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EXTENDED HAMILTONIAN FORMALISM OF FIELD THEORIES: VARIATIONAL ASPECTS AND OTHER TOPICS

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Abstract. We consider Hamiltonian systems in first-order multisymplectic field theories. In particular, we introduce Hamiltonian systems in the *extended multimomentum bundle*. The resulting *extended Hamiltonian formalism* is the generalization to field theories of the extended (symplectic) formalism for non-autonomous mechanical systems. In order to derive the corresponding field equations, a variational principle is stated for these extended Hamiltonian systems and, after studying the geometric properties of these systems, we establish the relation between this extended formalism and the standard one.

1. Introduction

It is well known that the structure of autonomous Hamiltonian dynamical systems is especially suitable for analyzing certain kinds of problems concerning these systems, such as: symmetries and related topics (existence of conservation laws and reduction), integrability (including numerical methods), and quantization. Geometrically, many of the characteristics of these systems arise from the existence of a "natural" geometric structure in the phase space: the *symplectic form*. The dynamic information is carried out by the *Hamiltonian function*, which is "independent" of the geometry.

We wish to generalize the structure of Hamiltonian systems in autonomous mechanics to first-order multisymplectic field theories. In these models, multisymplectic forms play the same role as symplectic forms in autonomous mechanics [2, 4–8, 11, 12, 14].

In order to establish the Hamiltonian formalism, the first problem to be considered is the choice of a suitable multimomentum bundle. Thus we can take:

- The *restricted multimomentum bundle* $J^1\pi^*$, which does not have a canonical multisymplectic form. Hamiltonian systems can be introduced on this bundle by means of Hamiltonian sections (carrying the physical information), which allow us to construct the geometric structure.
- The *extended multimomentum bundle* $\mathbb{M}\pi$, which is endowed with a canonical multisymplectic form. Hamiltonian systems can be introduced on this bundle as in autonomous mechanics, by means of suitable closed one-forms (and certain kinds of Hamiltonian multivector fields). The resultant *extended Hamiltonian formalism* is the generalization to field theories of the extended formalism of non-autonomous mechanical systems [9].
  Hamiltonian systems in $\mathbb{M}\pi$ have been recently introduced in [13], where their local properties are studied for the first time. Now, we generalize those ideas, and carry out a deeper geometric study of these kinds of systems.

All manifolds are real, paracompact, connected and C^∞. All maps are C^∞. Sum over crossed repeated indices is understood.

2. Preliminaries

2.1. Multivector Fields and Multisymplectic Manifolds

Let $\mathbb{M}$ be a n-dimensional differentiable manifold. Sections of $\Lambda^k(\mathrm{T}\mathbb{M})$ are called **k-multivector fields** in $\mathbb{M}$ (they are the contravariant skew-symmetric tensors of order k in $\mathbb{M}$). We denote by $\mathfrak{X}^k(\mathbb{M})$ the set of k-multivector fields in $\mathbb{M}$. A multivector field $\mathbf{X} \in \mathfrak{X}^k(\mathbb{M})$ is *locally decomposable* if for every $p \in \mathbb{M}$ there exists an open neighbourhood $U_p \subset \mathbb{M}$, and $X_1, \dots, X_k \in \mathfrak{X}(U_p)$ such that $\mathbf{X}|_{U_p} = X_1 \wedge \dots \wedge X_k$.

A non-vanishing multivector field $\mathbf{X} \in \mathfrak{X}^k()$ and an m-dimensional distribution $D \subset \mathrm{T}\mathbb{M}$ are *locally associated* if there exists a connected open set $U \subseteq \mathbb{M}$ such that $\mathbf{X}|_U$ is a section of $\Lambda^k D|_U$. A non-vanishing, locally decomposable multivector field $\mathbf{X} \in \mathfrak{X}^k(\mathbb{M})$ is *integrable* (or *involutive*) if so is its associated distribution. If $\pi\colon \mathbb{M} \to M$ is a fiber bundle, we are interested in the case where the integral manifolds of integrable multivector fields in $\mathbb{M}$ are sections of π. Thus, $\mathbf{X} \in \mathfrak{X}^k(\mathbb{M})$ is said to be *π-transverse* if at every point $y \in \mathbb{M}$ $(\imath(\mathbf{X})(\pi^*\beta))_y \neq 0$ for every differential form $\beta \in \Omega^k(M)$ with $\beta(\pi(y)) \neq 0$. Then, if $\mathbf{X} \in \mathfrak{X}^k(\mathbb{M})$ is integrable, it is π-transverse if and only if its integral manifolds are local sections of $\pi\colon \mathbb{M} \to M$.

