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EXACTLY SOLVABLE PERIODIC DARBOUX q-CHAINS

SERGEY SMIRNOV

Department of Higher Geometry and Topology, Moscow State University
Vorobévy Gory, 119992, Moscow, Russia

Abstract. A difference q-analogue of the dressing chain is considered in this paper. The relation between the discrete and continuous models is also discussed.

1. Introduction

Let $L_1, L_2, \ldots$ be self-adjoint differential operators acting on $\mathbb{R}$. They form a **Darboux chain** if they satisfy the relation

$$L_j = A_j A_j^+ - \alpha_j = A_{j-1}^+ A_{j-1} \tag{1}$$

where $A_j = -\frac{\mathrm{d}}{\mathrm{d}x} + f_j(x)$ are first order differential operators. A Darboux chain is called **periodic** if $L_{j+r} = L_j$ for some r and for all $j = 1, 2, \ldots$. The number r is called period of a Darboux chain. In the particular case $r = 1$ the operator $L_1 + \frac{\alpha}{2}$ appears to be the harmonic oscillator and it is known that it has a discrete spectrum consisting of the geometric sequence $\lambda_k = \dfrac{2k+1}{2}\alpha$, where $k = 1, 2 \ldots$. Eigenfunctions of the harmonic oscillator are expressed in terms of the Hermite polynomials and therefore they form a complete family in the Hilbert space $\mathcal{L}_2(\mathbb{R})$.

Periodic Darboux chain leads to the following integrable system of differential equations:

$$(f_j + f_{j-1})' = f_j^2 - f_{j-1}^2 - \alpha_j\,, \qquad j = 1, 2, \ldots \tag{2}$$

where $f_{j+r} \equiv f_j$. Sometimes this system is referred to as **dressing chain** which has been thoroughly examined in [8]. The cases $\alpha = 0$ and $\alpha \neq 0$, where $\alpha = \sum_{j=1}^{r} \alpha_j$, are cardinally different. The operators of a periodic

Darboux chain are finite-gap if $\alpha = 0$. If $\alpha \neq 0$, then the equations for f_j lead to the Painlevé equations or their higher analogues and exactly as in the case of the harmonic oscillator, relations (1) also define the discrete spectrum of the chain operators L_j. The spectrum of each of these operators consists of r arithmetic sequences and can be found using the Darboux scheme:

$$\lambda_{j,0} = 0, \qquad \lambda_{j+1,k+1} = \lambda_{j,k} + \alpha_j, \qquad \lambda_{j+r,k} = \lambda_{j,k}$$

(see [8] for the details). Hence, for any r the chain operators are unbounded. Eigenfunctions of the chain operators can also be obtained using the Darboux scheme:

$$\psi_{j+1,k+1} = A_j^+ \psi_{j,k}, \qquad \psi_{j+r,k} = \psi_{j,k} \tag{3}$$

where the ground states $\psi_{j,0}$ are defined by $A_{j-1}\psi_{j,0} = 0$.

The main difference between the cases $r = 1, 2$ and $r \geq 3$ is that in the case $r = 1, 2$ the dressing chain can be integrated explicitly and there are still many unanswered questions concerning the dressing chain for $r \geq 3$:

1. Do the chain operators L_j exist for arbitrary $\alpha_1, \ldots, \alpha_r$? If they do, are they non-singular?
2. Do the Darboux eigenfunctions (3) belong to the Hilbert space $\mathcal{L}_2(\mathbb{R})$? If they do, do they form a complete family in $\mathcal{L}_2(\mathbb{R})$?

At the time being it is only known that the answer to all these questions is "yes, they do", if r is odd and $(\alpha_1, \ldots, \alpha_r)$ is a small perturbation of $\alpha_1 = \cdots = \alpha_r$.

2. Periodic q-chains with Shift

Consider the following q-analogue of a Darboux chain:

$$L_j = A_j A_j^+ - \alpha_j = q A_{j-1}^+ A_{j-1} \tag{4}$$

where $A_j = a_j + b_j T$ are difference operators on one-dimensional lattice $\mathbb{Z}$ and T is the **shift operator**: $(A_j f)(n) = a_j(n) f(n) + b_j(n) f(n+1)$, $a_j(n), b_j(n) \in \mathbb{R}\backslash\{0\}$. Without loss of generality we may assume that $b_j(n) > 0$ for all j, n. One of the important features of difference operators in comparison to differential ones is the possibility to define a periodic chain in various ways. According to [2], we will say that a chain (4) is periodic with period r and shift s if the relation

$$L_{j+r} = T^{-s} L_j T^s \tag{5}$$

holds for all $j \geq 1$. Only the case $s = 0$ has been studied thoroughly. However, the possibility of factorization in "the reverse order" was mentioned in [3, 4], i. e. operators $A_j = a_j + b_j T$ can be replaced by operators of the form

$A_j = a_j T^+ + b_j$. Such factorization with $s = 0$ is, in fact, equivalent to the factorization “in the right order” for $s = r$ (the operator L_j is replaced by $T^{-j} L_j T^j$). Apparently, the general formulation with an arbitrary s has been considered only recently in [2, 6], while the following special cases had been studied in the literature before.

