Fifth International Conference on
Geometry, Integrability and Quantization
June 5–12, 2003, Varna, Bulgaria
Ivaïlo M. Mladenov and Allen C. Hirshfeld, Editors
SOFTEX, Sofia 2004, pp 178–188

THE ZEROS OF POLYNOMIALS ORTHOGONAL WITH RESPECT TO q-INTEGRAL ON SEVERAL INTERVALS IN THE COMPLEX PLANE

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Abstract. We construct the sequence of orthogonal polynomials with respect to an inner product defined in the sense of q-integration over several intervals in the complex plane. For such introduced polynomials we prove that all zeros lie in the smallest convex hull over the intervals in the complex plane. The results are stated precisely in some special cases, as some symmetric cases of equal lengths, angles and weights.

1. Introduction

We will start with well-known facts from q-calculus [1], [2], where q is a real number from the interval $(0, 1)$. The basic number $[x]_q$ is given by

$$[x]_q = \frac{1-q^x}{1-q} \qquad (x \in \mathbb{R})$$

and factorial of q-natural numbers

$$[0]_q! = 1, \quad [n]_q! = [n]_q[n-1]_q! = [n]_q[n-1]_q \cdots [1]_q, \quad n \in \mathbb{N}.$$

We define q-shifted factorials by

$$(a;q)_0 = 1, \quad (a;q)_n = \prod_{k=1}^{n}(1-aq^{k-1}), \quad (a;q)_\infty = \prod_{k=1}^{\infty}(1-aq^{k-1})$$

and q-shifted factorial of a vector is

$$(a_1, a_2, \dots, a_r; q)_n = (a_1; q)_n (a_1; q)_n (a_2; q)_n \cdots (a_r; q)_n.$$

Obviously,

$$[n]_q! = \frac{(q;q)_n}{(1-q)^n}, \quad n \in \mathbb{N}.$$

Generalization of gamma function $\Gamma(x)$ is given by

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty} (1-q)^{1-x}.$$

One of main properties of gamma function holds on, i.e.

$$\Gamma_q(x+1) = [x]_q \, \Gamma_q(x).$$

Basic hypergeometric function is defined by

$$ {}_r\Phi_s \left(\begin{matrix} a_1, a_2, \dots, a_r \\ b_1, b_2, \dots, b_s \end{matrix} \middle| q; z \right) = \sum_{k=0}^{\infty} \frac{(a_1, \dots, a_r; q)_k}{(b_1, \dots, b_s; q)_k} (-1)^{(1+s-r)k} q^{(1+s-r)\binom{k}{2}} \frac{z^k}{(q;q)_k}.$$

We assume that the convergence of all series in further discussion is assured.
Let us define a ***q*-integral** by

$$\int_a^b F(z)\, \mathrm{d}_q z := (b-a)(1-q) \sum_{k=0}^{\infty} F\left(a + (b-a)q^k\right) q^k \quad (0 < q < 1,\ a, b \in \mathbb{C}).$$

An inner product on *beam* of m-intervals $[a_s, b_s]$ $(s = 0, 1, \dots, m-1)$ is defined by

$$\langle F, G \rangle = \sum_{s=0}^{m-1} \frac{1}{b_s - a_s} \int_{a_s}^{b_s} F(z) \overline{G(z)} |W_s(z)| \, \mathrm{d}_q z \tag{1}$$

where $W_s(z)$ is a weight function on the interval $[a_s, b_s]$.
The previous inner product can be written in the form

$$\langle F, G \rangle = (1-q) \sum_{s=0}^{m-1} \sum_{k=0}^{\infty} q^k F(z) \overline{G(z)} |W_s(z)| \Big|_{z = a_s + (b_s - a_s)q^k}. \tag{2}$$

This product is positive-definite because of $\|F\|^2 = (F, F) > 0$, except for $F(x) \equiv 0$.
It implies existence of the sequence orthogonal polynomials $\{P_N(z)\}$ which satisfies

$$\langle P_M, P_N \rangle = \delta_{MN} \|P_N\|^2 \quad (M, N \in \mathbb{N}_0).$$

We can construct this sequence by the Gram-Schmidt orthogonalization.

