k} = 2r+1-k.$$

Here E_{kn} is an $2r \times 2r$ matrix whose matrix elements are $(E_{kn})_{ij} = \delta_{ik}\delta_{nj}$. Then the "dressed" potential have the form

$$Q_{(1)}(x,t) = Q_{(0)}(x,t) - (\lambda_1^+ - \lambda_1^-)[J, p(x,t)], \quad p(x,t) = P_1(x,t) - P_{-1}(x,t).$$

where

$$p(x,t) = \frac{2}{\langle m|n\rangle}\left(\sum_{k=1}^{r} h_k(x,t)H_{e_k} + \sum_{\alpha\in\Delta_+}(P_\alpha(x,t)E_\alpha + P_{-\alpha}(x,t)E_{-\alpha})\right)$$

$$\langle m|n\rangle = \sum_{k=1}^{r}(n_{0k}m_{0k}\mathrm{e}^{2a\nu_1 x + 16a\mu_1\nu_1 t} + n_{0\bar{k}}m_{0\bar{k}}\mathrm{e}^{-2a\nu_1 x - 16a\mu_1\nu_1 t})$$

$$h_k(x,t) = n_{0k}m_{0k}\mathrm{e}^{2a\nu_1 x + 16a\mu_1\nu_1 t} - n_{0\bar{k}}m_{0\bar{k}}\mathrm{e}^{-2a\nu_1 x - 16a\mu_1\nu_1 t}$$

and

$$P_\alpha(x,t) = (n_{0k}m_{0\bar{s}} - (-1)^{s+k}n_{0s}m_{0\bar{k}})\mathrm{e}^{-2ia\mu_1 x - 8ia(\mu_1^2-\nu_1^2)t}$$
$$P_{-\alpha}(x,t) = (n_{0\bar{s}}m_{0k} - (-1)^{s+k}n_{0\bar{k}}m_{0s})\mathrm{e}^{2ia\mu_1 x + 8ia(\mu_1^2-\nu_1^2)t}$$

for $\alpha = e_k + e_s$, and

$$P_\alpha(x,t) = n_{0k}m_{0s}\mathrm{e}^{2a\nu_1 x + 16a\mu_1\nu_1 t} - (-1)^{s+k}n_{0\bar{s}}m_{0\bar{k}}\mathrm{e}^{-2a\nu_1 x - 16a\mu_1\nu_1 t}$$
$$P_{-\alpha}(x,t) = n_{0\bar{k}}m_{0\bar{s}}\mathrm{e}^{-2a\nu_1 x - 16a\mu_1\nu_1 t} - (-1)^{s+k}n_{0s}m_{0k}\mathrm{e}^{2a\nu_1 x + 16a\mu_1\nu_1 t}$$

for $\alpha = e_k - e_s$, $k < s$.

Let us consider now the purely solitonic case, i.e., $Q_{(0)}(x) = 0$ and

$$\chi_{(0)}^+(x,t,\lambda_1^+) = \exp(-\mathrm{i}\lambda_1^+ Jx - 4\mathrm{i}\lambda_1^{+2}Jt).$$

Thus the one-soliton solution is

$$q_{jk} = -\frac{2ia\nu_1\left(n_{0j}m_{0\bar{k}} - (-1)^{j+k}n_{0k}m_{0\bar{j}}\right)\mathrm{e}^{-2ia\mu_1 x - 8ia(\mu_1^2-\nu_1^2)t}}{\sqrt{\varphi_1\varphi_2}\,\mathrm{ch}\left(2a\nu_1 x + 16a\nu_1\mu_1 t + \frac{1}{2}\ln\frac{\varphi_1}{\varphi_2}\right)}$$

where $(ij) \in \mathcal{J}$ and

$$\varphi_1 = \sum_{j=1}^{r} n_{0j} m_{0j}, \quad \varphi_2 = \sum_{j=1}^{r} n_{0\bar{j}} m_{0\bar{j}}, \quad \lambda_1^{\pm} = \mu_1 \pm \mathrm{i}\nu_1.$$

6. New Reductions of MNLS Equations

6.1. The Reduction Group

The reduction group G_R introduced by Mikhailov [25] provides a powerful tool for constructing new integrable equations [27, 6, 13, 14, 22, 23] starting from known ones. It is a finite group which preserves the Lax representation, i.e., it ensures that the reduction constrains are automatically compatible with the evolution. The main idea of the reduction group is to impose an invariance condition on the Lax operators (2) and (5). In particular this means that the dispersion law $f_{\mathrm{MNLS}}(\lambda) = -2\lambda^2 J$ must also be compatible with the reduction group action.