The couple $(\mathbb{M}, \Omega)$ with $\Omega \in \Omega^{m+1}(\mathbb{M})$ $(2 \leq m+1 \leq \dim \mathbb{M})$, is a **multisymplectic manifold** if Ω is closed and 1-nondegenerate, that is, for every $p \in \mathbb{M}$, and

$X_p \in \mathrm{T}_p\mathbb{M}$, we have that $\imath(X_p)\Omega_p = 0$ if and only if $X_p = 0$ (where $\imath(X_p)\Omega_p$ means the contraction of X_p with Ω_p).

If $(\mathbb{M}, \Omega)$ is a multisymplectic manifold, $\mathbf{X} \in \mathfrak{X}^k(\mathbb{M})$ is a **Hamiltonian k-multivector field** if $\imath(\mathbf{X})\Omega$ is an exact $(m+1-k)$-form; that is, there exists $\zeta \in \Omega^{m-k}(\mathbb{M})$ such that

$$\imath(\mathbf{X})\Omega = \mathrm{d}\zeta \tag{1}$$

where ζ is defined modulo closed $(m-k)$-forms, and it is called a **Hamiltonian form** for $\mathbf{X}$. Furthermore, $\mathbf{X}$ is a *locally Hamiltonian k-multivector field* if $\imath(\mathbf{X})\Omega$ is a closed $(m+1-k)$-form. In this case, for every point $x \in \mathbb{M}$, there is an open neighbourhood $W \subset \mathbb{M}$ and $\zeta \in \Omega^{m-k}(W)$ such that

$$\imath(\mathbf{X})\Omega = \mathrm{d}\zeta \qquad (\text{on } W)$$

and ζ are the *local Hamiltonian forms* for $\mathbf{X}$. Conversely, $\zeta \in \Omega^k(\mathbb{M})$ (resp. $\zeta \in \Omega^k(W)$) is a **Hamiltonian k-form** (respectively a *local Hamiltonian k-form*) if there exists a multivector field $\mathbf{X} \in \mathfrak{X}^{m-k}(\mathbb{M})$ (respectively $\mathbf{X} \in \mathfrak{X}^{m-k}(W)$) such that (1) holds (respectively on W).

2.2. Multimomentum Bundles

Let $\pi\colon E \to M$ be the *configuration bundle* of a field theory, ($\dim M = m$, $\dim E = n+m$), where M is an oriented manifold with volume form $\omega \in \Omega^m(M)$, and denote by (x^ν, y^A) ($\nu = 1, \ldots, m$, $A = 1, \ldots, n$) the natural coordinates in E adapted to the bundle, such that $\omega = \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^m \equiv \mathrm{d}^m x$. There are several multimomentum bundle structures associated with this bundle.

First we have $\Lambda_2^m \mathrm{T}^*E \equiv \mathbb{M}\pi$, which is the bundle of m-forms on E vanishing by the action of two π-vertical vector fields. It is called the **extended multimomentum bundle**, and its canonical submersions are denoted

$$\kappa\colon \mathbb{M}\pi \to E, \qquad \bar{\kappa} = \pi \circ \kappa\colon \mathbb{M}\pi \to M.$$

$\mathbb{M}\pi$ is a subbundle of $\Lambda^m \mathrm{T}^*E$, the multicotangent bundle of E of order m (the bundle of m-forms in E), and hence $\mathbb{M}\pi$ is endowed with canonical forms. First we have the "tautological form" $\Theta \in \Omega^m(\mathbb{M}\pi)$ which is defined as follows: let $(x, \alpha) \in \Lambda_2^m \mathrm{T}^*E$, with $x \in E$ and $\alpha \in \Lambda_2^m \mathrm{T}_x^*E$, then, for every $X_1, \ldots, X_m \in \mathrm{T}_{(x,\alpha)}(\mathbb{M}\pi)$, we have

$$\Theta((x,\alpha), X_1, \ldots, X_m) := \alpha(x, \mathrm{T}_{(x,\alpha)}\kappa(X_1), \ldots, \mathrm{T}_{(q,\alpha)}\kappa(X_m)).$$

