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CONFORMAL SCHWARZIAN DERIVATIVES AND DIFFERENTIAL EQUATIONS

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Abstract. We investigate the fundamental system of equations in the theory of conformal geometry, whose coefficients are considered as the conformal Schwarzian derivative. The integrability condition of the system is obtained in a simple method, which allow us to find a natural geometric structure on the solution space. From the solution spaces, using this geometric structure, we get a transformation whose Schwarzian derivative is equal to the given coefficients of the equation.

1. Introduction

Some years ago Sasaki and Yoshida [10] gave the fundamental system of linear equations, which is the key system connecting the theory of conformal connections and the uniformizing differential equations in the geometry of symmetric domains of type IV. It is a system of equations with n variables such that the maximal dimension of the solution space is $n+2$. The solutions naturally provide a map into the projective space whose image is contained in the hyperquadric, and accepts the conformal transformation group as its symmetry.

Sasaki and Yoshida considered the equations as a higher dimensional analogue of Gauss–Schwarz equation. In projective geometry of higher dimension, they defined Schwarzian derivatives as a difference of normal Cartan connections moved by a diffeomorphism. Using the Schwarzian derivatives, they got the system of linear equations such that the maximal dimension of the solution space is $n+1$ on n variables [9, 12]. In conformal geometry of higher dimension, the problem is much harder. As the Schwarzian derivatives we need more data than the difference of normal Cartan connections. They add some data derived from the difference of conformal metrics (see also [1]) and finally got the equation in [10]. From various viewpoints, several authors studied other generalized notions of Schwarzian derivative in conformal geometry (see e. g. [2, 7, 8]).

In a certain dimension, the equation gives a concrete description of Aomoto–Gel'fand hyper-geometric equation of type (3, 6) and is used by Matsumoto–Sasaki–Takayama–Yoshida to study the monodromy of the period map of some family of K3 surfaces [3–6].

But the introduction of the key equation in [10] is by a very roundabout way. They use the hypersurface theory of projective geometry and the theory of normal Cartan connection. In this paper, we introduce a simpler equation by a direct method showing that this is the only equation that we need. We use a simple calculation in conformal geometry and we get both a key equation and its integrability condition. By this simple method, we hope to find other key equation in other geometric structure that may extend the result of Seashi [11] to nilpotent geometries.

The following system of linear partial differential equations is fundamental for us:

$$\varphi_{ij} - \frac{1}{n} h_{ij} h^{ab} \varphi_{ab} = (\Gamma^c_{ij} - \frac{1}{n} h_{ij} h^{ab} \Gamma^c_{ab}) \varphi_c + (r_{ij} - \frac{1}{n} h_{ij} h^{ab} r_{ab}) \varphi \tag{1}$$

for each $i, j = 1, \ldots, n$, and we call (1) the **fundamental equation**. The summation over a, b and c run from 1 to n. Here φ is the unknown function on $\mathbb{R}^n$ (or $\mathbb{C}^n$), and φ_{ij} and φ_k are its partial derivatives, while $h_{ij} = h_{ji}$, $\Gamma^k_{ij} = \Gamma^k_{ji}$, $r_{ij} = r_{ji}$ are given functions on $\mathbb{R}^n$ (or $\mathbb{C}^n$), such that the matrix $\{h_{ij}\}$ is nondegenerate, and $\{h^{ab}\}$ is its inverse. We regard $h = \{h_{ij}\}$ as a pseudo-Riemannian metric, and note that the equation (1) is invariant under conformal changes of h, provided Γ^k_{ij} and r_{ij} are respectively the Christoffel symbol of h and $-1/(n-2)$ times the Ricci curvature.

Let ∇ denote the covariant derivative whose coefficients are $\{\Gamma^i_{jk}\}$. We assume that ∇ is the Weyl conformal connection of the metric h, that is, there exists a function f such that $\nabla h = \mathrm{d}f \otimes h$ holds.