If we define **moments** and **moment-determinants** by

$$\mu_{j,k} = \langle z^j, z^k \rangle, \quad \Delta_0 = 1, \quad \Delta_n = |\mu_{i,j}|_{i,j=0}^{n-1}, \quad n \geq 1$$

then these polynomials can be expressed in the form

$$P_0(z) = 1, \quad P_n(z) = \frac{1}{\Delta_n} \begin{vmatrix} \mu_{00} & \mu_{10} & \cdots & \mu_{n-1,0} & 1 \\ \mu_{01} & \mu_{11} & \cdots & \mu_{n-1,1} & z \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mu_{0,n-1} & \mu_{1,n-1} & \cdots & \mu_{n-1,n-1} & z^{n-1} \\ \mu_{0,n} & \mu_{1,n} & \cdots & \mu_{n-1,n} & z^n \end{vmatrix}, \quad n \geq 1.$$

This sequence is unique and the norms are

$$||P_n||^2 = \frac{\Delta_{n+1}}{\Delta_n}.$$

Especially, if all intervals start from the origin and q tends to 1, we get polynomials orthogonal on the radial rays in the complex plane which were introduced by G. Milovanović [4] and investigated in the papers of G. Milovanović, R. Rajković and Z. Marjanović [5, 6].

In the cases when all beams start from the real axis and make the zero or straight angle with it, we have standard case of orthogonal polynomials on interval or on a few segments of the real line.

2. Transformations of Beams and Polynomials

Here, we will discuss some transformations in the complex plane, such as scaling and rotating, of support of orthogonality and its repercussions to orthogonal polynomials.

Theorem 2.1. *Let σ be a complex number. The sequence $\{\pi_N^\sigma(z)\}_{N=0}^{+\infty}$ orthogonal with respect to the inner product*

$$\langle F, G \rangle_\sigma = \sum_{s=0}^{m-1} \frac{1}{\sigma(b_s - a_s)} \int_{\sigma a_s}^{\sigma b_s} F(z)\overline{G(z)}|W_s(z/\sigma)| \, \mathrm{d}_q z$$

can be expressed by

$$\pi_N^\sigma(z) = \pi_N(z/\sigma),$$

where $\{\pi_N(z)\}_{N=0}^{+\infty}$ is orthogonal with respect to (1).

Proof: Let $\{\pi_N(z)\}_{N=0}^{+\infty}$ be an orthogonal sequence with respect to (1) and $\tau_N(z) = \pi_N(z/\sigma)$. Then we have

$$\begin{aligned}
\langle \tau_K, \tau_N \rangle_\sigma &= \sum_{s=0}^{m-1} \int_{\sigma a_s}^{\sigma b_s} \tau_K(z)\overline{\tau_N(z)} |W_s(z/\sigma)| \,\mathrm{d}_q z \\
&= (1-q) \sum_{s=0}^{m-1} \sum_{k=0}^{\infty} q^k \tau_K(t)\overline{\tau_N(t)} |W_s(t/\sigma)| \Big|_{t=\sigma a_s + (\sigma b_s - \sigma a_s)q^k} \\
&= (1-q) \sum_{s=0}^{m-1} \sum_{k=0}^{\infty} q^k \pi_K(t/\sigma)\overline{\pi_N(t/\sigma)} |W_s(t/\sigma)| \Big|_{t=\sigma(a_s + (b_s - a_s)q^k)}.
\end{aligned}$$

Introducing a change $u = t/\sigma$, yields

$$\begin{aligned}
\langle \tau_K, \tau_N \rangle_\sigma &= (1-q) \sum_{s=0}^{m-1} \sum_{k=0}^{\infty} q^k \pi_K(u)\overline{\pi_N(u)} |W_s(u)| \Big|_{u=a_s + (b_s - a_s)q^k} \\
&= \langle \pi_K, \pi_N \rangle.
\end{aligned}$$