Thus we define the multisymplectic form

$$\Omega := -\mathrm{d}\Theta \in \Omega^{m+1}(\mathbb{M}\pi).$$

We can introduce natural coordinates in $\mathbb{M}\pi$ adapted to the bundle $\pi\colon E \to M$, which are denoted by (x^ν, y^A, p_A^ν, p), and such that $\omega = \mathrm{d}^m x$. Then the local expressions of these forms are

$$\Theta = p_A^\nu \mathrm{d}y^A \wedge \mathrm{d}^{m-1}x_\nu + p\mathrm{d}^m x, \quad \Omega = -\mathrm{d}p_A^\nu \wedge \mathrm{d}y^A \wedge \mathrm{d}^{m-1}x_\nu - \mathrm{d}p \wedge \mathrm{d}^m x$$

where $\mathrm{d}^{m-1}x_\nu := \imath\left(\frac{\partial}{\partial x^\nu}\right)\mathrm{d}^m x$.

Consider $\pi^*\Lambda^m \mathrm{T}^* M$, which is another bundle over E, whose sections are the π-semibasic m-forms on E, and denote by $J^1\pi^*$ the quotient $\Lambda_2^m \mathrm{T}^* E/\pi^*\Lambda^m \mathrm{T}^* M$. We have the natural submersions

$$\tau\colon J^1\pi^* \to E, \qquad \bar{\tau} = \pi \circ \tau\colon J^1\pi^* \to M.$$

Furthermore, the natural submersion $\mu\colon \mathbb{M}\pi \to J^1\pi^*$ endows $\mathbb{M}\pi$ with the structure of an affine bundle over $J^1\pi^*$, with $\tau^*\Lambda_1^m \mathrm{T}^* E$ as the associated vector bundle. $J^1\pi^*$ is usually called the **restricted multimomentum bundle** associated with the bundle $\pi\colon E \to M$. Natural coordinates in $J^1\pi^*$ (adapted to the bundle $\pi\colon E \to M$) are denoted by (x^ν, y^A, p_A^ν).

We have the diagram

$$\begin{tikzcd}
\mathbb{M}\pi \arrow[rr, "\mu"] \arrow[dr, "\kappa"] \arrow[ddr, "\bar{\kappa}"'] & & J^1\pi^* \arrow[dl, "\tau"'] \arrow[ddl, "\bar{\tau}"] \\
& E \arrow[d, "\pi"'] & \\
& M &
\end{tikzcd}$$

2.3. Hamiltonian Systems in $J^1\pi^*$

The Hamiltonian formalism in $J^1\pi^*$ that is presented here is based on the construction made in [2] (see also [3] and [4]).

Definition 1. *A section* $h\colon J^1\pi^* \to \mathbb{M}\pi$ *of the projection* μ *is called a* Hamiltonian section. *The differentiable forms* $\Theta_h := h^*\Theta$ *and* $\Omega_h := -\mathrm{d}\Theta_h = h^*\Omega$ *are called the* Hamilton–Cartan m and $(m+1)$ forms *of* $J^1\pi^*$ *associated with the Hamiltonian section* h. $(J^1\pi^*, h)$ *is said to be a* restricted Hamiltonian system.

In the natural coordinates we have that

$$h(x^\nu, y^A, p_A^\nu) \equiv (x^\nu, y^A, p_A^\nu, p = -\mathrm{h}(x^\gamma, y^B, p_B^\eta))$$

and $\mathrm{h} \in C^\infty(U)$, $U \subset J^1\pi^*$, is a *local Hamiltonian function*. Then we have

$$\Theta_h = p_A^\nu \mathrm{d}y^A \wedge \mathrm{d}^{m-1}x_\nu - \mathrm{h}\mathrm{d}^m x, \quad \Omega_h = -\mathrm{d}p_A^\nu \wedge \mathrm{d}y^A \wedge \mathrm{d}^{m-1}x_\nu + \mathrm{d}\mathrm{h} \wedge \mathrm{d}^m x.$$

The field equations for restricted Hamiltonian systems can be derived from a variational principle. In fact, first we state