We find the integrability condition of the equation (1) as follows: let R_{ij} and R be the Ricci and scalar curvature tensor of Γ^i_{jk}

$$r_{ij} = \left(\frac{R}{n(n-2)} + \frac{\mathrm{trace}_h\, r}{n} \right) h_{ij} - \frac{1}{n-2} R_{ij}$$

is one of the conditions, which is equivalent to $r_{ij} = -R_{ij}/(n-2)$ in the equation, and the other condition is the vanishing of the conformal Weyl curvature tensor of $\{h_{ij}\}$ (the Cotton tensor when the dimension of the manifold is 3).

Let us consider $\mathbb{R}^n$ (or $\mathbb{C}^n$) endowed with the linear pseudo-Riemannian metric h_0 of signature (p, q) $(p + q = n)$. For a local diffeomorphism f on this flat space, we denote by h_{ij} the pull back f^*h_0, and $\left\{ {k \atop ij} \right\}$ and R_{ij} the Christoffel symbol and Ricci curvature tensor of h_{ij}, respectively. In view of (1), we define the Schwarzian derivative of f as the following data:

$$\left\{ h_{ij}, \left\{ {k \atop ij} \right\} - \frac{1}{n} h_{ij} h^{ab} \left\{ {k \atop ab} \right\}, \quad \frac{1}{n-2} \left(R_{ij} - \frac{1}{n} h_{ij} h^{ab} R_{ab} \right) \right\}.$$

As we prove in Lemma 3, if the equation is normalized as indicated in Section 4, the solution space has a natural inner product. If the equation (1) is integrable, then the dimension of the solution space is equal to $n + 2$, and an orthonormal basis of the solution space is used to construct a map into the projective space whose image is contained in a hyperquadric, which is conformally equivalent to the standard flat conformal structure of $\mathbb{R}^n$ (or $\mathbb{C}^n$). Therefore we get a local diffeomorphism on this flat space.

Our main theorem concerns a close relation between the equation (1) and the resulting diffeomorphism via the Schwarzian derivative. We briefly state it as follows (for the precise statement see Theorem 1): *if h_{ij} is conformally flat, then the normalized fundamental equation is integrable and the resulting diffeomorphism has the Schwarzian derivatives equal to the coefficients of the equation. Conversely, given a diffeomorphism on a conformally flat space, the fundamental equation with coefficients equal to the Schwarzian derivatives of the diffeomorphism is integrable, and the resulting diffeomorphism is equal to the given diffeomorphism up to a translation in $O(p + 1, q + 1)$.*

2. Fundamental Equation

The Einstein convention of summation is assumed and all indices $a, b, \ldots$ will run from 1 to n.

Using the coefficient of the fundamental equation (1), we define an operator D by

$$D\varphi = \frac{1}{n} h^{ab} \varphi_{ab} - \frac{1}{n} h^{ab} \Gamma^c_{ab} \varphi_c - \frac{1}{n} h^{ab} r_{ab} \varphi = -\frac{1}{n}(\Delta\varphi + R) \tag{2}$$

where Δ is the Laplacian, and R is the scalar curvature. Then the equation (1) can be written as

$$\frac{\partial^2}{\partial x^i \partial x^j}\varphi = h_{ij} D\varphi + \sum_k \Gamma^k_{ij} \frac{\partial}{\partial x^k}\varphi + r_{ij}\varphi\,. \tag{3}$$

This form of the equation is suitable for the theory of hypersurfaces in the affine differential geometry, where $D\varphi$ serves as the Blaschke normal vector field.

Lemma 1. *Let S_i be the matrix defined bellow*

$$S_i = \begin{pmatrix} 0 & \delta^1_i & \cdots & \delta^n_i & 0 \\ r_{1i} & \Gamma^1_{1i} & \cdots & \Gamma^n_{1i} & h_{1i} \\ \vdots & \vdots & & \vdots & \vdots \\ r_{ni} & \Gamma^1_{ni} & \cdots & \Gamma^n_{ni} & h_{ni} \\ \beta_i & \gamma^1_i & \cdots & \gamma^n_i & \alpha_i \end{pmatrix}. \tag{4}$$