Because of the uniqueness of orthogonal polynomial sequence, we conclude that the statement is valid. □

Corollary 2.1. *The zeros of $\pi_N^\sigma(z)$ are obtained from the zeros of $\pi_N(z)$ by the multiplying with σ.*

Proof: Let ζ be a zero of the polynomial $\pi_N(z)$, i.e. $\pi_N(\zeta) = 0$. According the previous theorem, we find

$$\pi_N^\sigma(\sigma\zeta) = \pi_N(\zeta) = 0$$

i.e. $\sigma\zeta$ is a zero of the polynomial $\pi_N^\sigma(z)$. □

Remark. We can emphasize the next transforms obtained for some spacial values of σ:

1) rotation, when $\sigma = \mathrm{e}^{\mathrm{i}\alpha}$ ($\alpha \in \mathbb{R}$)
2) scaling, when $\sigma = R \in \mathbb{R}$.

3. Orthogonality on the Intervals with the Start Point in the Origin

In this section, we consider intervals which go from the origin to some points in the complex plane. Now, inner product can be written by

$$\langle F, G \rangle = \sum_{s=0}^{m-1} \frac{1}{b_s} \int_0^{b_s} F(z)\overline{G(z)} |W_s(z)| \,\mathrm{d}_q z \tag{3}$$

or

$$\langle F, G \rangle = (1-q) \sum_{s=0}^{m-1} \sum_{k=0}^{\infty} q^k F(t)\overline{G(t)} |W_s(t)| \Big|_{t=b_s q^k}. \tag{4}$$

Since b_s is a complex number, we can write it by

$$b_s = l_s \varphi_m(s)$$

where

$$l_s \in \mathbb{R}^+, \quad \varphi_m(s) = \exp(\mathrm{i}\theta_s), \quad \theta_s \in (-\pi, \pi], \qquad 0 \le s \le m-1, \ \mathrm{i}^2 = -1.$$

Theorem 3.1. *The polynomial $\pi_N(z)$ ($N > 0$) orthogonal with respect to* (3) *has all zeros in the smallest rectangle spread over the radial rays with edges parallel with axes,*

$$R = \{z : a_1 \le \mathrm{Re}(z) \le a_2 \wedge b_1 \le \mathrm{Im}(z) \le b_2\} \tag{5}$$

where

$$a_1 = \min\left\{0, \min_{0 \le s < m} l_s \cos\theta_s\right\}, \quad a_2 = \max\left\{0, \max_{0 \le s < m} l_s \cos\theta_s\right\}$$

and

$$b_1 = \min\left\{0, \min_{0 \le s < m} l_s \sin\theta_s\right\}, \quad b_2 = \max\left\{0, \max_{0 \le s < m} l_s \sin\theta_s\right\}.$$

Proof: Suppose that ζ is a zero of $\pi_N(z)$. Then we can write

$$\pi_N(z) = (z - \zeta) r_{N-1}(z), \quad r_{N-1}(z) \in \mathcal{P}_{N-1}.$$

Because of the orthogonality, we obtain

$$0 = \langle \pi_N, r_{N-1} \rangle = (1-q) \sum_{s=0}^{m-1} \sum_{k=0}^{\infty} q^k \left(b_s q^k - \zeta\right) |r_{N-1}(b_s q^k)|^2 \, |W_s(b_s q^k)|.$$

Hence, the real and imaginary part of the sum are equal to zero. Since b_s and ζ are complex numbers, we can write them as

$$b_s = l_s \cos\theta_s + \mathrm{i} l_s \sin\theta_s \quad (l_s \in \mathbb{R}^+, \ \theta_s \in (-\pi, \pi]), \quad \zeta = A + \mathrm{i}B, \quad A, B \in \mathbb{R}.$$

Now, we have

$$(1-q) \sum_{s=0}^{m-1} \sum_{k=0}^{\infty} q^k (q^k l_s \cos\theta_s - A) |r_{N-1}(b_s q^k)|^2 |W_s(b_s q^k)| = 0$$

and

$$(1-q) \sum_{s=0}^{m-1} \sum_{k=0}^{\infty} q^k (q^k l_s \sin\theta_s - B) |r_{N-1}(b_s q^k)|^2 |W_s(b_s q^k)| = 0.$$