Definition 2. *Let $(J^1\pi^*, h)$ be a restricted Hamiltonian system. Let $\Gamma(M, J^1\pi^*)$ be the set of sections of $\bar{\tau}$. Consider the map*

$$\begin{array}{rccc} \mathbf{H} : & \Gamma(M, J^1\pi^*) & \longrightarrow & \mathbb{R} \\ & \psi & \mapsto & \int_M \psi^*\Theta_h \end{array}$$

(where the convergence of the integral is assumed). The variational problem *for this system is the search of the critical sections of the functional* $\mathbf{H}$*, with respect to the variations of ψ given by $\psi_t = \sigma_t \circ \psi$, where $\{\sigma_t\}$ is a local one-parameter group of any compact-supported vector field $Z \in \mathfrak{X}^{V(\bar{\tau})}(J^1\pi^*)$ ($\bar{\tau}$-vertical vector fields in $J^1\pi^*$), that is*

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \int_M \psi_t^*\Theta_h = 0.$$

This is the so-called Hamilton–Jacobi principle *of the Hamiltonian formalism.*

Then the following fundamental theorem is proven (see [3]).

Theorem 1. *The following assertions on a section $\psi \in \Gamma(M, J^1\pi^*)$ are equivalent:*

1. *ψ is critical section for the variational problem posed by the Hamilton–Jacobi principle.*
2. *$\psi^* i(X)\Omega_h = 0$ for all $X \in \mathfrak{X}(J^1\pi^*)$.*
3. *ψ is an integral section of an integrable multivector field $\mathbf{X}_h \in \mathfrak{X}^m(J^1\pi^*)$ satisfying that*

$$i(\mathbf{X}_h)\Omega_h = 0, \qquad i(\mathbf{X}_h)(\bar{\tau}^*\omega) \neq 0, \qquad \bar{\tau}\text{-transversality}. \tag{2}$$

   (It is usual to take $i(\mathbf{X}_h)(\bar{\tau}^*\omega) = 1$).

4. *If $(U; x^\nu, y^A, p^\nu_A)$ is a system of coordinates in $J^1\pi^*$, then ψ satisfies in U*

$$\frac{\partial(y^A \circ \psi)}{\partial x^\nu} = \left.\frac{\partial \mathrm{h}}{\partial p^\nu_A}\right|_\psi, \qquad \frac{\partial(p^\nu_A \circ \psi)}{\partial x^\nu} = -\left.\frac{\partial \mathrm{h}}{\partial y^A}\right|_\psi \tag{3}$$

   which are the Hamilton–De Donder–Weyl equations *of the restricted Hamiltonian system.*

Definition 3. $\mathbf{X}_h \in \mathfrak{X}^m(J^1\pi^*)$ *is a* ***Hamilton–De Donder–Weyl*** *(HDW)* ***multivector field*** *for the system $(J^1\pi^*, h)$ if it is locally decomposable and verifies equations (2).*

The existence of HDW-multivector fields $\mathbf{X}_h$ for $(J^1\pi^*, h)$ is assured, and in a local system of coordinates they depend on $n(m^2-1)$ arbitrary functions. Nevertheless, they are not integrable necessarily, and hence the integrability conditions imply that the number of arbitrary functions will in general be less than $n(m^2-1)$.

3. Hamiltonian Systems in $\mathbb{M}\pi$

3.1. Extended Hamiltonian Systems

Definition 4. $(\mathbb{M}\pi, \Omega, \alpha)$ *is an* ***extended Hamiltonian system*** *if:*

1. $\alpha \in Z^1(\mathbb{M}\pi)$.
2. *There exists a locally decomposable multivector field* $\mathbf{X}_\alpha \in \mathfrak{X}^m(\mathbb{M}\pi)$ *satisfying that*

$$\imath(\mathbf{X}_\alpha)\Omega = (-1)^{m+1}\alpha, \qquad \imath(\mathbf{X}_\alpha)(\bar{\kappa}^*\omega) = 1, \qquad \bar{\kappa}\text{-transversality}. \tag{4}$$

If α is exact, $(\mathbb{M}\pi, \Omega, \alpha)$ *is an* extended global Hamiltonian system. *Then there exists* $\mathrm{H} \in C^\infty(\mathbb{M}\pi)$ *such that* $\alpha = \mathrm{dH}$, *which are called* Hamiltonian functions *of the system. (For an extended Hamiltonian system,* H *exist only locally, and they are called* local Hamiltonian functions).