Then the equation (3) is equivalent to

$$\frac{\partial}{\partial x^i}{}^t(\varphi, \varphi_1, \cdots \varphi_n, D\varphi) = S_i \cdot {}^t(\varphi, \varphi_1, \cdots \varphi_n, D\varphi) \tag{5}$$

if and only if

$$\begin{aligned} \alpha_i &= \frac{1}{n-1} h^{ab} R^{\infty}_{abi}, \quad \beta_i = \frac{1}{n-1} h^{ab} R^0_{abi} \\ \gamma^j_i &= \frac{1}{n-1}(h^{ab} R^j_{abi} + h^{ja} r_{ai} - h^{ab} r_{ab} \delta^j_i) \end{aligned} \tag{6}$$

where

$$\begin{aligned} R^{\infty}_{ijk} &= h_{ik,j} - h_{ij,k} + \Gamma^a_{ik} h_{aj} - \Gamma^a_{ij} h_{ak} \\ R^l_{ijk} &= \Gamma^l_{ik,j} - \Gamma^l_{ij,k} + \Gamma^a_{ik}\Gamma^l_{aj} - \Gamma^a_{ij}\Gamma^l_{ak} \\ R^0_{ijk} &= r_{ik,j} - r_{ij,k} + \Gamma^a_{ik} r_{aj} - \Gamma^a_{ij} r_{ak}\,. \end{aligned}$$

Proof: We are required to explain the derivatives $(D\varphi)_i$ of $D\varphi$ with respect to x^i in terms of $\{h_{ij}, \Gamma^k_{ij}, r_{ij}\}$. Differentiating (2) we get

$$\begin{aligned} n(D\varphi)_i &= h^{ab}_{,i}\left(\varphi_{ab} - \Gamma^c_{ab}\varphi_c - r_{ab}\varphi\right) + h^{ab}\left(\varphi_{ab} - \Gamma^c_{ab}\varphi_c - r_{ab}\varphi\right)_i \\ &= h^{ab}_{,i}\left(h_{ab} D\varphi + \Gamma^c_{ab}\varphi_c + r_{ab}\varphi - \Gamma^c_{ab}\varphi_c - r_{ab}\varphi\right) \\ &\quad + h^{ab}\left(\varphi_{aib} - \Gamma^c_{ab,i}\varphi_c - r_{ab,i}\varphi - \Gamma^c_{ab}\varphi_{ci} - r_{ab}\varphi_i\right). \end{aligned}$$

We substitute $\varphi_{aib} = \Big(h_{ai}D\varphi + \Gamma^d_{ai}\varphi_d + r_{ai}\Big)_b$ to get

$$
\begin{aligned}
&(n-1)\Big(D\varphi\Big)_i \\
&= \Big(h^{ab}_{,i}h_{ab} + h^{ab}h_{ai,b} + h^{ab}\Gamma^c_{ai}h_{cb} - h^{ab}\Gamma^c_{ab}h_{ci}\Big)D\varphi \\
&\quad + \Big(h^{ab}\Gamma^d_{ai,b} + h^{ab}\Gamma^c_{ai}\Gamma^d_{cb} + h^{ad}r_{ai} - h^{ab}\Gamma^d_{ab,i} - h^{ab}\Gamma^c_{ab}\Gamma^d_{ci} - h^{ab}r_{ab}\delta^d_i\Big)\varphi_d \\
&\quad + \Big(h^{ab}\Gamma^c_{ai}r_{cb} + h^{ab}r_{ai,b} - h^{ab}\Gamma^c_{ab}r_{ci} - h^{ab}r_{ab,i}\Big)\varphi\,.
\end{aligned}
$$

The last equation is equivalent to $(D\varphi)_i = \alpha_i D\varphi + \gamma^a_i\varphi_a + \beta_i\varphi$. □

The Ricci curvature tensor of $\{\Gamma^k_{ij}\}$ is by definition,

$$R_{ij} = R^a_{iaj}$$

which is also equal to

$$h^{ab}R_{aibj} = h^{ab}R_{iajb} = h^{ab}R^c_{ajb}h_{ci} = -h^{ab}R^c_{abj}h_{ci}\,. \tag{7}$$

Finally γ^j_i is equal to

$$\gamma^j_i = \frac{1}{n-1}\left(-h^{ja}R_{ai} + h^{ja}r_{ai} - \mathrm{trace}_h\, r\delta^j_i\right) \tag{8}$$

where the $\mathrm{trace}_h\, r$ denotes the sum $h^{ab}r_{ab}$.