It means that the functions

$$F(k) = \sum_{s=0}^{m-1} (q^k l_s \cos\theta_s - A) q^k |r_{N-1}(b_s q^k)|^2 |W_s(b_s q^k)|$$

and

$$G(k) = \sum_{s=0}^{m-1} (q^k l_s \sin\theta_s - B) q^k |r_{N-1}(b_s q^k)|^2 |W_s(b_s q^k)|$$

change their signs for some $k \in \mathbb{N}$. Hence we conclude that the zeros are in R.

Really, let us suppose that $A > \max_s \{l_s \cos\theta_s, 0\}$. Then, for every s and $k \in \mathbb{N}$, we will have $A > \max\{l_s \cos\theta_s, 0\} > q^k l_s \cos\theta_s$. But, because $A - q^k l_s \cos\theta_s > 0$, we will have also $F(k) < 0$ for all $k \in \mathbb{N}$, what is in contradiction with assumption.

Also, if we suppose that $A < \min_s \{l_s \cos\theta_s, 0\}$, then, for every s and $k \in \mathbb{N}$, we will have $A < \min\{l_s \cos\theta_s, 0\} < q^k l_s \cos\theta_s$. But, because $A - q^k l_s \cos\theta_s < 0$, we will have $F(k) > 0$ for all $k \in \mathbb{N}$, which again is in contradiction with our assumption. In this way we can prove all inequalities. □

Theorem 3.2. *All zeros of the polynomial $\pi_N(z)$ ($n \in \mathbb{N}$) orthogonal with respect to* (3) *lie in the smallest convex region which contains intervals.*

Proof: Let us consider the endpoints b_{k_0} and b_{k_1} of some intervals such that the line $b_{k_0} b_{k_1}$ does not have the intersection with any interval from the set $[0, b_j]$, $j = 0, 1, \ldots, m-1$. If we rotate the whole beam of intervals for the angle α_0 such that the line $b_{k_0}^{(\alpha_0)} b_{k_1}^{(\alpha_0)}$ is parallel to the real axes, then all zeros of the polynomial $\pi_N^{(\alpha_0)}(z)$ lie in rectangle $P_0^{(\alpha_0)}$ whose one edge contains $b_{k_0}^{(\alpha_0)} b_{k_1}^{(\alpha_0)}$. According to Corollary 2.1, all zeros of $\pi_N(z)$ lie in rectangle P_0 which is obtained from the rectangle $P_0^{(\alpha_0)}$ by rotation for the opposite angle of α.

In that way, taking any two endpoints of intervals of orthogonality including the origin 0, we can find rectangles P_j spread over the intervals which contains all zeros of $\pi_N(z)$. Their intersection is the smallest convex region which contains all zeros of $\pi_N(z)$. The vertex of the region are the tops of those rays which hold on its convexity. □

Corollary 3.1. *All zeros of the polynomial $\pi_N(z)$ orthogonal with respect to the inner product*

$$(f, g) = \int_0^{b_0} f(z)\overline{g(z)} |w_0(z)| \, \mathrm{d}_q z + \int_0^{b_1} f(z)\overline{g(z)} |w_1(z)| \, \mathrm{d}_q z$$

are in the triangle Ob_0b_1.