Here we consider two types of G_R reductions – like in [15] we will embed them as subgroup of $W_{\mathfrak{g}}$:

$$\textbf{Type I:} \quad \begin{aligned} &B^{-1}U^{\dagger}(x,t,\lambda^*)B = U(x,t,\lambda), \qquad B^{-1}JB = J \\ &B^{-1}V^{\dagger}(x,t,\lambda^*)B = V(x,t,\lambda) \end{aligned} \tag{27}$$

$$\textbf{Type II:} \quad \begin{aligned} &C^{-1}U^{*}(x,t,\lambda^*)C = -U(x,t,\lambda), \qquad C^{-1}JC = -J \\ &C^{-1}V^{*}(x,t,\lambda^*)C = -V(x,t,\lambda) \end{aligned} \tag{28}$$

where

$$U(x,t,\lambda) = Q(x,t) - \lambda J$$

and

$$V(x,t,\lambda) = -[Q, \mathrm{ad}_J^{-1} Q] + 2\mathrm{i}\,\mathrm{ad}_J^{-1} Q_x(x,t) + 2\lambda Q(x,t) - 2\lambda^2 J.$$

The automorphisms C and B must be of even order.

6.2. Example of $\mathbb{Z}_4$-Reduction

Let us impose the following $\mathbb{Z}_4$-reduction

$$B^{-1}(U^{\dagger}(x,t,\lambda^*))B = U(x,t,\lambda), \qquad U(x,t,\lambda) = Q(x,t) - \lambda J$$
$$B = w_{e_1-e_2} \cdot w_{e_2-e_3} \cdot w_{e_3-e_4}, \qquad B^4 = \mathbb{1}$$

where $w_{e_i-e_j}$ are the Weyl reflection with respect to the roots $e_i - e_j$ of the $\mathfrak{so}(10)$-algebra. Then $B(J) = J$ and the corresponding reduced potential $Q^{\mathrm{red}} \in \mathfrak{so}(10)$

takes the form

$$Q^{\rm red}=\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & q_{15} & q_{14} & 0 & q_{12} & 0\\
0 & 0 & 0 & 0 & 0 & q_{25} & 0 & q_{14} & 0 & q_{12}\\
0 & 0 & 0 & 0 & 0 & q_{15} & q_{12} & 0 & q_{14} & 0\\
0 & 0 & 0 & 0 & 0 & q_{25} & 0 & q_{12} & 0 & q_{14}\\
0 & 0 & 0 & 0 & 0 & 0 & q_{25} & -q_{15} & q_{25} & -q_{15}\\
q_{15}^* & q_{25}^* & q_{15}^* & q_{25}^* & 0 & 0 & 0 & 0 & 0 & 0\\
q_{14}^* & 0 & q_{12}^* & 0 & q_{25}^* & 0 & 0 & 0 & 0 & 0\\
0 & q_{14}^* & 0 & q_{12}^* & -q_{15}^* & 0 & 0 & 0 & 0 & 0\\
q_{12}^* & 0 & q_{14}^* & 0 & q_{25}^* & 0 & 0 & 0 & 0 & 0\\
0 & q_{12}^* & 0 & q_{14}^* & -q_{15}^* & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$