3. Maurer–Cartan Relation of the Fundamental Equation

In this section, we derive a necessary and sufficient condition for the fundamental equation (1) to be integrable, that is, to have an $n+2$ dimensional solution space.

Let S be the matrix valued 1-form defined by

$$S = S_a\,\mathrm{d}x^a$$

where the matrices S_a are given in (4). From Lemma 1, it follows that the fundamental equation (1) is integrable, if and only if S satisfies the Maurer–Cartan relation

$$\mathrm{d}S = S \wedge S\,.$$

The exterior derivative of S is equal to

$$\mathrm{d}S = \mathrm{d}(S_a dx^a) = \mathrm{d}S_a \wedge \mathrm{d}x^a = S_{a,b}\,\mathrm{d}x^b \wedge \mathrm{d}x^a$$

and the wedge product is equal to

$$S \wedge S = (S_a \, \mathrm{d}x^a) \wedge (S_b \, \mathrm{d}x^b) = S_a S_b \, \mathrm{d}x^a \wedge \mathrm{d}x^b \, .$$

Therefore the Maurer–Cartan relation can be read as

$$S_{i,j} - S_{j,i} + S_i S_j - S_j S_i = 0 \, .$$

We can derive the same equations in terms of $\{h_{ij}, \Gamma^k_{ij}, r_{ij}\}$ that is equivalent to the Maurer–Cartan relation. First of all the derivatives of S are given by

$$S_{i,j} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ r_{1i,j} & \Gamma^1_{1i,j} & \cdots & \Gamma^n_{1i,j} & h_{1i,j} \\ \vdots & \vdots & & \vdots & \vdots \\ r_{ni,j} & \Gamma^1_{ni,j} & \cdots & \Gamma^n_{ni,j} & h_{ni,j} \\ \beta_{i,j} & \gamma^1_{i,j} & \cdots & \gamma^n_{i,j} & \alpha_{i,j} \end{pmatrix} .$$

If we denote by S^l_{ijk} the matrix elements of $S_i S_j$, then they are equal to

$$S^0_{ij0} = r_{ij}, \quad S^l_{ij0} = \Gamma^l_{ij}, \quad S^{n+1}_{ij0} = h_{ij}$$
$$S^0_{ijk} = \Gamma^a_{ki} r_{aj} + h_{ki}\beta_j, S^l_{ijk} = r_{ki}\delta^l_j + \Gamma^a_{ki}\Gamma^l_{aj} + h_{ki}\gamma^l_j, S^{n+1}_{ijk} = \Gamma^a_{ki} h_{aj} + h_{ki}\alpha_j$$
$$S^0_{ij(n+1)} = \gamma^a_i r_{aj} + \alpha_i\beta_j, S^l_{ij(n+1)} = \beta_i\delta^l_j + \gamma^a_i\Gamma^l_{aj} + \alpha_i\gamma^l_j$$
$$S^{n+1}_{ij(n+1)} = \gamma^a_i h_{aj} + \alpha_i\alpha_j$$

$(i, j, k, l = 1, \ldots, n)$. Finally each entry of the Maurer–Cartan relation is as follows:

MC1 $h_{ij} = h_{ji}, \quad \Gamma^k_{ij} = \Gamma^k_{ji}, \quad r_{ij} = r_{ji};$

MC2 $r_{ki,j} - r_{kj,i} + \Gamma^a_{ki} r_{aj} - \Gamma^a_{kj} r_{ai} + h_{ki}\beta_j - h_{kj}\beta_i = 0;$

MC3 $r_{ki}\delta^l_j - r_{kj}\delta^l_i - R^l_{kij} + h_{ki}\gamma^l_j - h_{kj}\gamma^l_i = 0;$