Corollary 3.2. *The monic polynomial $\pi_N(z)$ ($N > 0$) orthogonal with respect to the inner product over real intervals*

$$(f, g) = \sum_{s=0}^{\nu-1} \int_{-l_s}^{0} f(x)\overline{g(x)} |w(x)| \, \mathrm{d}_q x + \sum_{s=\nu}^{m-1} \int_0^{l_s} f(x)\overline{g(x)} |w(x)| \, \mathrm{d}_q x$$

has all zeros in the interval $(-h_1, h_2)$, *where*

$$h_1 = \max_{0 \le s \le \nu-1} l_s, \quad h_2 = \max_{\nu \le s \le m-1} l_s.$$

4. Some Symmetric Cases

In this section, we study the case of equal angles between successive intervals $[0, b_s]$, i.e.

$$b_s = l_s \varphi_m(s), \quad \text{where} \quad \varphi_m(s) = \exp\left(\mathrm{i}\frac{2\pi s}{m}\right), \quad s = 0, 1, \ldots, m-1. \tag{6}$$

Lemma 4.1. *The function* $s \mapsto \varphi_m(s)$ *has the following properties:*

1. $\varphi_m(ms) = 1$, $\overline{\varphi_m(s)} = \varphi_m(-s)$
2. $\varphi_m(s+r) = \varphi_m(s) + \varphi_m(r)$, $\varphi_m^N(s) = \varphi_m(Ns)$
3. $\sum\limits_{s=0}^{m-1} \varphi_m((mn+\nu)s) = \sum\limits_{s=0}^{m-1} \varphi_m(\nu s) = \begin{cases} m, & \nu = 0 \\ 0, & 1 \le \nu \le m-1. \end{cases}$

Theorem 4.1. *Suppose that* $m = pr$ $(p, r \in \mathbb{N})$ *and that the lengths and the weights of the radial rays are repeated periodically with period p, i.e.*

$$\begin{gathered} l_{s+jp} = l_s, \quad |w_{s+jp}(\varphi_m(jp)z)| = |w_s(z)| \quad z \in [0, b_s] \\ 0 \le s \le p-1, \quad j = 1, 2, \ldots, r-1. \end{gathered} \tag{7}$$

Then the polynomial $\pi_N(z)$ *orthogonal with respect to* (3) *has the property*

$$\pi_N(\varphi_m(p)z) = \varphi_m(Np)\pi_N(z).$$

Proof: Let $\pi_N(z)$ be the member of the set of the monic polynomials orthogonal with respect to the inner product (3) and $Q_N(z) = \pi_N(\varphi_m(p)z)$. By using known properties of $\varphi_m(s)$ from Lemma 4.1, for $M = 0, 1, \ldots, N$, we have

$$\begin{aligned} &\langle Q_N, z^M \rangle \\ &= (1-q) \sum_{s=0}^{m-1} \sum_{k=0}^{\infty} q^k Q_N(t) \overline{t^M} |w_s(t)| \Big|_{t=b_s q^k} \\ &= (1-q) \sum_{s=0}^{m-1} \sum_{k=0}^{\infty} q^k \pi_N(\varphi_m(p) b_s q^k) \overline{(b_s q^k)^M} |w_s(b_s q^k)| \\ &= (1-q) \sum_{s=0}^{m-1} \sum_{k=0}^{\infty} q^k \pi_N(q^k l_s \varphi_m(s+p)) l_s^M q^{kM} \varphi_m(-Ms) |w_s(b_s q^k)| \\ &= \varphi_m(Mp)(1-q) \end{aligned}$$

$$\sum_{s=0}^{m-1}\sum_{k=0}^{\infty} q^k\pi_N(q^k l_{s+p}\varphi_m(s+p))l_{s+p}^M q^{kM}\varphi_m(-M(s+p))|w_{s+p}(b_{s+p}q^k)|$$

$$= \varphi_m(Mp)(1-q)\sum_{j=p}^{p+m-1}\sum_{k=0}^{\infty} q^k\pi_N(q^k l_j\varphi_m(j))l_j^M q^{kM}\varphi_m(-Mj)|w_j(b_jq^k)|$$

$$= \varphi_m(Mp)(1-q)\sum_{j=0}^{m-1}\sum_{k=0}^{\infty} q^k\pi_N(q^k l_j\varphi_m(j))l_j^M q^{kM}\varphi_m(-Mj)|w_j(b_jq^k)|$$

$$= \varphi_m(Mp)(1-q)\sum_{j=0}^{m-1}\sum_{k=0}^{\infty} q^k\pi_N(q^k b_j)\overline{(b_jq^k)^M}|w_j(b_jq^k)|.$$