Thus one derives the following four component MNLS system related to $\mathbf{D}_5$-algebra (here the independent fields are: $q_{12}(x,t), q_{14}(x,t), q_{15}(x,t)$ and $q_{25}(x,t)$)

$$\begin{aligned}
&\mathrm{i}a\frac{\partial q_{12}}{\partial t}+\frac{\partial^2 q_{12}}{\partial x^2}+2q_{12}(|q_{12}|^2+2|q_{14}|^2+|q_{15}|^2+|q_{25}|^2)\\
&\qquad\qquad+2q_{14}(|q_{15}|^2+|q_{25}|^2)+2q_{14}^2q_{12}^*=0\\
&\mathrm{i}a\frac{\partial q_{14}}{\partial t}+\frac{\partial^2 q_{14}}{\partial x^2}+2q_{14}(|q_{14}|^2+2|q_{12}|^2+|q_{15}|^2+|q_{25}|^2)\\
&\qquad\qquad+2q_{12}(|q_{15}|^2+|q_{25}|^2)+2q_{12}^2q_{14}^*=0\\
&\mathrm{i}a\frac{\partial q_{15}}{\partial t}+\frac{\partial^2 q_{15}}{\partial x^2}+2q_{15}(2|q_{15}|^2+2|q_{25}|^2+|q_{12}|^2+|q_{14}|^2)\\
&\qquad\qquad+2q_{12}q_{15}q_{14}^*+2q_{14}q_{15}q_{12}^*=0\\
&\mathrm{i}a\frac{\partial q_{25}}{\partial t}+\frac{\partial^2 q_{25}}{\partial x^2}+2q_{25}(2|q_{25}|^2+2|q_{15}|^2+|q_{12}|^2+|q_{14}|^2)\\
&\qquad\qquad+2q_{14}q_{25}q_{12}^*+2q_{12}q_{25}q_{14}^*=0.
\end{aligned}\tag{29}$$

The Hamiltonian of (29) is obtained from the general expression (14) by imposing the reduction constraints

$$\begin{aligned}
H&=H_{\rm kin}+H_{\rm int}\\
H_{\rm kin}&=\frac{2}{a}\int_{-\infty}^{\infty}\mathrm{d}x\left(\frac{\partial q_{15}}{\partial x}\frac{\partial q_{15}^*}{\partial x}+\frac{\partial q_{14}}{\partial x}\frac{\partial q_{14}^*}{\partial x}+\frac{\partial q_{12}}{\partial x}\frac{\partial q_{12}^*}{\partial x}+\frac{\partial q_{12}}{\partial x}\frac{\partial q_{12}^*}{\partial x}\right)\\
H_{\rm int}&=-\frac{2}{a}\int_{-\infty}^{\infty}\mathrm{d}x\,(|q_{15}|^2+|q_{25}|^2+|q_{12}|^2+|q_{14}|)^2\\
&\quad-\frac{2}{a}\int_{-\infty}^{\infty}\mathrm{d}x\,(|q_{15}|^2+|q_{25}|^2+q_{12}^*q_{14}+q_{12}q_{14}^*)^2.
\end{aligned}$$

6.3. Example of $\mathbb{Z}_6$-Reduction

Let us impose the following $\mathbb{Z}_6$-reduction

$$B^{-1}(U^\dagger(x,t,\lambda^*))B = U(x,t,\lambda), \qquad U(x,t,\lambda) = Q(x,t) - \lambda J$$
$$B = w_{e_1-e_2}\cdot w_{e_2-e_3}\cdot w_{e_3-e_4}\cdot w_{e_4-e_5}\cdot w_{e_5-e_6}, \qquad B^6 = \mathbb{1}$$

where $w_{e_i-e_j}$ are the Weyl reflection with respect to the roots $e_i - e_j$ of the $\mathfrak{so}(12)$-algebra. Then $B(J) = J$ and the corresponding reduced potential $Q^{\rm red} \in \mathfrak{so}(12)$ takes the form

$$Q^{\rm red} = \begin{pmatrix} 0 & \mathbf{q} \\ \mathbf{q}^\dagger & 0 \end{pmatrix}$$

where

$$\mathbf{q} = \begin{pmatrix} q_{12} & -q_{13} & q_{14} & q_{13} & q_{12} & 0 \\ -q_{13} & q_{14} & q_{13} & q_{12} & 0 & q_{12} \\ q_{14} & q_{13} & q_{12} & 0 & q_{12} & -q_{13} \\ q_{13} & q_{12} & 0 & q_{12} & -q_{13} & q_{14} \\ q_{12} & 0 & q_{12} & -q_{13} & q_{14} & q_{13} \\ 0 & q_{12} & -q_{13} & q_{14} & q_{13} & q_{12} \end{pmatrix}.$$