In addition, the integrability of $\mathbf{X}_\alpha$ must be also considered.

Not every closed form $\alpha \in \Omega^m(\mathbb{M}\pi)$ defines an extended Hamiltonian system. In fact, a simple calculation in coordinates leads to the following

Proposition 1. *If* $(\mathbb{M}\pi, \Omega, \alpha)$ *is an extended Hamiltonian system, then* $\imath(Y)\alpha \neq 0$, *for all* $Y \in \mathfrak{X}^{\mathrm{V}(\mu)}(\mathbb{M}\pi)$, $Y \neq 0$. *In particular, for every system of natural coordinates* $(x^\nu, y^A, p^\nu_A, p))$ *in* $\mathbb{M}\pi$ *adapted to the bundle structure (with* $\omega = \mathrm{d}^m x$*)*

$$\imath\left(\frac{\partial}{\partial p}\right)\alpha = 1.$$

As a consequence, if $(\mathbb{M}\pi, \Omega, \alpha)$ *is an extended Hamiltonian system, locally* $\alpha = \mathrm{d}p + \beta$, *where β is a closed and μ-basic local one-form in* $\mathbb{M}\pi$.

In a system of natural coordinates we have that

$$\alpha = \mathrm{d}p + \mathrm{d}\tilde{\mathrm{h}}(x^\nu, y^A, p^\nu_A) = \mathrm{d}(p + \tilde{\mathrm{h}}(x^\nu, y^A, p^\nu_A)) \equiv \mathrm{dH}$$

where $\tilde{\mathrm{h}} = \mu^*\mathrm{h}$, for some $\mathrm{h} \in C^\infty(\mu(U))$, $U \subset \mathbb{M}$.

Then, it can be proven [3] that

Theorem 2. *Let* $\alpha \in Z^1(\mathbb{M}\pi)$ *satisfying the condition stated in Proposition* 1. *Then there exist locally decomposable multivector fields* $\mathbf{X}_\alpha \in \mathfrak{X}^m(\mathbb{M}\pi)$ *(not necessarily integrable) satisfying equations* (4) *(and hence* $(\mathbb{M}\pi, \Omega, \alpha)$ *is an extended*

Hamiltonian system). In a local system of coordinates the above solutions depend on $n(m^2-1)$ arbitrary functions.

A simple calculation leads to the following local expression

$$\mathbf{X}_\alpha = \bigwedge_{\nu=1}^{m}\left(\frac{\partial}{\partial x^\nu} + \tilde{F}_\nu^A \frac{\partial}{\partial y^A} + \tilde{G}_{A\nu}^\rho \frac{\partial}{\partial p_A^\rho} + \tilde{g}_\nu \frac{\partial}{\partial p}\right)$$

where

$$\tilde{F}_\nu^A = \frac{\partial \mathrm{H}}{\partial p_A^\nu} = \frac{\partial \tilde{\mathrm{h}}}{\partial p_A^\nu}, \qquad A=1,\dots,n,\ \nu=1,\dots,m \tag{5}$$

$$\tilde{G}_{A\mu}^\mu = -\frac{\partial \mathrm{H}}{\partial y^A} = -\frac{\partial \tilde{\mathrm{h}}}{\partial y^A}, \qquad A=1,\dots,n \tag{6}$$

$$\tilde{g}_\nu = -\frac{\partial \tilde{\mathrm{h}}}{\partial x^\nu} + \frac{\partial \tilde{\mathrm{h}}}{\partial p_A^\nu}\tilde{G}_{A\eta}^\eta - \frac{\partial \tilde{\mathrm{h}}}{\partial p_A^\eta}\tilde{G}_{A\nu}^\eta, \qquad A=1,\dots,n,\ \eta\neq\nu. \tag{7}$$

Definition 5. $\mathbf{X}_\alpha \in \mathfrak{X}^m\mathbb{M}\pi)$ *is an* ***extended Hamilton–De Donder–Weyl multivector field*** *for* $(\mathbb{M}\pi,\Omega,\alpha)$ *if it is a solution to equations.* (4).