At last, we get

$$\langle Q_N, z^M\rangle = \varphi_m(Mp)\langle \pi_N\,,\, z^M\rangle, \quad M = 0,1,\ldots,N.$$

Since the polynomials $\pi_N(z)$ are orthogonal, we conclude that $\langle Q_N, z^M\rangle = 0$ for $M = 0,1,\ldots,N-1$. Because of the uniqueness of the sequence of the orthogonal polynomials, it must be valid

$$Q_N(z) = K\pi_N(z),$$

where K is the constant. On the other hand,

$$\langle Q_N, z^N\rangle = \langle K\pi_N, z^N\rangle = K\langle \pi_N, z^N\rangle,$$

wherefrom we get $K = \varphi_m(Np)$. □

Theorem 4.2. *Under the assumptions of Theorem 4.1, if ξ is a zero of $\pi_N(z)$, then the zeros are also $\xi\varphi_m(jp)$, $j = 1,\ldots,m/p$.*

Now, we will discuss the cases when the zeros stay on the rays.

Theorem 4.3. *Under the assumptions of Theorem 4.1, if $p = 1$ or $p = 2$, then the polynomial $\pi_N(z)$ has all zeros contained by the intervals of support.*

Proof: Let $\zeta_0 = \rho e^{i\alpha}$ be a zero of $\pi_N(z)$. According to a previous theorem, the zeros are also $\zeta_j = \zeta_0\varphi_m(jp)$, $j = 1,\ldots,r-1$ ($r = m/p$). Denoting by

$$s_k = \sum_{j=0}^{r-1}\zeta_j^k = \zeta_0^k\sum_{j=0}^{r-1}\varphi_m(pkj)$$

we get $s_k = 0$ ($k < r$) and $s_r = r\zeta_0^r$. By Newton's formulas, we have

$$\prod_{j=0}^{r-1}(z-\zeta_j) = z^r - a_1z^{r-1} - \cdots - a_{r-1}z - a_r$$

where $a_1 = s_1$, $ka_k = s_k - a_1 s_{k-1} - a_2 s_{k-2} - \cdots - a_{k-1}s_1$ $(k = 2, \ldots, r)$. Now, since $a_k = 0$ $(k < r)$ and $a_r = \zeta_0^r$, we have

$$\prod_{j=0}^{r-1}(z - \zeta_j) = z^r - \zeta_0^r.$$

Then, the polynomial $\pi_N(z)$ we can write in the form

$$\pi_N(z) = \tau_{N-r}(z)(z^r - \zeta_0^r), \quad \tau_{N-r}(z) \in \mathcal{P}_{N-r}.$$

Because of orthogonality, we have further

$$0 = \langle \pi_N, \tau_{N-r} \rangle = \sum_{s=0}^{m-1} \frac{1}{b_s} \int_0^{b_s} (z^r - \zeta_0^r)\tau_{N-r}(z)\overline{\tau_{N-r}(z)}|w_s(z)|\, \mathrm{d}_q z,$$

i.e.

$$0 = (1-q) \sum_{s=0}^{m-1} \sum_{k=0}^{\infty} q^k \left(z^r - \zeta_0^r\right) |\tau_{N-r}(z)|^2 |w_s(z)| \Big|_{z=q^k b_s}.$$

Separating the real and imaginary part of the previous integral, we get

$$(1-q) \sum_{s=0}^{m-1} \sum_{k=0}^{\infty} q^k \Big(q^{kr} l_s^r \cos(2\pi rs/m) - \rho^r \cos(r\alpha)\Big) |\tau_{N-r}(q^k b_s)|^2 |w_s(q^k b_s)| = 0$$

and

$$(1-q) \sum_{s=0}^{m-1} \sum_{k=0}^{\infty} q^k \Big(q^{kr} l_s^r \sin(2\pi rs/m) - \rho^r \sin(r\alpha)\Big) |\tau_{N-r}(q^k b_s)|^2 |w_s(q^k b_s)| = 0.$$