MC4 $h_{ki,j} - h_{kj,i} + \Gamma^a_{ki} h_{aj} - \Gamma^a_{kj} h_{ai} + h_{ki}\alpha_j - h_{kj}\alpha_i = 0;$

MC5 $\beta_{i,j} - \beta_{j,i} + \gamma^a_i r_{aj} - \gamma^a_j r_{ai} + \alpha_i\beta_j - \alpha_j\beta_i = 0;$

MC6 $\gamma^l_{i,j} - \gamma^l_{j,i} + \beta_i\delta^l_j - \beta_j\delta^l_i + \gamma^a_i\Gamma^l_{aj} - \gamma^a_j\Gamma^l_{ai} + \alpha_i\gamma^l_j - \alpha_j\gamma^l_i = 0;$

MC7 $\alpha_{i,j} - \alpha_{j,i} + \gamma^a_i h_{aj} - \gamma^a_j h_{ai} = 0.$

4. Normalization of the Fundamental Equation

We normalize the fundamental equation (1) as follows:

$$\Gamma^k_{ij} = \left\{ {k \atop ij} \right\}, \qquad r_{ki} = -\frac{1}{n-2} R_{ki} \tag{9}$$

where $\left\{ {k \atop ij} \right\}$ denotes the Christoffel symbol with respect to $\{h_{ij}\}$

$$\left\{ {k \atop ij} \right\} = \frac{1}{2} h^{ka}\Big(h_{ai,j} + h_{ja,i} - h_{ij,a}\Big) .$$

First, we remark that the equality $\Gamma^k_{ij} = \left\{ {k \atop ij} \right\}$ holds, if and only if $\{h_{ij}\}$ is parallel with respect to the connection ∇^Γ; $0 = \nabla^\Gamma_k h_{ij} = h_{ij,k} - h_{ia}\Gamma^a_{jk} - h_{ja}\Gamma^a_{ik}$.

Remark 1.

1. *The condition* $0 = \nabla^\Gamma h_{ij}$ *seems to be essential for us. We don't know whether it is possible to replace it by other conditions or not.*
2. *We normalize the fundamental equation* (1) *as* $\nabla^\Gamma h = 0$ *in place of* $\nabla^\Gamma h = f \otimes h$*, since the equation* (1) *is invariant under the conformal change of the metric.*

By taking the trace of (8), we get

$$0 = \mathrm{trace}_h\, r + \frac{R}{n-1} + \mathrm{trace}\,\gamma \tag{10}$$

where R denotes the scalar curvature of $\{\Gamma^k_{ij}\}$

$$R = h^{ab} R_{ab} .$$

Letting $l = i$ run from 1 to n in the equality (MC3), we get

$$\gamma^k_i = \mathrm{trace}\,\gamma\delta^k_i + h^{ka}R_{ai} + (n-1)h^{ka}r_{ai} . \tag{11}$$

From (8), (10) and (11), we get

$$r_{ki} = \left(\frac{R}{n(n-2)} + \frac{\mathrm{trace}_h\, r}{n}\right) h_{ki} - \frac{1}{n-2} R_{ki} \tag{12}$$

and also

$$\begin{aligned} \gamma^k_i &= \left(\frac{R}{n(n-1)(n-2)} - \frac{\mathrm{trace}_h\, r}{n}\right)\delta^k_i - \frac{1}{n-2} h^{ka}R_{ai} \\ &= -\frac{1}{n}\left(\frac{R}{n-1} + 2\,\mathrm{trace}_h\, r\right)\delta^k_i + h^{ka}r_{ai} . \end{aligned} \tag{13}$$

Remark that, even though the coefficients Γ^k_{ij} and r_{ij} are replaced by $\bar\Gamma^k_{ij} = \Gamma^k_{ij} + h_{ij}\mu^k$ and $\bar r_{ij} = r_{ij} + \lambda h_{ij}$, the fundamental equation (1) does not change at all. Therefore we equivalently normalize the fundamental equation by

$$r_{ij} = -\frac{1}{n-2} R_{ij} . \tag{14}$$

as well as (12). Then the equality $\mathrm{trace}_h\, r = -R/(n-2)$ holds, and also, by (10), the equality $\mathrm{trace}\,\gamma = R/(n-1)(n-2)$ holds. Therefore, from (13), it follows that

$$\gamma^i_j = \frac{R}{(n-1)(n-2)}\delta^i_j - \frac{1}{n-2}h^{ia}R_{aj} = \frac{R}{(n-1)(n-2)}\delta^i_j + h^{ia}r_{aj}\,. \quad (15)$$