As above, the integrability of $\mathbf{X}_\alpha$ makes that the number of arbitrary functions is, in general, less than $n(m^2-1)$.
If $\mathbf{X}_\alpha$ is integrable and $\tilde{\psi}(x)$ is an integral section of $\mathbf{X}_\alpha$, then

$$\frac{\partial(y^A\circ\tilde{\psi})}{\partial x^\nu} = \tilde{F}_\nu^A \circ \tilde{\psi}, \qquad \frac{\partial(p_A^\nu\circ\tilde{\psi})}{\partial x^\nu} = \tilde{G}_{A\nu}^\nu \circ \tilde{\psi}, \qquad \frac{\partial(p\circ\tilde{\psi})}{\partial x^\nu} = \tilde{g}_\nu \circ \tilde{\psi}$$

and equations (5), (6) and (7) give PDE's for the integral sections of $\mathbf{X}_\alpha$.

3.2. Geometric Properties of Extended Hamiltonian Systems

Proposition 2. *Let* $(\mathbb{M}\pi,\Omega,\alpha)$ *be an extended Hamiltonian system, and* D_α *the* characteristic distribution *of* α. *Then:*

1. D_α *is a* μ*-transverse involutive distribution of corank equal to one.*
2. *The integral submanifolds* S *of* D_α *are one-codimensional and* μ*-transverse local submanifolds of* $\mathbb{M}\pi$.
3. *If* S *is an integral submanifold of* D_α, *then* $\mu|_S\colon S\to J^1\pi^*$ *is a local diffeomorphism.*
4. *For every integral submanifold* S *of* D_α, *and* $p\in S$, *there exists* $W\subset\mathbb{M}\pi$, *with* $p\in W$, *such that* $h=(\mu|_{W\cap S})^{-1}$ *is a local Hamiltonian section of* μ *defined on* $\mu(W\cap S)$.

Proof: See [3]. □

As $\alpha = \mathrm{dH} = \mathrm{d}(p + \mu^*\mathrm{h})$ (locally), every local Hamiltonian function H is a constraint defining locally the integral submanifolds of D_α. Thus the local Hamiltonian sections associated with these submanifolds are expressed as

$$h(x^\nu, y^A, p_A^\nu) = (x^\nu, y^A, p_A^\nu, p = -\mathrm{h}(x^\gamma, y^B, p_B^\eta)).$$

As a straighforward consequence of the last proposition we obtain

Proposition 3. *Every extended HDW $\mathbf{X}_\alpha \in \mathfrak{X}^m(\mathbb{M}\pi)$ for $(\mathbb{M}\pi, \Omega, \alpha)$ is tangent to every integral submanifold of D_α. As a consequence, if $\mathbf{X}_\alpha$ is integrable, then its integral sections are contained in the integral submanifolds of D_α.*

Using the local expressions of α and $\mathbf{X}_\alpha$, and equations (5) and (6), the tangency condition leads to equations (7), which are just consistency conditions. (See also the comment in Remark 1.)

3.3. Relation between Extended and Restricted Hamiltonian Systems

Theorem 3. *Let $(\mathbb{M}\pi, \Omega, \alpha)$ be an extended Hamiltonian system, and $(J^1\pi^*, h)$ a restricted Hamiltonian system such that $\mathrm{Im}\, h = S$ is an integral submanifold of D_α. For every $\mathbf{X}_\alpha \in \mathfrak{X}^m(\mathbb{M}\pi)$ solution to the equations* (4)

$$\imath(\mathbf{X}_\alpha)\Omega = (-1)^{m+1}\alpha, \qquad \imath(\mathbf{X}_\alpha)(\bar{\kappa}^*\omega) = 1$$

there exists $\mathbf{X}_h \in \mathfrak{X}^m(J^1\pi^*)$, such that $\Lambda^m h_* X_h = X_\alpha|_S$, which is a solution to the equations (2)

$$\imath(\mathbf{X}_h)\Omega_h = 0, \qquad \imath(\mathbf{X}_h)(\bar{\tau}^*\omega) = 1.$$

Furthermore, if $\mathbf{X}_\alpha$ is integrable, then $\mathbf{X}_h$ is integrable too, and the integral sections of $\mathbf{X}_h$ are recovered from those of $\mathbf{X}_\alpha$ as follows: if $\tilde{\psi} \colon M \to \mathbb{M}\pi$ is an integral section of $\mathbf{X}_\alpha$, then $\psi = \mu \circ \tilde{\psi} \colon M \to J^1\pi^*$ is an integral section of $\mathbf{X}_h$.