We will discuss the next two cases:

1) If $r = m$, then we have

$$\sum_{s=0}^{m-1} \sum_{k=0}^{\infty} q^k \left(q^{km} l_s^m - \rho^m \cos(m\alpha)\right) |\tau_{N-r}(q^k b_s)|^2 |w_s(q^k b_s)| = 0$$

and

$$\rho^m \sin(m\alpha) \sum_{s=0}^{m-1} \sum_{k=0}^{\infty} q^k |\tau_{N-r}(q^k b_s)|^2 |w_s(q^k b_s)| = 0.$$

From the last identity, we conclude that $\sin(m\alpha) = 0$. Therefore, there exists an nonnegative integer j such that $\alpha = j\pi/m$. But, for j odd, the first relation yields the form

$$\sum_{s=0}^{m-1} \sum_{k=0}^{\infty} q^k \left(q^{km} l_s^m + \rho^m\right) |\tau_{N-r}(q^k b_s)|^2 |w_s(q^k b_s)| = 0$$

that is impossible because of the positivity of the sum. So, it must be $\alpha = 2\pi j/m$, $j \in \mathbb{N}_0$. Now, including statement of Theorem 4.1, the zero $\zeta_0 = \rho e^{i\alpha}$ lies on some interval of support.

2) In the case $r = m/2$, we find $\sin(\alpha m/2) = 0$, wherefrom it exists $j \in \mathbb{N}_0$ such that $\alpha = 2\pi j/m$. It guarantees that the zero ζ is on the support. □

5. The Case of Equal Lengths, Angles and Weights

Let us suppose

$$l_s = l, \quad \varphi_m(s) = \exp\left(\frac{2\pi s\mathrm{i}}{m}\right), \quad |w_s(x\varphi_m(s))| = w(x)$$

for $x \in (0, l)$ and $0 \leq s \leq m-1$. Then the inner product (1) yields the form

$$(f, g) = \int_0^l \left(\sum_{s=0}^{m-1} f(x\varphi_m(s))\overline{g(x\varphi_m(s))} \right) w(x)\, \mathrm{d}_q x. \tag{8}$$

About the zeros of $\pi_N(z)$ we can prove:

Theorem 5.1. *All zeros of the polynomial $\pi_N(z)$, orthogonal with respect to* (6), *are simple and located on the support, with possible exception of a multiple zero at the origin $z = 0$ of order ν, if $N \equiv \nu \pmod{m}$.*

Proof: In the Theorem 4.2 we proved that the zeros of $\pi_N(z)$ are on the support. In the paper [4] it was proved that the polynomial $\pi_N(z)$ can be expressed in the form $\pi_N(z) = z^\nu q_n^{(\nu)}(z^m)$, $\nu \in \{0, 1, \ldots, m-1\}$, where $q_n^{(\nu)}(t)$ is orthogonal on $(0, l)$ with respect to a positive weight. It is well known that the zeros of $q_n^{(\nu)}(t)$ are real and distinct and are located in $(0, l)$. Let $\tau_k^{(n,\nu)}$, $k = 1, \ldots, n$ denote the zeros of $q_n^{(\nu)}(t)$ in increasing order

$$\tau_1^{(n,\nu)} < \tau_2^{(n,\nu)} < \cdots < \tau_n^{(n,\nu)}.$$

Each zero $\tau_k^{(n,\nu)}$ generates m zeros

$$z_{k,s}^{(n,\nu)} = \sqrt[m]{\tau_k^{(n,\nu)}}\, \mathrm{e}^{\mathrm{i}\frac{2\pi s}{m}}, \quad s = 0, \ldots, m-1$$

of $\pi_N(z)$.
On every interval

$$|z_{1,s}^{(n,\nu)}| < |z_{2,s}^{(n,\nu)}| < \cdots < |z_{n,s}^{(n,\nu)}|, \quad s = 0, \ldots, m-1.$$

Also, $\pi_{N+1}(z)$ and $\pi_N(z)$ separate their zeros in the intervals. □

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