Thus one derives the following three-component MNLS system related to $\mathbf{D}_6$-algebra (here the independent fields are: $q_{12}(x,t)$, $q_{13}(x,t)$, and $q_{14}(x,t)$)

$$\begin{aligned}
&\mathrm{i}a\frac{\partial q_{12}}{\partial t} + \frac{\partial^2 q_{12}}{\partial x^2} - 2q_{12}(3|q_{12}|^2 + 2|q_{13}|^2 + 2|q_{14}|^2) + 4q_{14}(|q_{13}|^2 - |q_{12}|^2) \\
&\qquad + 2q_{13}^2 q_{12}^* - 2q_{12}^2 q_{14}^* - 2q_{13}^2 q_{14}^* - 2q_{14}^2 q_{12}^* = 0 \\
&\mathrm{i}a\frac{\partial q_{13}}{\partial t} + \frac{\partial^2 q_{13}}{\partial x^2} - 2q_{13}(3|q_{13}|^2 + 2|q_{12}|^2 + 2|q_{14}|^2) + 2q_{12}^2 q_{13}^* + 2q_{14}^2 q_{13}^* \\
&\qquad + 4q_{13}q_{12}q_{14}^* + 4q_{13}q_{14}q_{12}^* - 4q_{12}q_{14}q_{13}^* = 0 \\
&\mathrm{i}a\frac{\partial q_{14}}{\partial t} + \frac{\partial^2 q_{14}}{\partial x^2} - 2q_{14}(|q_{14}|^2 + 4|q_{12}|^2 + 4|q_{13}|^2) + 4q_{12}(2|q_{13}|^2 - |q_{12}|^2) \\
&\qquad + 4q_{13}^2 q_{14}^* - 4q_{13}^2 q_{12}^* - 4q_{12}^2 q_{14}^* = 0.
\end{aligned} \tag{30}$$

The Hamiltonian of (30) is obtained from the general expression (14) by imposing the reduction constraints

$$
\begin{aligned}
H &= H_{\rm kin} + H_{\rm int} \\
H_{\rm kin} &= \frac{3}{a}\int_{-\infty}^{\infty} {\rm d}x \left(\frac{\partial q_{12}}{\partial x}\frac{\partial q_{12}^*}{\partial x} + 2\frac{\partial q_{13}}{\partial x}\frac{\partial q_{13}^*}{\partial x} + 2\frac{\partial q_{14}}{\partial x}\frac{\partial q_{14}^*}{\partial x} \right) \\
H_{\rm int} &= \frac{3}{a}\int_{-\infty}^{\infty} {\rm d}x \left(6|q_{12}|^4 + 6|q_{13}|^4 + |q_{14}|^4 + 8|q_{14}|^2(|q_{12}|^2 + |q_{13}|^2) \right) \\
&\quad + \frac{6}{a}\int_{-\infty}^{\infty} {\rm d}x \, (q_{12}^2 q_{14}^{*2} + q_{14}^2 q_{12}^{*2} - q_{13}^2 q_{14}^{*2} - q_{14}^2 q_{13}^{*2} - q_{12}^2 q_{13}^{*2} - q_{13}^2 q_{12}^{*2}) \\
&\quad + \frac{12}{a}\int_{-\infty}^{\infty} {\rm d}x \, ((|q_{12}|^2 - 2|q_{13}|^2)(q_{12}q_{14}^* + q_{14}q_{12}^*)) \\
&\quad + \frac{12}{a}\int_{-\infty}^{\infty} {\rm d}x \, (q_{12}q_{14}q_{13}^{*2} + q_{13}^2 q_{14}^* q_{12}^*).
\end{aligned}
$$

The soliton solutions of the reduced MNLS require additional efforts. The problem is that the generic expression for the dressing factor (26) does not satisfy the reduction conditions.