By using (6) or (MC2), we get

$$\begin{aligned}\beta_i &= \frac{1}{n-1}\Big(\nabla^\Gamma_a(h^{ab}r_{bi}) - \nabla^\Gamma_i(h^{ab}r_{ba})\Big)\\ &= -\frac{1}{(n-1)(n-2)}\Big(\nabla^\Gamma_a(h^{ab}R_{bi}) - R_i\Big)\end{aligned}$$

where $R_i = \dfrac{\partial R}{\partial x^i}$. Since Bianchi identity implies $2\nabla^\Gamma_a(h^{ab}R_{bi}) = R_i$, we obtain

$$2\beta_i = \frac{R_i}{(n-1)(n-2)}\,. \quad (16)$$

Finally we remark that the condition $\nabla^\Gamma h_{ij} = 0$ implies

$$\alpha_i = 0\,. \quad (17)$$

5. Integrability and Conformal Flatness

The equation (1) is supposed to be normalized according (14) and (15). Then $\{\alpha_i, \gamma_{ij}, \beta_i\}$ satisfy (15), (16) and (17). We denote by ∇ the covariant derivative defined by $\{\Gamma^k_{ij}\}$.

Lemma 2. *If the Weyl conformal curvature tensor and the Cotton tensor of the metric $\{h_{ij}\}$ vanishes, then all equalities from (MC1) to (MC7) hold.*

Proof: (MC1), (MC5) and (MC7) hold, because h_{ij} and the Ricci curvature tensor R_{ij} are symmetric, and because β_i is given by (16).
(MC3) follows from the vanishing of the Weyl conformal curvature tensor

$$\begin{aligned}C^l_{ijk} =& R^l_{ijk} + \frac{1}{n-2}(R_{ij}\delta^l_k - R_{ik}\delta^l_j + h_{ij}R^l_k - h_{ik}R^l_j)\\ &- \frac{R}{(n-1)(n-2)}(h_{ij}\delta^l_k - h_{ik}\delta^l_j)\,.\end{aligned}$$

Actually, if we substitute for r_{ij} and γ^i_j the formulae from (14) and (15), we find that (MC3) is exactly the same as $C^l_{ijk} = 0$.

(MC2) and (MC6) follow from the vanishing of Cotton tensor

$$C_{ijk} = \nabla_i R_{jk} - \nabla_j R_{ik} + \frac{R_i}{2(n-1)} h_{jk} - \frac{R_j}{2(n-1)} h_{ik} \,.$$

For this, we note that (MC2) is equivalent to

$$\nabla_j r_{ki} - \nabla_i r_{kj} + h_{ki}\beta_j - h_{kj}\beta_i = 0$$

and that (MC6) is equivalent to

$$\nabla_j \gamma_i^l - \nabla_i \gamma_j^l + \beta_i \delta_j^l - \beta_j \delta_i^l = 0 \,.$$

Notice that $\alpha_i = 0$. If we substitute for r_{ij}, γ_j^i and β_i the formulae in (14), (15) and (16), we easily see that the equalities (MC2) and (MC6) comes from $C_{ijk} = 0$. □