1. $\alpha = 0$, $q = 1$, r is arbitrary [4]. Operators L_j are finite-gap.
2. $\alpha > 0$, $q = 1$, $r = 1$ (difference analogue of the harmonic oscillator) [4]. In this case, the operator relation (4) is equivalent to the following system of difference equations:

$$a(n) = \text{const}$$
$$b^2(n) - b^2(n-1) = \alpha\,.$$

   Thus, symmetric operator L acting on the space of functions on the lattice $\mathbb{Z}$ does not exist. Nevertheless, there exists a solution on the “half-line” $\mathbb{Z}_{>0}$. The spectrum of the operator $L + \frac{\alpha}{2}$ is exactly the same as that of the harmonic oscillator $\lambda_k = \alpha(k + \frac{1}{2})$, the eigenfunctions are expressed in terms of the Charlier polynomials and therefore, form a complete family in $\mathcal{L}_2(\mathbb{Z}_{>0})$.
3. $r = 1$, $\alpha > 0$, $0 < q < 1$ (or $\alpha < 0$, $1 < q$) [4, 7] (q-oscillator). In this case, the operator relation (4) is equivalent to the following system of difference equations:

$$a^2(n) + b^2(n) = q(a^2(n) + b^2(n-1)) + \alpha$$
$$a(n) = qa(n-1)\,.$$

   The discrete spectrum of the operator L lies in the interval $[0, \frac{q\alpha}{1-q})$ (or in $(0, \frac{q\alpha}{q-1})$ if $q > 1$) and it forms a “q-arithmetic sequence”. Note that, in this case, the operator L is unbounded (since its coefficients $a(n)$ tend to ∞), though its discrete spectrum is bounded. In [4] it is conjectured that L has continuous spectrum in the interval $\left(\frac{q\alpha}{1-q}, \infty\right)$.
4. Another version of the q-oscillator is considered in [1]. It is presented by a difference operator on the whole “line” $\mathbb{Z}$. In our settings, this version of the q-oscillator can be interpreted as the case $s = 1$, $r = 2$, $\alpha_1 = \alpha_2$, (α and q are the same as in 3). A particular solution that is symmetric about the origin was found in [1]. This case is specific because here the operator L is bounded and has no continuous spectrum.

We claim here that the same property holds for a q-chain of an arbitrary even period r with the shift $s = r/2$ (the idea to consider q-chains with the shift twice smaller than the period belongs to I. Dynnikov). The chain operators are bounded and have no continuous spectrum. We also provide explicitly the general solution of the problem in the case $s = 1$, $r = 2$.

3. Main Results

Theorem 1. *Suppose r is even, $\alpha_1, \ldots, \alpha_r$ are positive, q satisfies the inequality $0 < q < 1$, and we have $s = r/2$. Then the system (4), (5) has an r-parametric family of solutions. The operator L_j is bounded for each j and its spectrum $\{\lambda_{j,0}, \lambda_{j,1}, \ldots\}$ is discrete and is contained in the interval $[0, \|L_j\|)$. It can be found by using the Darboux scheme:*

$$\lambda_{j,0} = 0, \qquad \lambda_{j+1,k+1} = q(\lambda_{j,k} + \alpha_j), \qquad \lambda_{j+r,k} = \lambda_{j,k}\,.$$

For each j, the eigenfunctions of the operator L_j can also be obtained by using the Darboux scheme:

$$A_{j-1}\psi_{j,0} = 0, \qquad \psi_{j+1,k+1} = A_j^+ \psi_{j,k}$$

and these eigenfunctions form a complete family in $\mathcal{L}_2(\mathbb{Z})$.
A similar assertion holds for $\alpha_1, \ldots, \alpha_r < 0$, $1 < q$ (in this case, the point 0 is not included in the spectrum of L_j).

Now we present an explicit form of the operators A_1, A_2 for $r = 2$.