7. Reductions and the Scattering Data

The reduction conditions (27) and (28) imposed on the potential of the Lax operator (2) induce an invariance conditions for the corresponding fundamental analytical solutions

$$
\textbf{Type I:} \quad B^{-1}(\chi^+(x,t,\lambda^*))^\dagger B = \begin{pmatrix} c^-(\lambda) & 0 \\ 0 & a^-(\lambda) \end{pmatrix} (\chi^-(x,t,\lambda))^{-1} \tag{31}
$$

$$
\textbf{Type II:} \quad C^{-1}(\chi^\pm(x,t,\lambda^*))^* C = \chi^\pm(x,t,\lambda) \tag{32}
$$

and for the scattering matrix (11)

$$
\textbf{Type I:} \quad B^{-1}T^\dagger(t,\lambda^*)B = (T(t,\lambda))^{-1} \tag{33}
$$

$$
\textbf{Type II:} \quad C^{-1}T^*(t,\lambda^*)C = T(t,\lambda). \tag{34}
$$

If we represent the internal automorphisms S_α of **type I** in the form

$$
S_\alpha = \begin{pmatrix} ccB_+ & 0 \\ 0 & B_- \end{pmatrix}
$$

the reduction conditions on the matrix blocks $a^\pm(\lambda)$, $b^\pm(\lambda)$, $c^\pm(\lambda)$ and $d^\pm(\lambda)$ in the scattering matrix and its inverse (11) read

$$
\begin{aligned}
&B_+(a^+(\lambda^*))^\dagger B_+^{-1} = c^+(\lambda), \qquad B_+(b^+(\lambda^*))^\dagger B_-^{-1} = d^-(\lambda) \\
&B_-(b^-(\lambda^*))^\dagger B_+^{-1} = d^+(\lambda), \qquad B_-(a^-(\lambda^*))^\dagger B_-^{-1} = c^-(\lambda).
\end{aligned} \tag{35}
$$

For the other reductions of **type II** the internal automorphisms preserving J up to a sign has off-diagonal block structure as follows

$$S_\alpha = \begin{pmatrix} 0 & C_+ \\ C_- & 0 \end{pmatrix}$$

and the matrix blocks of $T(\lambda)$ are constrained by

$$C_+(a^-(\lambda^*))^* C_+^{-1} = a^+(\lambda), \qquad C_-(a^+(\lambda^*))^* C_-^{-1} = a^-(\lambda)$$
$$C_+(b^+(\lambda^*))^* C_-^{-1} = -b^-(\lambda), \qquad C_-(b^-(\lambda^*))^* C_+^{-1} = -b^+(\lambda).$$

Similar reduction constraints could be written also for the inverse $\hat{T}(\lambda)$ to the scattering matrix $T(\lambda)$.

Our last remark here is that the reductions described above can not be applied to any generic NLEE. Indeed, equation (12) will be compatible with the reduction only if the dispersion law $f(\lambda)$ satisfies

$$B_+^{-1}\,(f_+(\lambda^*))^\dagger\, B_+ = f_+^{-1}(\lambda), \qquad B_-^{-1}\,(f_-(\lambda^*))^\dagger\, B_-^{-1} = f_-^{-1}(\lambda).$$

Any generic NLEE is compatible with reduction of Type II if the dispersion law complies with

$$C_+\,(f_-(\lambda^*))^*\, C_+^{-1} = f_+(\lambda), \qquad C_-\,(f_+(\lambda^*))^*\, C_-^{-1} = f_-(\lambda).$$

8. Conclusions

We have described new systems of MNLS type obtained as $\mathbb{Z}_4$ and $\mathbb{Z}_6$-reductions of the MNLS related to a **D.III** type symmetric space. The Hamiltonian formalism and the theory of Λ-operators for MNLS related to the relevant simple Lie algebras are briefly discussed. We show how the method, presented in [13] for the N-wave equations and their gauge equivalent systems can be extended to MNLS type systems [14]. The reduction of the multi-component nonlinear Schrödinger (NLS) equations on symmetric space **C.I** $\simeq \mathrm{Sp}(2p)/\mathrm{U}(p)$ for $p = 2$ is related to spinor model of Bose–Einstein condensate. Other interesting reductions of MNLS type equations were reported in [15] and a systematic study of the problem is on the way.

These results can be extended and the reductions of MNLS-type equations related to other symmetric and homogeneous spaces can be explored. As a result one can systematically obtain and classify new integrable systems of MNLS type. The method is explicitly gauge covariant and can also be applied to their gauge equivalent systems of Heisenberg ferromagnet type. Such research would entail a voluminous calculations and will be continued in subsequent publications.

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