6. Inner Product on the Solution Space

We suppose that the equation (1) is normalized as in Section 4. Then we have

$$\begin{aligned}(D\varphi \cdot \psi + \varphi \cdot D\psi)_i &= (\gamma_i^a \varphi_a + \beta_i \varphi)\psi + \varphi(\gamma_i^a \psi_a + \beta_i \psi) + D\varphi \cdot \psi_i + \varphi_i \cdot D\psi \\ &= \gamma_i^a(\varphi_a \psi + \varphi \psi_a) + 2\beta_i \varphi\psi + D\varphi \cdot \psi_i + \varphi_i \cdot D\psi\end{aligned}$$

and

$$\begin{aligned}(h^{ab}\varphi_a\psi_b)_i &= h^{ab}_{,i}\varphi_a\psi_b + h^{ab}(h_{ai}D\varphi + \Gamma^c_{ai}\varphi_c + r_{ai}\varphi)\psi_b \\ &\quad + h^{ab}\varphi_a(h_{bi}D\psi + \Gamma^c_{bi}\psi_c + r_{bi}\psi) \\ &= D\varphi \cdot \psi_i + \varphi_i \cdot D\psi + h^{ab} r_{bi}(\varphi\psi_a + \varphi_a\psi) \,.\end{aligned}$$

The subtraction of the above equations gives

$$\begin{aligned}(h^{ab}\varphi_a\psi_b)_i - (D\varphi \cdot \psi + \varphi \cdot D\psi)_i &= (h^{ab} r_{bi} - \gamma_i^a)(\varphi\psi_a + \varphi_a\psi) - 2\beta_i\varphi\psi \\ &= -\frac{R}{(n-1)(n-2)}(\varphi\psi_i + \varphi_i\psi) - 2\beta_i\varphi\psi.\end{aligned}$$

Here we used the normalization (14) and (15). As a conclusion, we find that

$$\langle \varphi, \psi \rangle = h^{ab}\varphi_a\psi_b - D\varphi \cdot \psi - \varphi \cdot D\psi + \frac{R}{(n-1)(n-2)}\varphi\psi$$

is a constant for any solutions φ, ψ of the fundamental equation. Thus we have proved

Lemma 3. *Let (p, q) be the signature of the pseudo-riemannian metric $\{h_{ij}\}$. On the solution space of the fundamental equation (1), the inner product defined by*

$$\langle \varphi, \psi \rangle = -D\varphi \cdot \psi - \varphi \cdot D\psi + \frac{R}{(n-1)(n-2)} \varphi\psi + \sum h^{ab} \varphi_a \psi_b \quad (18)$$

is nondegenerate and of signature $(p + 1, q + 1)$, provided (1) satisfies the integrability condition.

7. Main Theorem

The following theorem explains the relation between the fundamental equation (1) and the conformal geometry.

Theorem 1. *Suppose that the fundamental equation (1) with coefficients $\{h_{ij}, \Gamma^k_{ij}, r_{ij}\}$ is normalized as in (9), and that the pseudo-Riemannian metric h_{ij} is conformally flat. Then the equation (1) is integrable, and for any orthonormal basis $(\varphi^0, \cdots, \varphi^{n+1})$ of the solution space, the metric h_{ij} coincides with the pull back of the standard pseudo-Riemannian metric δ_{ij} by the map $\varphi : (x^1, \cdots, x^n) \mapsto (\varphi^1/\varphi^0, \cdots, \varphi^n/\varphi^0)$. Especially, in the case that $h_{ij} = \delta_{ij}$ and $\Gamma^k_{ij} = R_{ij} = 0$, the map φ is a linear transformation from $O(p+1, q+1)$.*
Conversely, let h_{ij} be the pull back of the standard metric δ_{ij} by a diffeomorphism $\psi = (\psi_1, \cdots, \psi_n)$, and let Γ^k_{ij} and R_{ij} be Weyl conformal connection and the Ricci curvature of h_{ij}. Then the fundamental equation (1) with coefficients $\{h_{ij}, \Gamma^k_{ij}, -R_{ij}/(n-2)\}$ is thus integrable, and the diffeomorphism φ obtained as above by an orthonormal basis of the solution space coincides with ψ up to a linear transformation from $O(p + 1, q + 1)$. Actually, $\{1, \psi^1, \cdots, \psi^n, \frac{1}{2}\sum_a (\psi^a)^2\}$ is an orthonormal basis of the solution space.