Proposition 1. *For $r = 2 = 2s$, $\alpha_1, \alpha_2 > 0$, $0 < q < 1$, the general solution of the problem (4), (5) has the form $a_1(n) = \epsilon\sqrt{\xi_{2n}}$, $b_1(n) = \sqrt{\eta_{2n+1}}$, $a_2(n) = \epsilon\sqrt{\xi_{2n-1}}$, $b_2(n) = \sqrt{\eta_{2n}}$, where $\epsilon = \pm 1$,*

$$\begin{aligned}
\xi_n &= \frac{1}{2}\,\frac{c_n - 2\kappa q^{-n-\varphi-\frac{1}{2}} + c_{n+1} q^{-2n-2\varphi-1}}{(1 - q^{-2(n+\varphi)})(1 - q^{-2(n+\varphi+1)})} \\
\eta_n &= \frac{1}{2}\,\frac{c_{n+1} - 2\kappa\, q^{n+\varphi+\frac{1}{2}} + c_n q^{2n+2\varphi+1}}{(1 - q^{2(n+\varphi)})(1 - q^{2(n+\varphi+1)})} \\
c_n &= \frac{\alpha_1 + \alpha_2}{1 - q} + (-1)^n \frac{\alpha_1 - \alpha_2}{1 + q}
\end{aligned} \tag{6}$$

$\varphi \in \mathbb{R}$ is arbitrary, the parameter κ satisfies the restrictions $c_{[\varphi]} q^{-\theta} + c_{[\varphi]-1} q^{\theta} < 2\kappa < \min(c_{[\varphi]} q^{\theta+1} + c_{[\varphi]-1} q^{-\theta-1}, c_{[\varphi]} q^{\theta-1} + c_{[\varphi]-1} q^{-\theta+1})$ if $\varphi \notin \mathbb{Z}$ and $2\kappa = c_\varphi q^{\frac{1}{2}} + c_{\varphi-1} q^{-\frac{1}{2}}$ if $\varphi \in \mathbb{Z}$. In this case, in fractions (6), one has to cancel the factor $(1 - q^{2(n+\varphi)})$ for $n \equiv \varphi \mod 2$ and the factor $(1 - q^{2(n+\varphi+1)})$ for $n \equiv (\varphi + 1) \mod 2$. Here, we set $\theta = \varphi - [\varphi] - \frac{1}{2}$ and $[\varphi]$ stands for the integral part of φ.

See [6] for the detailed proofs. The explicit formulae (6) were obtained by I. Dynnikov.

4. Discussion

4.1. Observation

In the case $r = 2$, $\alpha_1 = \alpha_2$, operators $L_j + \frac{\alpha_j}{2}$ constructed from the above solutions converge to the harmonic oscillator as $q \to 1$ in the following sense: take $\varphi = 0$, $\epsilon = -1$, $q = \exp(-\frac{\alpha_1}{4}h^2)$, $x = nh$, $T = \exp(h\frac{\mathrm{d}}{\mathrm{d}x})$ and assume that n is real in formulae (6) and that the operator L_j is a difference operator on $\mathbb{R}$. Then for any $f \in C^2(\mathbb{R})$ we have

$$\Big(L_{1,2} + \frac{\alpha_1}{2}\Big)f(x) = \Big(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{\alpha_1^2}{4}x^2\Big)f(x) + o(h)\,.$$

If $\alpha_1 \neq \alpha_2$, then the operator L_j converges in the same sense to

$$-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{(\alpha_j + \alpha_{j+1})^2}{16}x^2 - \frac{\alpha_j}{2} - \frac{(\alpha_j - \alpha_{j+1})(\alpha_j + 3\alpha_{j+1})}{4(\alpha_j + \alpha_{j+1})^2 x^2}$$

where $\alpha_{j+2} = \alpha_j$.

In the cases 2 and 3 mentioned above there is no link of that kind between the discrete and the continuous models. Thus, the considered case $s = r/2$ gives, in a certain sense, a proper discretization of a Darboux chain.

We hope to prove that a q-chain converges in the same way to an ordinary Darboux chain for an arbitrary even r. The numerical experiment confirms that a q-chain of the period 6 converges to an ordinary Darboux chain of the period 3 if $\alpha_j = \alpha_{j+3}$.