Proof: See [3]. □

Definition 6. *Given an extended Hamiltonian system $(\mathbb{M}\pi, \Omega, \alpha)$, and considering all the Hamiltonian sections $h \colon J^1\pi^* \to \mathbb{M}\pi$ such that $\mathrm{Im}\, h$ are integral submanifolds of D_α, we have a family $\{(J^1\pi^*, h)\}_\alpha$, which is called the class of **restricted Hamiltonian systems** associated with $(\mathbb{M}\pi, \Omega, \alpha)$.*

In addition, if $\{(J^1\pi^*, h)\}_\alpha$ is the class of restricted Hamiltonian systems associated with an extended Hamiltonian system $(\mathbb{M}\pi, \Omega, \alpha)$, then the submanifolds $\{(S_h, \jmath_{S_h}^*\Omega\})$ are premultisymplectomorphic (where $S_h = \mathrm{Im}\, h$, for every Hamiltonian section h in this class, and $\jmath_{S_h} \colon S_h \hookrightarrow \mathbb{M}\pi$ is the natural embedding).

Conversely, we have also

Proposition 4. *Given a restricted Hamiltonian system* $(J^1\pi^*, h)$, *let* $\jmath_S\colon S = \mathrm{Im}\, h \hookrightarrow \mathbb{M}\pi$ *be the natural embedding. Then, there exists a unique local form* $\alpha \in \Omega^1(\mathbb{M}\pi)$ *such that:*

1. $\alpha \in Z^1(\mathbb{M}\pi)$ (*it is a closed form*).
2. $\jmath_S^*\alpha = 0$.
3. $\imath(Y)\alpha \neq 0$, *for every non-vanishing* $Y \in \mathfrak{X}^{\mathrm{V}(\mu)}(\mathbb{M}\pi)$ *and, in particular, such that* $\imath\left(\dfrac{\partial}{\partial p}\right)\alpha = 1$, *for every system of natural coordinates* $(x^\nu,\, y^A,\, p_A^\nu,\, p)$ *in* $\mathbb{M}\pi$, *adapted to the bundle structure* (*with* $\omega = \mathrm{d}^m x$).

Proof: See [3]. □

Definition 7. *Given a restricted Hamiltonian system* $(J^1\pi^*, h)$, *if* $\alpha \in \Omega^1(\mathbb{M}\pi)$ *satisfies the above conditions, then* $(\mathbb{M}\pi, \Omega, \alpha)$ *is called the local* ***extended Hamiltonian system*** *associated with* $(J^1\pi^*, h)$.

3.4. Variational Principle and Field Equations

As in the case of restricted Hamiltonian systems, the field equations for extended Hamiltonian systems can be derived from a suitable variational principle.

Let $\Gamma_\alpha(M, \mathbb{M}\pi)$ be the set of sections of $\bar{\kappa}$ which are integral submanifolds of D_α, and consider

$$\mathfrak{X}_\alpha^{\mathrm{V}(\bar{\kappa})}(\mathbb{M}\pi) = \{Z \in \mathfrak{X}(\mathbb{M}\pi)\,;\; \imath(Z)\alpha = 0,\; Z \text{ is } \bar{\kappa}\text{-vertical}\}.$$

Here Z are the $\bar{\kappa}$-vertical vector fields of $\mathbb{M}\pi$ wich are tangent to the integral submanifolds of D_α.

Definition 8. *Let* $(\mathbb{M}\pi, \Omega, \alpha)$ *be an extended Hamiltonian system. Consider*

$$\begin{array}{rccc} \tilde{\mathbf{H}}_\alpha \;:\; & \Gamma_\alpha(M, \mathbb{M}\pi) & \longrightarrow & \mathbb{R} \\ & \tilde{\psi} & \mapsto & \int_U \tilde{\psi}^*\Theta \end{array}$$

(*where the convergence of the integral is assumed*). *The* variational problem *for this system is the search for the critical sections of the functional* $\tilde{\mathbf{H}}_\alpha$, *with respect to the variations of* $\tilde{\psi} \in \Gamma_\alpha(M, \mathbb{M}\pi)$ *given by* $\tilde{\psi}_t = \sigma_t \circ \tilde{\psi}$, *where* $\{\sigma_t\}$ *is the local one-parameter group of any compact-supported vector field* $Z \in \mathfrak{X}_\alpha^{\mathrm{V}(\bar{\kappa})}(\mathbb{M}\pi)$, *that is*

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \int_U \tilde{\psi}_t^*\Theta = 0.$$

This is the extended Hamilton-Jacobi principle.