Proof: Let H be the $(n + 2) \times (n + 2)$-matrix defined by

$$H = \begin{pmatrix} R/(n-1)(n-2) & 0 & -1 \\ 0 & h^{ij} & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

Then the inner product (18) is given by

$$\langle \varphi, \psi \rangle = (\varphi, \varphi_1, \cdots, \varphi_n, D\varphi) H^t (\psi, \psi_1, \cdots, \psi_n, D\psi).$$

For a set of linearly independent solutions $(\varphi^0, \cdots, \varphi^{n+1})$ of the fundamental equation (1), let Φ denotes the Wronskian

$$\Phi = \begin{pmatrix} \varphi^0 & \cdots & \varphi^{n+1} \\ \varphi^0_1 & \cdots & \varphi^{n+1}_1 \\ \cdots & \cdots & \cdots \\ \varphi^0_n & \cdots & \varphi^{n+1}_n \\ D\varphi^0 & \cdots & D\varphi^{n+1} \end{pmatrix}.$$

By Lemma 3, we find that

$$x \mapsto {}^t\Phi(x)H(x)\Phi(x)$$

is a constant map.

Since our inner product is nondegenerate and of signature $(p+1, q+1)$, we can choose a set of solutions $\varphi^0, \cdots, \varphi^{n+1}$ so that

$${}^t\Phi(x)H(x)\Phi(x) = \begin{pmatrix} 0 & 0 & -1 \\ 0 & \delta_{ij} & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

Let I be the matrix of the right hand of the above. Since the matrix satisfies the equality $I^2 = I$, we find

$$\Phi I {}^t\Phi = \Phi I(I\Phi^{-1}H^{-1}) = H^{-1} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & h_{ij} & 0 \\ -1 & 0 & -R/(n-1)(n-2) \end{pmatrix}.$$

In view of the $(1,1)$-entry of this matrix, we see that these solutions satisfies

$$\sum_a (\varphi^a)^2 - 2\varphi^0\varphi^{n+1} = 0\,.$$

From this, it follows that the image of the map

$$x \mapsto [\varphi^0(x); \cdots; \varphi^{n+1}(x)] \in P^{n+1}$$

is contained in a hyperquadric of P^{n+1}. The $(i+1, j+1)$-entry reads as

$$\sum_{a=1}^{n} \varphi^a_i\varphi^a_j - \varphi^0_i\varphi^{n+1}_j - \varphi^0_j\varphi^{n+1}_i = h_{ij}\,.$$

On the other hand it holds that

$$\begin{aligned}
(\varphi^0)^4 \sum_a \left(\frac{\varphi^a}{\varphi^0}\right)_i \left(\frac{\varphi^a}{\varphi^0}\right)_j &= \sum_a \left(\varphi_i^0 \varphi^a - \varphi^0 \varphi_i^a\right)\left(\varphi_j^0 \varphi^a - \varphi^0 \varphi_j^a\right) \\
&= \varphi_i^0 \varphi_j^0 \sum_a (\varphi^a)^2 - \varphi^0 \varphi_i^0 \sum_a \varphi^a \varphi_j^a - \varphi^0 \varphi_j^0 \sum_a \varphi^a \varphi_i^a + (\varphi^0)^2 \sum_a \varphi_i^a \varphi_j^a \\
&= \varphi_i^0 \varphi_j^0 2(\varphi^0 \varphi^{n+1}) - \varphi^0 \varphi_i^0 (\varphi^0 \varphi^{n+1})_j - \varphi^0 \varphi_j^0 (\varphi^0 \varphi^{n+1})_i + (\varphi^0)^2 \sum_a \varphi_i^a \varphi_j^a \\
&= (\varphi^0)^2 \Big(- \varphi_i^0 \varphi_j^{n+1} - \varphi_j^0 \varphi_i^{n+1} + \sum_a \varphi_i^a \varphi_j^a \Big) .
\end{aligned}$$

Therefore we get

$$\sum_a \left(\frac{\varphi^a}{\varphi^0}\right)_i \left(\frac{\varphi^a}{\varphi^0}\right)_j = (\varphi^0)^{-2} h_{ij}$$

from which we find that the metric h_{ij} and the pull back of the standard metric by the map $x \mapsto (\varphi^1(x)/\varphi^0(x), \ldots, \varphi^n(x)/\varphi^0(x))$ are conformally related. □

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