Another interesting question is integrability of the discrete q-chain: since corresponding continuous system is integrable, it would have been rather natural to think about integrability (in a certain sense) of the discrete model. The periodic q-chain (4), (5) is equivalent to the following system of difference equations:

$$\begin{aligned} \xi_j(n-1) + \eta_j(n) &= q(\xi_{j-1}(n) + \eta_{j-1}(n-1)) + \alpha_j \\ \xi_j(n)\eta_j(n-1) &= q^2\xi_{j-1}(n-1)\eta_{j-1}(n) \end{aligned} \tag{7}$$

where $j = 1, \ldots, r$ is a cyclic index, $\xi_j(n)$ are defined only if $j + n$ is odd and $\eta_j(n)$ are defined only if $j + n$ is even:

$$\begin{aligned} \xi_{2i+1}(2k) &= a_{2i+1}^2(k+i) \qquad & \eta_{2i+1}(2k+1) &= b_{2i+1}^2(k+i) \\ \xi_{2i}(2k-1) &= a_{2i}^2(k+i-1) \qquad & \eta_{2i}(2k) &= b_{2i}^2(k+i-1) \end{aligned} \tag{8}$$

Thus, the core of the problem is to find sufficient number of independent integrals of the system (7). So, one has to seek for some values which do not change as n changes.

According to the above theorem the problem (4), (5) has an r-parametric family of solutions and therefore the system (7) also has an r-parametric family of solutions (here $\xi_j(n)$, $\eta_j(n)$ are non-negative for all possible j, n). It follows from the proof of the theorem (see [6]) that for each of these solutions the coordinates have the following asymptotic:

$$\begin{array}{ll} \xi_{2i}(2k-1) \to 0, & \xi_{2i+1}(2k) \to 0 \\ \eta_{2i}(2k) \to c_{2i}, & \eta_{2i+1}(2k+1) \to c_{2i+1} \end{array} \quad \text{as } n \to +\infty \tag{9}$$

$$\begin{array}{ll} \eta_{2i+1}(2k+1) \to 0, & \eta_{2i}(2k) \to 0 \\ \xi_{2i+1}(2k) \to c_{2i+1}, & \xi_{2i}(2k+1) \to c_{2i} \end{array} \quad \text{as } n \to -\infty \tag{10}$$

for all i.

Our idea is to extract dynamic values which do not change as n changes out of this limiting procedure, that is, to examine the rate of convergence of dynamic variables ξ_j, η_j. Actually we have

Lemma 1. *For each $j = 1, \dots, r$*

$$\frac{\xi_j(j+1+2n)}{q^{4n}} = O(1), \quad \frac{c_j - \eta_j(j+2n)}{q^{2n}} = O(1) \qquad \textit{as } n \to +\infty$$

$$\frac{\eta_j(j+2n)}{q^{4n}} = O(1), \quad \frac{c_j - \xi_j(j+1+2n)}{q^{2n}} = O(1) \qquad \textit{as } n \to -\infty\,.$$

We will say that a sequence $z(n)$ *tends to an s-cycle $(\rho_1, \dots, \rho_s)$ as $n \to +\infty$* if the subsequence $z(j+sn)$ tends to ρ_j as $n \to +\infty$ for each $j = 1, \dots, s$. The numerical experiment shows that, for example, the sequence $(c_1 - \eta_1(1+2n))/q^{2n}$ tends to a certain s-cycle $(\rho_1, \dots, \rho_s)$, where $s = r/2$. The other sequences $(c_j - \eta_j(j+2n))/q^{2n}$ also tend to some s-cycles; moreover, the experiment shows that the sequences $(c_j - \xi_j(j+1+2n))/q^{2n}$ tend to some s-cycles similarly. These results encourage us to make the following conjecture.

Conjecture 1. *Suppose r is even, $\alpha_1, \dots, \alpha_r$ are positive, q satisfies the inequality $0 < q < 1$ and we have $s = r/2$. Then r different sequences*

$$\frac{c_1 - \eta_1(1+2n)}{q^{2n}}, \ \frac{c_2 - \eta_2(2+2n)}{q^{2n}}, \ \dots, \ \frac{c_r - \eta_r(r+2n)}{q^{2n}}$$

tend to the same s-cycle $(\rho_1, \rho_2, \dots, \rho_s)$ as $n \to +\infty$. Similarly, r different sequences

$$\frac{c_1 - \xi_1(2n)}{q^{2n}}, \ \frac{c_2 - \xi_2(1+2n)}{q^{2n}}, \ \dots, \ \frac{c_r - \xi_r(r-1+2n)}{q^{2n}}$$

also tend to the same s-cycle $(\nu_1, \nu_2, \ldots, \nu_s)$. *Moreover, the parameters* $\rho_1, \ldots, \rho_s$, $\nu_1, \ldots, \nu_s$ *uniquely define the solution of the system* (7).

It is not yet clear how to obtain explicit formulae for the parameters $\rho_1, \ldots, \rho_s$, $\nu_1, \ldots, \nu_s$ in the general case. Nevertheless, the formulae (6) allow us to do this in the most simple case $r = 2s = 2$. Besides this, the parameters ρ and ν are independent in this case in the sense that they define the solution of the system (7) uniquely.

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