Then the following fundamental theorem is proven

Theorem 4. *The following assertions on* $\tilde{\psi} \in \Gamma_\alpha(M, \mathbb{M}\pi)$ *are equivalent:*

1. *$\tilde{\psi}$ is a critical section for the Hamilton–Jacobi principle.*
2. *$\tilde{\psi}^* \imath(X)\Omega = 0$ for every $X \in \mathfrak{X}_\alpha(\mathbb{M}\pi)$.*
3. *$\tilde{\psi}$ is an integral section of an integrable multivector field $\mathbf{X} \in \mathfrak{X}^m(\mathbb{M}\pi)$ which is a solution to the equations* (4)

$$\imath(\mathbf{X}_\alpha)\Omega = (-1)^{m+1}\alpha, \qquad \imath(\mathbf{X}_\alpha)(\bar{\kappa}^*\omega) = 1.$$

4. *If $(U; x^\nu, y^A, p_A^\nu, p)$ is a natural system of coordinates in $\mathbb{M}\pi$, then $\tilde{\psi}$ satisfies the following system of equations in U*

$$\frac{\partial(y^A \circ \tilde{\psi})}{\partial x^\nu} = \frac{\partial \tilde{\mathrm{h}}}{\partial p_A^\nu}\Big|_{\tilde{\psi}}, \quad \frac{\partial(p_A^\nu \circ \tilde{\psi})}{\partial x^\nu} = -\frac{\partial \tilde{\mathrm{h}}}{\partial y^A}\Big|_{\tilde{\psi}}, \quad \frac{\partial(p \circ \tilde{\psi})}{\partial x^\nu} = -\frac{\partial(\tilde{\mathrm{h}} \circ \tilde{\psi})}{\partial x^\nu} \tag{8}$$

where $\tilde{\mathrm{h}} = \mu^*\mathrm{h}$, for some $\mathrm{h} \in C^\infty(\mu(U))$, is any function such that $\alpha|_U = \mathrm{d}p + \mathrm{d}\tilde{\mathrm{h}}(x^\nu, y^A, p_A^\nu)$. These are the extended Hamilton–De Donder–Weyl equations of the extended Hamiltonian system.

Proof: See [3]. □

Remark 1. The last group of equations in (8) are consistency conditions with respect to the hypothesis on the sections $\tilde{\psi}$. In fact, this group of equations leads to $p \circ \tilde{\psi} = -\tilde{\mathrm{h}} \circ \tilde{\psi} + \mathrm{const}$, that is, $\tilde{\psi} \in \Gamma_\alpha(M, \mathbb{M}\pi)$. (See also the comment after Proposition 3). The rest of the equations (8) are just the Hamilton–De Donder–Weyl equations (3) of the restricted case.

4. Conclusions and Outlook

1. $J^1\pi^*$ is the "natural" multimomentum phase space for field theories, but it has not a natural multisymplectic structure.
   Restricted Hamiltonian systems are defined by Hamiltonian sections which are also used to construct the multisymplectic form Ω_h in $J^1\pi^*$. Both the geometry and the "physical information" are coupled in Ω_h.
2. $(\mathbb{M}\pi, \Omega)$ is a canonical multisymplectic manifold.
   Extended Hamiltonian systems are defined by closed one-forms, $\alpha \in Z^1(\mathbb{M}\pi)$, which must be μ-transverse. The geometry (Ω) and the "physical information" (α) are not coupled. Field equations are analogous to the dynamical equations of autonomous mechanical Hamiltonian systems.
3. Every extended Hamiltonian system is associated with a family of restricted Hamiltonian systems and, conversely, every restricted Hamiltonian system is associated with an extended Hamiltonian systems (at least locally).
4. The definitions of restricted and extended Hamiltonian systems for submanifolds of $J^1\pi^*$ and $\mathbb{M}\pi$ (satisfying suitable conditions) can be achieved in

order to include the almost-regular field theories in this framework [10]. Their properties are analogous to the former case.

5. We hope that some problems could be studied in the extended formalism in an easier way than in the restricted case. For instance: multisymplectic reduction by symmetries, integrability, or quantization of field theories.

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